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A
TREATISE
OF
MECHANICS,
THEORETICAL, PRACTICAL,
AND
DESCRIPTIVE.

BY OLINTHUS GREGORY, LL.D.

OF THE ROYAL MILITARY ACADEMY, WOOLWICH;
HONORARY MEMBER OF THE PHILOSOPHICAL SOCIETIES OF LONDON AND
NEWCASTLE-UPON-TYNE,
AND OF THE NEWCASTLE ANTIQUARIAN SOCIETY.

THE THIRD EDITION,
CORRECTED AND IMPROVED.

VOL. I.
CONTAINING
THE THEORY OF STATICS, DYNAMICS, HYDROSTATICS, HYDRODYNAMICS, AND PNEUMATICS.

——Philosophia mater omnium bonarum artium est. CICERO.

——Meta autem Scientiarum vera et legitima, non alia est, quam ut dotetur vita humana novis inventis, et copiis. BACON. *Nov. Org.*

LONDON:
PRINTED FOR F. C. AND J. RIVINGTON; G. WILKIE; J. RICHARDSON; LACKINGTON AND CO.; G. KEARSLEY; LONGMAN, HURST, REES, ORME, AND BROWN; J. TAYLOR; BALDWIN, CRADOCK, AND JOY; AND WALKER AND EDWARDS.

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TO THE RIGHT HONOURABLE

JOHN JOSHUA PROBY,

EARL OF CARYSFORT;

KNIGHT OF THE MOST ILLUSTRIOUS ORDER OF ST. PATRICK,
&c. &c. &c.

MY LORD,

ALTHOUGH the promotion of the arts and sciences has been ever deemed the proper province of the great and noble, yet it has not always been the good fortune of those who employ their time and exertions in the dissemination of knowledge, to meet with a patron who is at once distinguished by his rank, his talents, and his zeal in the encouragement of useful performances. I cannot, therefore, but deem it a high honour, and a source of considerable gratification, that I am permitted to present a Treatise of Mechanics to the public, under your lordship's protection: and I have been doubly solicitous that the work which you have thus been pleased to encourage, may not be found altogether unworthy the approbation of such a patron.

When I reflect upon your lordship's eminent and accurate acquaintance with many of the subjects discussed in the following sheets, my diffidence might be justly alarmed; but I throw myself upon the candour and kindness I have so often experienced: and though I am not vain enough to imagine that my undertaking will be found free from faults and errors, or that these can escape your notice; yet I have the satisfaction to know, notwithstanding, that whatever will stand the test of a correct and solid judgment, or may be found calculated to enlarge the stock of human knowledge, or more widely to diffuse an acquaintance with the principles of a branch of science of acknowledged utility, will not fail to receive your lordship's commendation and support.

Allow me, my lord, publicly to thank you for the numerous favours you have conferred upon me in different periods of my life, and to subscribe myself, with unfeigned respect, and the sincerest gratitude,

My lord,

Your lordship's most obliged

and most obedient Servant,

OLINTHUS G. GREGORY.

PREFACE.

IT has of late been a too prevailing opinion in this country, that a man may become celebrated as a natural philosopher, and be very successful in the application of his knowledge to practical purposes, particularly in the construction of machinery, while he is completely unacquainted with the principles of mathematics. Among a variety of unpleasant consequences which have resulted from this erroneous opinion, may be reckoned, the rapid decline of the mathematical sciences in Britain, the dissemination of superficial and vague notions on physical topics, and the absolute necessity of having recourse to foreign publications for profound and extensive information on those subjects, which we should have supposed, had not experience convinced us to the contrary, Englishmen would have been proud to cultivate; since they were first placed upon an unshaken basis in the Principia of our own countryman, the illustrious Newton. It has been asserted by a competent judge, Dr. Robison, and it is mortifying to reflect that the assertion is true, "that while the continent has supplied us with most elaborate and useful treatises on various articles in physical astronomy, practical mechanics, hydraulics, and optics, there has not appeared in Britain half a dozen treatises worth consulting for the last forty years." We have, it is readily admitted, many popular works on mechanics, hydrostatics, &c. in which the principles are exhibited, and their application to the useful purposes in life illustrated by examples; but since their authors have scarcely ever aimed at demonstration, those who rest satisfied with the information they can gather from these treatises, can at best acquire but a kind of second-hand knowledge of unproved principles loosely floating in the mind; and must generally hesitate in every case presenting a novel aspect, for want of that confidence in principles and results which the irresistible evidence accompanying the deductions of pure science alone can inspire. I would not be understood as intending to cast any censure upon the authors of familiar treatises on scientific subjects: I only wish it to be recollected that these are much fitter for children than for men; that they may be useful as introductory books for pupils, but of little or no immediate service in the active concerns of a more advanced state; and that, because they merely excite a thirst for knowledge which they cannot gratify. Nor would I wish to insinuate that there are not in our language some excellent works on the *theory* of mechanics; but I cannot help being apprehensive that most of these (chiefly on account of the small encouragement likely to be given them) are too confined in their object, and for that reason fail in being so beneficial to the public, as the talents of their authors might otherwise lead us naturally to expect.

For some years I have seen (or *thought* I have seen), and often regretted, that a forbidding distance and awkward jealousy seem to subsist between the theorists and the practical men engaged in the cultivation of mechanics in this country: and it is a desire to shorten this distance, and to eradicate this jealousy, that has been a principal stimulant in the execution of the following performance. I have by long habit, combined perhaps with early acquired prejudices, been much delighted with the investigations of theorists: but while I prize the deductions of sound theory as highly as any person, and rest as firmly upon them; yet am I desirous not to forget, that, as all general principles imply the exercise of abstraction, it would be highly injudicious not to regard them in thei
practical applications as approximations, the defects of which mu
be supplied, as indeed the principles themselves are deduced, fr
experience. Habits of abstraction and theorizing may be car
to excess; and crude experience without reflection will never
productive of essential good. But, as an eminent philosoph
(for whose talents and virtues I entertain great respect) rem
" Care should be taken to guard against both these extreme
" to unite habits of abstraction with habits of business, in

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* Professor Dugald Stewart; Elements of the Philosophy of the Huma
p. 231, &c.

“ manner as to enable men to consider things either in general or
“ in detail, as the occasion may require. Whichever of these
“ habits may happen to gain an undue ascendant over the mind,
“ it will necessarily produce a character limited in its powers, and
“ fitted only for particular exertions.—When theoretical knowledge
“ and practical skill are happily combined in the same person, the
“ intellectual power of man appears in its full perfection, and fits
“ him equally to conduct with a masterly hand the details of ordinary business, and to contend successfully with the untried difficulties of new and hazardous situations. In conducting the
“ former, mere experience may frequently be a sufficient guide:
“ but experience and speculation must be combined together to
“ prepare us for the latter.” ‘ Expert men,’ says Lord Bacon,
‘ can execute and judge of particulars, *one by one;* but the *general*
‘ counsels, and the plots, and the marshalling of affairs, come best
‘ from those that are learned.’

Admitting the truth of these observations—and their truth I think cannot well be denied—it will thence follow, that theoretical and practical men will most effectually promote their mutual interests, not by affecting to despise each other, but by blending their efforts: and further, that an essential service will be done to mechanical science, by endeavouring to make all the scattered rays of light they have separately thrown upon this region of human knowledge converge to one point. In conformity with these ideas, I have undertaken a task, attended, I am aware, with so many difficulties, that even failure will not be disgraceful; while, on the other hand, success, if my labours should fortunately be crowned with it, will be doubly gratifying; as I shall then flatter myself with having contributed in some measure to the union of accurate theory and judicious practice, and thus, ultimately, to the promotion of arts and manufactures.

The Treatise of Mechanics I now presume to lay before the public is comprised in two large volumes, besides a volume of plates. Of these the first volume is devoted chiefly to the theory: the second is practical and descriptive. The theoretical part is divided into five books, and these are subdivided into chapters, as the nature of the several subjects seemed to require.

Book I. is appropriated to the subject of STATICS, and is subdivided into six chapters. It commences, as indeed do all the books, with definitions and preliminary remarks; which are succeeded by a statement and illustration of the Newtonian laws of motion and rest, which are assumed as fundamental principles to guide our subsequent enquiries. The nature of statical equilibrium, with the composition and resolution of forces, are then discussed; forces being considered, 1. as disposed in one plane, and concurring in the same point: 2. as directed to one point, but not confined to one plane: 3. as situated in one plane, but applied to different points of a body: 4. as not confined to one plane, and directed to various points of a body. This portion of the work occupies rather more space than has been commonly assigned to it in treatises of mechanics; but this circumstance will, I hope, be found to facilitate the application of the parallelogram of forces to the investigations which follow. The remaining chapters in this book treat of the centre of gravity, and the centrobaryc method; the simple machines, or, as they are usually called, the mechanical powers; the strength and stress of timber and other materials; and the equilibrium, tension, and pressure of cords, arches, and domes. The minuter topics connected with these general points of enquiry cannot well be specified here.

The second book relates to DYNAMICS. It comprehends six chapters; treating of, motion, uniform and variable; the descent and ascent of heavy bodies in vertical lines; the motion of projectiles, with observations on ricochet-firing; descents along inclined planes and curves; the vibrations of pendulums, and the curve of swiftest descent; central forces; the rotation of bodies about fixed axes, and in free space, with theorems relative to the centres of oscillation, gyration, percussion, spontaneous rotation, &c.; the physico-mathematical theory of percussion, nearly as first delivered in Don George Juan's Examen Maritimo; and the motion of machines, and their maximum effects; closing with some remarks that shew in what points of view machines ought to be considered by those who would labour beneficially for their improvement.

Book III. is devoted to the subject of HYDROSTATICS. It is divided into four chapters—on the pressure of non-elastic fluids; the determination of the specific gravities of solid and fluid bodies; the construction of hydrometers or areometers; the equilibrium, stability, and oscillations of floating bodies, with a particular reference to ships and canal boats; and, on the phenomena of attraction in capillary tubes. This book contains an extensive, and, I trust, correct table of specific gravities.

In the fourth book I have given some of the most useful propositions and theorems relative to HYDRODYNAMICS. But as this is a very complex and intricate subject, in which the most elaborate theory can do but little, independent of numerous experiments, it seemed prudent, rather than to delude the student with a display of mathematical accuracy where it is so far from being attained, to curtail the theoretical part with a view to the introduction of some valuable experimental results. This book comprises, then, only four chapters. 1. On the discharge of fluids through apertures in the bottom and sides of vessels, and on spouting fluids: 2. an account of experiments made by different philosophers (as Bossut, Venturi, Eytelwein, Young, Vince, &c.) on the discharge of water through apertures and tubes; and the practical deductions from those experiments: 3. on the effect of water upon the motion of water-wheels: 4. an account of Mr. Smeaton's experiments on water-wheels, and some of those of M. Bossut.

The fifth book, which terminates the first volume, is on PNEUMATICS, and contains six chapters. The subjects stated and discussed in this book are, the equilibrium of elastic fluids; the measurement of altitudes with the barometer and thermometer; the motion of air when the equilibrium of pressure is removed; the theory of air-pumps, and pumps for raising water; the resistance of fluids to bodies moving in them; and results of experiments on the resistance of fluids.

The second volume of this work, which, as before mentioned, is chiefly practical, commences with an introduction occupying 84 pages, and comprising general remarks, rules, and directions, —on the construction and simplification of machinery; on rotatory, rectilinear, and reciprocating motions; on bevel-geer, and proportioning the number of teeth; on friction and the rigidity of cords, with the experiments of Vince, Coulomb, &c. and an example of the power of the capstan, allowing for friction and the stiffness of cords; on water and wind as movers of machinery, with Smeaton's rules relative to windmills; on the strength of fired gun-powder; on steam as a mover of machinery, with the theorems and results of Bettancourt and Dalton; and, on the strength of men and horses, according to the best and most accurate observations. The remaining part of this volume is appropriated to the description of a variety of mechanical contrivances, in number exceeding an hundred and fifty. These are arranged alphabetically, that they may be consulted with most convenience; and several of them relate to machines and instruments which have never before been publicly described; and for accounts of which I am indebted to some of our most celebrated civil engineers, whose names will appear with the respective articles. A general enumeration in this preface would be tiresome, especially as the articles are separately mentioned in the Table of Contents to the second volume. It may, however, be proper just to observe, that the greater part of the contrivances there described are such as are highly useful in various arts and manufactures, and the most important purposes of human life: among them will be found descriptions of air-pumps, balances, bark-mills, barometers, bellows, clocks, cranes in great variety, file-cutters, fire-engines, flax-mills, flour-mills, foot-mills, hand-mills, hydraulic engines of various kinds, hygrometers, kneading-mills, lathes, and the most curious turning apparatus, locks, oil-mills, ordnance boring machines, contrivances for what are called parallel motions, pendulums, pile-engines, planing machinery, presses, and pressure-engines, several pumps, pyrometers, Ramsden's dividing machines, saw-mills, scapements, steam-engines, curious and accurate steel-yards, stream-measurers, telegraphs, thermometers, thrashing-mills, tide-mills, watches, water-mills, weighing apparatus, wind-mills, yarn-mills, &c. In some instances where the subject seemed very important, and not admitting of sufficiently copious discussion, among so many other particulars, I have inserted catalogues, arranged chronologically, of the chief writings on that respective subject, whether smaller dissertations, or full and extensive treatises: these catalogues may be seen under the words *hydraulic-engines*, and *mills*. In a few cases too, where the theory of a machine had not been correctly exhibited in any English work with which I am acquainted, I have given such theory in the second volume, for the sake of more readily deducing the most useful practical results: this has been dor with respect to *Barker's mill*, the *Persian-wheel*, and *Archimedes's screw*.

In the composition of the first volume of this treatise, I have derived material assistance from the labours of several of my predecessors in this department of science; though I have not, perhaps, so frequently cited my authorities as some readers may be apt to expect: but this will not, I trust, on consideration, be thought a culpable omission; for although I have not, for example, ascribed to Prony what I found in succession in the writings of Varignon, Belidor, Bezout, and D'Alembert, nor to Parkinson, or Atwood, what had previously appeared in the writings of Galileo, Wilkins, Wallis, Desaguliers, or Emerson, esteeming whatever I found, in such circumstances, as common property, to be adopted without hesitation; yet, in all cases where I could speak confidently of the *original* author, and particularly where the matter quoted had been but seldom published, I have not failed to make the corresponding reference. As to the second volume, it is professedly a compilation; and I have no other merit to claim respecting it, than that of having employed much labour and pains in consulting a great many volumes of journals, transactions, acts, encyclopedias, theatres of machines, &c. published in England, France, and Germany, and having selected from these numerous and often voluminous works, such particulars as were most likely to be serviceable to my countrymen, when presented to them (separate from every thing extraneous) in a convenient though copious single volume.

In the descriptive part I have but seldom given accounts of machinery for which the inventors have taken out *patents*. In this measure I have been guided solely by views of utility: when a patent machine appeared to exhibit some peculiar ingenuity in its construction, or when the specification developed some general methods which are not merely serviceable with respect to the inventions to which they are now applied, but may after the expiration of the patent be advantageously appropriated to other machinery, I have thought it right to insert a description of such contrivances; taking care at the same time to mention the exclusive right secured for the term of the patent. An indiscriminate account of what are called *inventions*, in this fruitful age for the revival of old contrivances, would have added to the bulk of the work; but it must necessarily have operated as a dead weight to sink its respectability.

It may, probably, be imagined by some, that considering the advantages for that purpose, with which my situation at Woolwich furnishes me, I should insert descriptions of the various kinds of machinery used in the *artillery* service. This, however, I have not done: chiefly because such descriptions, by Muller and other respectable authors, are already easily procured in works appropriated exclusively to that important object, and where the accounts are given with far greater copiousness than they could be in a performance where so many other objects are handled.

In the exposition of the theory I have generally proceeded by a series of connected propositions and dependent corollaries; those in each chapter having a manifest relation to each other, and flowing naturally from the same source: and I have at the same time endeavoured to attain a just medium between that fatiguing prolixity of detail which leaves nothing to be struck out by the ingenuity of the pupil, and that obscurity which commonly results from the suppression of intermediate ideas. I have, therefore, occasionally introduced such general scholia and more limited remarks, as seemed calculated to clear up a point in the history, or a doubt in the principle, or to suggest an application: and it was my first intention to have these printed in a different type from the propositions and corollaries; but I relinquished that design, on finding that it would much enhance the price of these volumes, a circumstance which I have been all along solicitous to prevent.

I fear many persons who may take up this work will be dissatisfied with it on account of the frequent introduction of the fluxional analysis into the theoretical part: this dissatisfaction is engendered in some by a prejudice to which I have already adverted in the beginning of my preface; and in others, by an undue attachment to the geometry of the ancients. I would beg the attention of the latter class of objectors to a few observations. I am very ready to allow that in the study of geometry, the diagrams to which our attention is directed serve as a continual check upon our reasoning powers; they, besides, exhibit to the senses a variety of relations which the language of algebra by no means suggests, and which it is perhaps too general to express: still, as every geometrical investigation *may* be expressed algebraically, and as, moreover, there have been excellent geometricians who were blind and never *saw* a geometrical figure; the circumstance of the diagrams does not constitute so essential a difference between geometry and algebra as some may be apt to suppose. In many physical investigations, it is true, where a geometrical process can be applied, it carries with it a degree of luminousness certainly not to be exceeded by a process conducted according to the modern analysis; and it must further be acknowledged, that this analysis, without a cautious exercise of the judgment, and a frequent recurrence to first principles, may terminate in an absurd conclusion: yet, on the other hand, it must be recollected, that physical enquiries conducted geometrically by the most eminent men, have sometimes led to very erroneous results; and, that in numerous cases, the geometrical method can be applied with difficulty, if at all, in which an investigation carried on by the modern analysis has issued in some very important discoveries. If, with regard to the fluxional calculus in particular, it be recollected that it was invented by *an Englishman* who will be the immortal honour of his country, and that this is neither the least brilliant nor the least valuable of his discoveries, it cannot but be considered as singular, that many among his countrymen, who profess themselves the friends of science, should discountenance the introduction of his analysis into our scientific works. Philosophical men, when contemplating the progress of knowledge, have been delighted to remark that "the discoveries which in one age were confined to the studious and enlightened few, become in the next the established creed of the learned; and, in the third, form part of the elementary principles of education;" how completely different from this the progress must be, when the discoveries of a century back are disregarded, not because of their being useless or obsolete, but because of their *difficulty*, is too palpable to need more than an allusion, and cannot be dwelt upon without regret. If what has been here advanced should not be thought sufficient to justify the occasional adaption of the fluxional analysis (I do not contend for the exclusive use of it), let it be recollected that, in our best treatises on mensuration, gauging, &c. it is allowed, and for this sufficient reason, because such treatises would be very deficient, imperfect, and confined in their utility without it: unless, therefore, a similar occasion will not justify a similar mode of proceeding in a treatise of mechanics, I shall hope to be excused, even by those I am now addressing, when it is considered that some of the most important branches of mechanical enquiry, and especially great part of what relates to the *maximum* effects of machines, must either be entirely omitted, or treated by the fluxional calculus, or some modern calculus analogous to it. In consistency with the preceding observations, I have followed a kind of mixed course, and have deduced the chief truths commonly developed in our elementary books of mechanics, either after the geometrical manner, or by introducing the algebraical notation in aid of the deductions from geometry; and in the more curious and abstruse enquiries which occur in various parts of the work, I have had recourse to the fluxional analysis, not for the sake of displaying my mathematical attainments (being well aware how scanty those attainments are), but in order to render the treatise in some degree complete: thus giving it such latitude and comprehensiveness, that those who consult it may draw from it some such notions of the science of mechanics in general as may be learned respecting an existing structure, by examining a model shewing its several parts in due proportion.

It may happen, after all, that the arguments adduced in favour of the method I have chosen may fail in their impression, on account of their not being stated with sufficient force: let me then fortify my opinion, and shelter this part of my plan from censure by high mathematical authority. Mr. *Simpson*, in the preface to his TRACTS, speaking of the problems in mechanics and physical astronomy, says, " In treating of this subject I have chiefly adhered to " the *analytic method* of investigation, as being the most direct and " extensive, and best adapted to these abstruse kinds of speculations. " Where a *geometrical* demonstration could be introduced, and " seemed preferable, I have given one: but, though a problem " *sometimes*, by this last method, acquires a degree of perspicuit " and elegance not easily to be arrived at any other way; yet " cannot be of the opinion of those who affect to shew a dislike " every thing performed by means of symbols and an algebraic " process; since, so far is the synthetic method from having t " advantage in all cases, that there are innumerable enquiries in

" nature, as well as in abstracted science, where it cannot be at " all applied to any purpose. Sir Isaac Newton himself (who per- " haps extended it as far as any *man* could) has, even in the most " simple case of the lunar orbit (*Princip*. b. 3. prop. 28.), been " obliged to call in the assistance of algebra; which he has also " done in treating of the motion of bodies in resisting mediums; " and in various other places. And it appears clear to me, that it " is by a diligent cultivation of the *modern analysis* that *foreign* " *mathematicians* have, of late, been able to push their researches " further in many particulars than Sir Isaac Newton and his fol- " lowers here have done: though it must be allowed, on the other " hand, that the same neatness and accuracy of demonstration is " not every where to be found in these authors; owing, in some " measure, perhaps, to too great a disregard for the geometry of " the ancients."

When treating of the theory of mechanics, I have not attempted to explain the *nature* of gravity, impulse, or the other sources of the motion of bodies: but this, I hope, will not be considered as a defect, when it is recollected that the general definition of the term *force* in a mechanical sense, supersedes the necessity of enquiring into the essence of the various kinds of forces which may operate upon matter. Those who carefully contemplate the process of that gradual refinement of language, which results from the necessary demands occasioned by the progress of civilisation, will see how requisite it is to appropriate terms originally of a laxer or of a grosser signification, to some peculiar modification of thought; and hence, that such words as *power* and *force*, primarily used to denote animal energy, are now, by a natural extension, grounded upon an obvious analogy, employed to express *efficiency* in general. In the philosophical acceptation, then, we define *force or power to be that*, whatever it be, *which causes a change in the state of a body, whether that state be rest or motion:* and this definition does not require entering into any metaphysical disquisitions relative to the nature of causes, or the connection of cause and effect: that every effect is brought about by some cause, is a truth which I think none will be disposed to deny; but what is the agency, or where it actually resides, we can seldom know, except perhaps in the case of our own voluntary actions. It is not, then, the business of the mechanist, strictly speaking, to enquire into the *modus operandi*: we learn from universal experience, that the muscular energy of animals, the operation of gravity, electricity, pressure, impact, &c. are sources of motion, or of modifications of motion; and hence, without pretending to know the essence of either of these, we do not hesitate to call them mechanical forces; because it is incontrovertible that bodies exposed to the free action of either, are put into motion, or have the state of their motion changed. Forces, therefore, being known to us only by their effects, can only be measured by the effects they produce in like circumstances, whether those effects be creating, accelerating, retarding, deflecting, or preventing motions: and it is by comparing these effects, or by referring them to some common measure of ready appreciation, not by ascertaining the essential nature of any forces, that mechanics is made one of the mathematical sciences.

Besides, what is meant by the *nature* of any thing? As we are ignorant of its essence, or what makes it that thing and no other thing, we must content ourselves with the discovery of its qualities or properties: and it is the assemblage of these which is commonly called its *nature*: yet this is very inaccurate, since these are only the *consequences* of the essence. Hence, we can give no definition of even the *simplest* of things which comprehends its real essence; and it is justly observed by Locke *, that *nominal essences* only, which are the creatures of our own minds, can be properly comprehended by us, or can be properly defined; and even of these there are many too simple in their nature to admit of definition. The various properties of a body which fall under our observation, offer themselves to our notice simultaneously; while we have the power of an intellectual separation, by which we can form different branches, each constituting a particular object of research: thus, the extension, the materiality, the impenetrability, the colour, &c. are coexistent qualities of a body; but our minds being too limited to examine these all at once, we divide and assort them, and consider them singly, or in kindred groups; hence arise the sciences of geometry, mechanics, chromatics, &c. It would seem, then, that to the weakness of our reason is to be ascribed the origin of the sciences: and this, if it is not very gratifying to our pride, may teach us at least to moderate our expectations when we attempt to investigate those ultimate questions, which, sooner or later, occur in all objects of human enquiry. I trust it will not be supposed that I mean to discourage an examination of physical and metaphysical problems; on the contrary, I am of opinion, that by looking attentively into such subjects, our minds are opened and enlarged: an enquiry, properly conducted, will lead to at least one of two good effects; though it should not make us more knowing, it may make us modest; though it may not preserve us from error, it may from the *spirit* of error; and may teach us to be aware of dogmatism, when so much labour may terminate in so much uncertainty.

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* This great philosopher, so far from conceiving, as many of the moderns have done, that the nature of a mechanical operation or event was sufficiently explained by reducing it to some supposed case of impulsion, considered even the connexion between impulse and motion as inexplicable, as a *conjunction* gathered solely from experience, and not as a consequence deducible from any reasoning *à priori*. The following passages on this subject are highly deserving of the attention of the student. "Another idea we have of body, is the power of "communicating motion by impulse; and of our souls, the power of exciting "motion by thought. These ideas, the one of body, the other of our minds, "every day's experience clearly furnishes us with: but, if here again we enquire "how this is done, we are equally in the dark. For, in the communication of "motion by impulse, wherein as much motion is lost to one body as is got to the "other, which is the ordinariest case, we can have no other conception but of "the passing of motion out of the one into another; which, I think, is as obscure and inconceivable as how our minds move or stop our bodies by thought, "which we every moment find they do." Again: "Sensation convinces us "that there are solid extended substances; and reflection, that there are think-

The preceding observations may serve to suggest the advantage of a mathematical process, in which we have a clear perception, not of the things themselves, but of the agreement or disagreement of our ideas at every step of the enquiry, over other processes in which there is much that is conjectural, or gratuitously assumed, and in which a mass of error may be produced in the result, proportioned to the length to which the chain of reasoning is extended:

" ing ones: experience assures us of the existence of such beings; and that the " one hath a power to move body by impulse, and the other by thought. If " we would enquire further into their nature, causes, and manner, we perceive " not the nature of extension clearer than we do of thinking. If we would explain them any further, one is as easy as the other; and there is no more " difficulty to conceive how a substance we know not should by thought set " body in motion, than how a substance we know not should by impulse set " body into motion." Locke's Essay, book ii. chap. 23.

although in the mathematics the relations of quantity are extremely various, yet, as they are all along susceptible of exact mensuration, these, if any, are the sciences in which reason may triumph, while in others its trophies are comparatively inconsiderable. On these accounts I have been careful in the composition of the following sheets, to associate nothing with the theory of mechanics but the results of important experiments, and the descriptions of ingenious and useful machinery; omitting all those discussions which, in my judgment, appeared to be merely conjectural and speculative.

Hitherto I have been using the language of justification, with which I have, unawares, detained the reader much longer than I at first intended: it is time to adopt the tone of apology. In the execution of this work, I have experienced the wide difference between forming a project and accomplishing the object I had in view: although I have sometimes pleased myself with imagining I have struck out a more satisfactory illustration, or a more convincing demonstration, than has been usually given, I have too frequently found it necessary to rest below that perfection which I had previously fancied within my reach. I am conscious of some deficiencies and of some mistakes in arrangement, especially in the second volume, which were discovered too late to be now rectified. I hope, however, none of them will be found of much consequence. Other mistakes, it may be apprehended, will be detected by the acute reader; I shall rejoice if these are but few and of slight importance. Every exertion has been made on my part to render the performance correct, perspicuous, and useful; and if the *candid* examiner should find, that in the main it possesses these properties, I shall hope to escape severe censure for those imperfections which my want of abilities, not want of care, may have permitted to meet the eye of the public.

Royal Military Academy, Woolwich.
December, 1805.

*** In this third edition the work has been carefully revised, and in various places considerably enlarged. Some new theoretical propositions are added, and several descriptions of new mechanical inventions. M. Biot's researches into the dilatations of liquids, which did not arrive early enough for insertion in the 5th book of vol. i. are given in vol. ii. as a supplement to the article THERMOMETER, to the improvement of which they tend.

July, 1815.

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MECHANICS.

BOOK I.

STATICS.

Introductory Definitions and Remarks.

1. MECHANICS, the subject of this work, is a mixed mathematical science, which treats of forces or powers, and their effects upon bodies, either with or without the intervention of machines.

The *theory* of Mechanics, next to that of Geometry, is justly reckoned the most certain; for next to that it proposes the fewest properties of matter as objects of contemplation. Generally speaking, this science proposes for consideration, *time, force*, or *power*, and the following properties of bodies: viz. *extension, figure, impenetrability, mass, mobility*, and *inertia.* The application of this science to the various practical purposes in human life, leads to many other branches of enquiry; such as, the nature of machinery, the advantages, and disadvantages, of different materials, the effects of friction, &c. Each of which will be brought under discussion, after the several parts of the theory are laid down.

2. *Matter* is a term by which we denote that substance of which every thing our senses perceive is imagined to be composed. So far as relates to Mechanics, its essence may be regarded as consisting in *extension, impenetrability*, and *inertness.*

3. *Body* is such a collected quantity of matter as is palpable, or obvious to some of the senses. We say that a body is *solid*, when it is composed of particles, or moleculæ, so adhering the one to the other that they cannot be separated without effort: such are metals, stone, wood, &c. The term *fluid* we apply to such substances as are composed of particles adhering very slightly, and which, yielding to any small effort, are easily moved among each other: such are water, wine, air, &c.

All bodies, it is manifest, are extended, and therefore are found existing under figure, or shape, which is the boundary of extension. It appears also essential to matter, that it prevents all other substances *of the same kind* from occupying its place; and that it requires the exertion of something *ab extra* to remove it from its situation, or to change its state. The former of these properties is called *Solidity*, or *Impenetrability*: the latter, *inertness* or *inertia*; of which we shall have occasion to say more. (18.)

4. All bodies are *divisible* either in reality or in imagination. By art they may be divided into parts of surprising minuteness; as by the operations of grinding, hammering, wiredrawing, &c. In some chemical solutions this may be carried farther still. And the constituent particles of the solids and fluids of the myriads of animalculæ which are discovered by the microscope, must be small beyond conception: so that we can scarcely help imagining the capacity of divisibility to be without limit. What, however are called mathematical demonstrations of the *infinite* divisibility of matter, fail in their object: they merely prove the infinite divisibility of extension, in a geometrical sense. It is probable that, with respect to matter, the actual division may admit of being carried to a different extent in different bodies: but the inquiry is of little, if any, consequence to the mechanician. Those who wish to peruse a copious dissertation on this topic are referred to *Keil's Introduct. ad Ver. Physic. Lect.* 3, 4, *and* 5.

5. *Space*, is commonly defined as the order of things which coexist: in this sense it is a mere abstract idea, arising from our notion of the actual or possible situation of things amongst themselves. We rather choose to call *space* an extension considered as without bounds, immoveable but penetrable by matter: this may be regarded as *absolute space*.

6. *Relative space* is that variable dimension or measure of absolute space which our senses define by its relation to bodies within it.

7. *Place*, or *absolute place*, is that part of infinite space which a body possesses. *Relative place* is the space a body occupies considered with relation to other objects.

8. *Mobility* is a property of body, by which it is capable of being transferred from one place to another, or of existing in different parts of space.

9. All bodies are porous: whence, together with the extreme minuteness of their particles, it happens, that fluids will insinuate themselves into all bodies; that sometimes a mixture of two fluids will be less in bulk than when they are separate; and that the same bulk may contain different quantities of matter, or *masses*.

10. *Density*, strictly speaking, denotes vicinity or closeness of particles. But in mechanical science it is used as a term of comparison expressing the proportion of the number of equal moleculæ, or the quantity of matter in one body, to the number of

equal moleculæ in the same bulk of another body: density, therefore, is directly as the quantity of matter, and inversely as the magnitude of the body.

Since it may be shewn experimentally that the quantity of matter in different bodies is proportional to their weight, it will thence follow that the density of any body is directly as its weight, and inversely as its magnitude: or, the inverse ratio of the magnitude of two bodies, having experimentally equal weights (in the same place) constitutes the ratio of their densities.

11. *Motion* is a simple idea, and therefore admits not of definition: when we say that it is a continual and successive change of place, we describe it in a periphrasis, by its sensible effects. Or, by another circumlocution, motion may be described as that state of a body which is not consistent with its continuance in the same place; or in which it is not, in two successive instants of duration, at the same distance from divers fixed points in space: this state is opposed to that of *rest*.

Thus, conceiving in space three planes in known and fixed positions, not parallel to each other; if from any material point perpendiculars are demitted on these three planes, we say the point is in motion when it does not constantly retain these distances, but when in *any* two successive instants, one of these perpendiculars changes its magnitude.

12. The motion of bodies is considered either as *absolute* or *relative*. A body is said to be in *absolute motion* while it is actually passing from one point in fixed space to another; and to be in *relative motion* while its position is varying with respect to other bodies.

It is obvious that these two kinds of motion can only coincide when the bodies, to which the reference is made, are fixed: in other cases a body in relative motion may or may not be in absolute motion. The determination of the absolute motions, by means of observations on the relative motions, is always a matter of great difficulty, nay, is generally absolutely impossible. Thus, when a ball is discharged from a piece of ordnance, it is possible, by means of the ballistic pendulum, and other contrivances of ingenious men, to ascertain its relative motion, that is, its motion with respect to that place on the earth's surface from which it is projected; but in order to determine its absolute motion, the diurnal and annual motions of the earth about the sun, and probably the motion of that luminary about the centre of some more extensive system, must be taken into the account; so that on the whole this apparently simple enquiry becomes sufficiently complex to baffle the proudest efforts of human intelligence.

13. The consideration of motion necessarily involves that of *time*; for no motion can be instantaneous.

14. *Absolute time* is a portion of duration whose quantity is only known by a comparison with another portion; and consequently the relation between any two parts of absolute time is not to be discovered. *Relative time* is a part of duration which elapses during any motion of body, or any succession of external appearances.

There is a striking analogy between the affections of space and time: hence it is, that time may be represented by lines and measured by motions. Hence also, we say that an *instant* is the boundary between any two contiguous portions of time, as a point is the boundary of any contiguous lines. A *moment* is any small portion of time. To render time susceptible of mathematical discussion, it must be conceived as measureable; and to this end it is necessary to recur to some event which we imagine uniformly requires equal times for its accomplishment. We are furnished with such an event in the complete rotation of the earth upon its axis, which marks out a natural day as an apt and obvious *unit* of time. This is divided into 24 equal parts called hours, each of these into 60 equal parts called minutes, and each of these again into 60 equal parts called seconds. A *second* is the unit of time generally employed in mechanical disquisitions.

15. *Velocity*, or *celerity*, is that affection of motion which determines its quantity: it is the name expressing the relation between the space described by a moving body and the time which elapses during its description; and it is measured by the space *uniformly described* in a given time.

A body is said to move with a *uniform, accelerated,* or *retarded* velocity, according as its rate of motion continues the same, increases or decreases: when the increase or decrease of velocity is the same in any equal times, the acceleration or retardation is said to be uniform: and when this increase or decrease of velocity itself increases or decreases in any equal times, the acceleration or retardation increases or decreases in the same ratio. These circumstances will be brought more fully into consideration as we proceed.

16. The *direction of a motion* is the position of the line along which it is performed: thus if a body move from a point A to another point B along the straight line which joins these points, AB is called the direction of the body: if the body move from B to A along the same right line, BA is its direction. If a body move along a curve line, its *direction* is continually changing; it may, however, in any given point be regarded as coinciding with the tangent to the curve at that point.

17. *Force* or *power*, in a mechanical sense, is that which causes a change in the state of a body, whether that state be rest or motion.

We speak here of proximate causes, for it is not the business of mechanics to search into the essential and hidden causes of motion. The enquiry whether they are material or spiritual may exercise the talents of ingenious speculatists, and may, perhaps, be of some importance in a moral point of view; but certainly forms no part of the principles of mechanical science. The muscular power of animals, as likewise pressure, impact, gravity, electricity, &c., are by us looked upon as forces, or sources of motion; for it is an incontrovertible fact that bodies exposed to the free action of either of these are put into motion, or have the state of their motion changed. All forces, however various, are measured by the effects they produce in like circumstances; whether the effects be creating, accelerating, retarding, or deflecting motions: the effect of some general and commonly observed force is taken for unity; and with this any others may be compared, and their proportions represented by numbers or by lines: in this point of view they are considered by the mathematician; all else falls within the province of the universal philosopher or the metaphysician. When we say that a force is represented by a right line AB, it is to be understood that it would cause a material point situated at rest in A, to run over the line AB (which we name the *direction* of the power) so as to arrive at B, at the end of a given time; while another power would cause the same point to have moved a greater or less distance from A in the same time.

18. Among other forces it has been customary to speak of the *vis inertiæ*, or *inert force* of matter; applying the term to that property of bodies by which they tend to retain their present state (3.), or are indifferent to motion or rest. But while we admit that much of the language which relates to powers, forces, actions, &c., is metaphorical, we must object to such use of it in the present case; this property being improperly called a force: 1st. Because were it actually such it must be of some definite quantity in a given body, and therefore an impressed force less than that would not move the body; whereas any impressed force, however small, will move any body however great. 2dly. Because it seems to indicate an active power resident in matter; or rather, it implies an absolute contradiction, namely, that a body should be both active and inactive at the same time. It is desirable, therefore, that only the term inertia, or inertness, should be retained: for this term will imply, as it ought to do, that matter is a merely passive thing. A fact which needs no laboured proof: for this inertia presents itself immediately in all

our observations and experiments upon matter, and is inseparable from it, even in idea. When we consider any of the active powers of nature, as they are called, such as gravitation, magnetism, electricity, or the attractions and repulsions which take place in the cohesions and separations of the small particles of natural bodies, and endeavour to resolve them into some higher and simpler principles, the *inertia* is always the common basis upon which we endeavour to erect our solutions. For the active party which is supposed to generate the gravitation, magnetism, &c. in the passive one, must have a motion, and inertia, whereby it continues in that motion, else it could have no power; and by parity of reason, the passive party must be inert also, else it could not re-act against the active party, nor impress motion on a foreign body. And this by the way, if the reader will pardon *one* slight digression, suggests a brief but cogent argument for the immateriality of the Supreme Being. For, as the acute *Hartley* observes, " let us proceed as far as " we please in a series of successive solutions, we shall always " find an inertia inherent in matter, and a motion derived to it " from some foreign cause. If this cause be supposed matter " always, we shall be carried on to an infinite series of solutions, " in each of which the same precise difficulty will recur, without our at all approaching to the removal of it. Whence, " according to the mathematical doctrine of ultimate ratios, " not even an infinite series, were that possible in this case, " could remove it. We must, therefore, stop somewhere, and " suppose the requisite motion to be imparted to the supposed " subtle matter, by something which is not matter; *i. e.* since " God is the ultimate author of all motion, we must suppose " him to be immaterial."

19. *Equilibrium* may be defined generally, as an equality of weights, powers, or forces, of any sort. Bodies at *rest* are in a state of equilibrium, when they are solicited by various forces in different directions in such a manner as to be completely balanced, and have no tendency to move in any direction. Bodies in *motion* are in a state of equilibrium, when the resistance to motion and the power producing it are so adjusted, that the result shall be *uniform motion.* An accurate knowledge of both kinds of equilibrium is indispensably necessary, in order that the theory may be applied to good practical purposes.

The equilibrium of bodies at rest is sometimes characterised in two different ways; equilibrium of *rotation,* and equilibrium of *translation:* the former denoting that the tendency to a rotatory motion is counteracted; the latter, that the tendency to a rectilinear motion is prevented; and *rest* in both cases ensured.

20. We may now define somewhat more distinctly the

Science of Mechanics: it comprises the doctrine of the rest, the equilibrium, and the motion of bodies. It may be divided into *Mechanics*, properly so called, and *Hydraulics:* the first comprises *Statics*, which has for its object the balanced rest of solid bodies; and *Dynamics*, which considers the motion of solid bodies and their force during motion: the latter comprises *Hydrostatics*, which relates to the resting equilibrium of liquids or non-elastic fluid bodies; and *Hydrodynamics*, which treats of such bodies in motion. To these must likewise be added *Pneumatics*, which comprehends the doctrine of the weight, pressure, and effects of elastic fluids, as *air*, &c. According to this division we shall arrange the subjects of our first five books.

CHAPTER I.

Axioms, or Laws of Motion and Rest.

21. In order that the doctrine of Mechanics may be brought within the boundaries of mathematical investigation, it is necessary not only that the quantities it proposes for discussion should be measurable either in themselves or in their effects; but also that some general principles should be exhibited, the truth of which should be incontrovertible, and to which the student may at all times appeal in the course of his researches. Such general principles were first distinctly proposed by Sir Isaac Newton in his *Principia;* they have since his time been received as Mechanical Axioms, or, as they are commonly called, Laws of Motion. They are, in reality, intermediate propositions between geometry and philosophy, through which mechanics becomes a mathematical branch of physics, and are as follows:

I. Every body continues in its state of rest, or of uniform motion in a right line, until a change is effected by the agency of some mechanical force.

II. Any change effected in the quiescence or motion of a body is in the direction of the force impressed, and is proportional to it in quantity.

III. Reaction is always equal and contrary to action; or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Of these axioms, the first has respect to the continuance of bodies in a state of repose or of motion, without *any alteration,* except so far as subsequent causes operate; the second assigns the quantity and nature of such alterations; and the third has respect to the mutual circumstances of the patient, which suffers alteration from any cause, and of the agent producing the alteration. So that these principles, when established, have a manifest tendency to facilitate the study of the science in which they are proposed, not as self-evident truths, but as truths which result by legitimate induction from the testimony of our senses.

22. The evidences from which our assent to these axioms is derived, are generally stated as threefold. 1. From the constant observation of our senses, which tend to suggest the truth of

them in the ordinary motion of bodies, as far as the experience of mankind extends. 2. From experiments, properly so called. 3. From arguments *à posteriori*. But besides these it has been suggested that they are, in fact, laws of human thought resulting from the nature of things independent of experience. That the student may, without hesitation, yield his assent to these principles, it is thought expedient to state briefly the nature of the different kinds of evidence thus afforded.

23. I. *From constant observation.* Here with respect to the first axiom, that a body at rest cannot put itself into motion, is supposed to be known from universal experience: but that a body in motion would continue to move uniformly in a rectilinear path, though equally true, seems less apparent. Since, however, motion is continued the longer, in proportion as obstructions become less, as a vibrating pendulum moves longer in air than in water, longer yet in an exhausted receiver, it may be reasonably inferred that, if all obstacles could be removed, motion once communicated to any body would never cease. The principle may be farther confirmed by a reference to common occurrences. Thus, when a stone is whirled round in a sling, on being set at liberty, it will continue to move with the force it has acquired. Again, when a vessel, containing a quantity of water, is moved along upon an horizontal plane, the water obeying its tendency to rest, will at first rise up in a direction contrary to that in which the moving force acts: when the motion of the vessel is communicated to the water, it will persevere in this state; and if the vessel be suddenly stopped, the water by its inertia opposing the change from motion to rest, it will rise up on the opposite side of the vessel. In like manner, if a horse, which was standing still, suddenly start forward, the rider will be in danger of being thrown backwards: if the horse when in motion stop suddenly, the rider will be thrown forwards. In all these and numerous other instances which might be adduced there is strong evidence that matter has a tendency to continue in its *present state;* and this is what the first axiom asserts.

As to the second, the inferences from experience are equally powerful. Thus a ball moving with a double or triple velocity, generates in another, by impulse, a double or triple velocity, and the ball loses the same proportions of its own velocity. Two bodies meeting with equal quantities of motion mutually stop each other. Two forces, which, by acting similarly during equal times, produce equal velocities in some third body, are found, by acting together during the same length of time to produce a double velocity. If a new force be impressed upon a body in motion, in the direction in which it moves, its motion is increased proportionally to the new force impressed: if this force act in a direction contrary to that in which the body moves, it is found to lose a proportional part of its motion: if the direction of this force be oblique to the direction of the moving body, it gives it a new direction compounded of both. A force which we know to act equably produces equal increments of velocity in equal times. In all these examples we may trace a strict conformity to the axiom.

The third principle is likewise observable in all the motions of nature, obtaining indeed throughout the whole solar system with the utmost precision: all which will be obvious enough when it is well understood what this axiom affirms. In the communication of pressure upon any immoveable plane, whether arising from the protrusion, gravity, or impact of a body, the meaning of this principle is, that the resistance of the plane, and an opposite force equal to that producing the pressure, have precisely the same effect, as either of them only destroys the force of protrusion, gravity, or impact. In the communication of motion by visible impact the axiom asserts that the quantities of motion lost and gained are equal when estimated in opposite directions. In the communication of motion by unknown means, as magnetism, electricity, &c., it affirms that the body attracting or repelling moves in an opposite direction to that of the body attracted or repelled, and with an equal quantity of motion. Thus, to propose an instance in the case of attractions: When a loadstone and a piece of iron, equal in weight, float in water upon equal and similar pieces of cork, they are found to approach each other with equal velocities; and when they meet, or are kept asunder by any obstacle, they sustain each other by equal and opposite pressures. Again, when a force of any kind is counterpoised, whether by the exertion of animal strength, the pressure of a fluid, the force of a spring, &c., each of these opposed forces will be equal to some given weight, which if it be substituted so as to act instead of the force, will preserve the equilibrium unaltered. Even an immoveable obstacle exerts a definite force equal and contrary to that which is applied in any manner or degree to alter its position: for if such obstacle be taken away, a force equal and contrary to that by which it was before urged must be substituted in its place to keep the whole at rest.

24. II. *Evidence from experiments.* Common experience, however, it must be admitted, is not entirely sufficient to establish the truth of these general principles; for, as we have already remarked (12.), we know not what is the actual motion of a body, or when it is absolutely at rest; so that those observations which have been adduced as made upon bodies appa-

rently at rest, or in rectilinear motion, do not satisfactorily determine the point they are intended to establish. Recourse must, therefore, be had to experiments of a more refined and artificial complexion, and such may readily be found. Although we cannot appreciate the absolute motions of bodies, we can observe and accurately measure their relative motions. For instance, if it can be shewn experimentally, that bodies shew equal tendencies to resist the augmentation and the diminution of their relative motions, they, of consequence, would exhibit equal tendencies to resist such augmentation or diminution in their absolute motions: and numerous experiments, it is evident, might be conducted which show this very clearly. And in like manner experiments might be contrived tending to establish the other axioms: and it would undoubtedly be found that in proportion as all impediments were removed, the more perfectly would the experiments coincide with what these axioms might lead us to expect. Many experiments might easily be proposed; but our business here is not to exhibit them, but to suggest the argument which flows from them.

25. III. *From arguments à posteriori.* The conviction with which we are furnished in this manner is very cogent: for, as Mr. Atwood remarks, " Let a proposition be assumed as true " even without evidence of any kind, if by strict and logical " reasoning, various conclusions are deduced, which upon exa" mination are found consistent among themselves, and with " experience, this will be a presumptive proof in favour of the " principle assumed; and our assent to it will be the more " strongly enforced in proportion as the conclusions inferred " and the comparison of them with experience have been more " extensive. From the Newtonian axioms assumed as true, a " system has been adduced and compared with phenomena in " numberless cases: it has been applied to the motion of the " planets and comets, to that of bodies on the earth's surface; " even to the motion of those minute particles which compose " both solid and fluid substances. A *perfect agreement* between " these consequences of the axioms and matter of fact has been " the result, *no one instance excepted.* These and other similar " arguments, upon the whole amounting to evidence scarcely " inferior to mathematical demonstration, are the grounds on " which the laws of motion are received as axioms, from " which the various theorems concerning the effects of forces " are systematically demonstrated." *Treatise on Motion.* Page 360.

26. IV. Lest the combined effects of the preceding arguments should after all be insufficient, let us enquire whether the truth of these general principles cannot be made evident, from

the laws of human judgment independent of all experiment? Now with respect to the first axiom, let it be recollected that we know nothing of forces of any kind but from their effects, and that we constantly infer the agency of a force from its changing the state (as to motion or rest) of bodies on which it acts. When we witness a change of motion we ascribe it to some force; and when there is no change of motion we conclude there is no changing force. On the other hand, when we suppose the action of a changing force we suppose a change of motion: and when we do not think of a changing force, it is because no change of motion suggests such change. We suppose both rest and motion to be *states* or *conditions* of a body, from the one of which it is no more likely to be converted into the other *without cause*, than it is to change spontaneously from a cubical to a pyramidal or globular, or any other form. If this be admitted, as we trust it will, it follows that a body *must continue in its existing state of rest or uniform motion, until that state is changed by some mechanical force.* The matter has been stated a little differently, thus: " Suppose a body in motion, " and that the instantaneous action of the moving force is ade- " quate to make the body describe a certain space; then, since " after the first instant the moving force ceases, and the motion " still continues, it must be uniform, because the body cannot " of itself either accelerate or retard it; the motion will more- " over be rectilinear, for there is no reason why the body should " deviate to the right rather than to the left; and hence, in this " case, where the body is capable of moving, during a certain " time and independently of the moving cause, the motion will " be uniform and rectilinear. A body which can move itself " uniformly, and in a straight line during a certain time, will " perpetually move after the same manner:—for, suppose a " body capable of describing uniformly a straight line, of which " the two extremities are A and B (Pl. I. fig. 1.); between A " and B take two points C and D, then the body at D is pre- " cisely in the same state as when at C, excepting that it is in " a different place. Therefore the same ought to happen to the " body as when at C: but, by hypothesis, when at C it can " move itself uniformly to B; therefore when at D it will be " liable to move itself uniformly to a point G, taking DG=CB; " and so on for ever." *Monthly Rev.* N. S. vol. xxviii. page 316.

Again, with respect to the second axiom, it may almost be considered as an identical proposition, considering force as we do (17.), merely as the cause of motion, or of a change in motion: the law is in fact equivalent to saying that " we take the " changes of motion as the measures of the changing forces,

" and the direction of the change for the indication of the di-" rection of the forces."

The third axiom is indeed a universal fact: but we conceive it is likewise a necessary truth. To assert the contrary is, we think, to maintain an absurdity: for if action and reaction are not equal, the greater either acts against nothing and is therefore not action (contrary to hypothesis), or exists without a cause, which if once admitted, we know not what can be denied.

27. These axioms relate immediately to the actions of particles of matter upon each other, in free space: or to those cases in which the whole mass may be conceived as collected into a point.

When bodies move on fixed axes, the energy of the moving force and resistance of the body moved will depend on the distance from the axis: in these cases the inertia of the parts of the system, by which they oppose motion, being calculated, and a mass of equivalent inertia being substituted for the given system; the quantity of the moving force when moving with the same velocity as the matter moved, is likewise computed. The moving force and mass being thus ascertained, the resulting motions may be calculated, and it will thence appear, that in rotatory motions on fixed axes, or on moveable axes in free space, as well as in direct rectilinear motions, action is always equal to reaction. But this is merely hinted by the way; it will be developed more fully when we come to DYNAMICS. (Book ii. ch. 4.)

CHAPTER II.

On Statical Equilibrium, and the Composition and Resolution of Forces.

28. Equilibrium, according to our definition (19)., is of two kinds, the one relating to bodies at rest, the other relating to bodies in uniform motion: the former may be properly called *Statical equilibrium*, and may, therefore, whenever referred to in our first book (whose object is Statics) be characterised by the term equilibrium alone; the latter may be called *Dynamical equilibrium*, and will be brought under consideration in subsequent parts of the work.

29. Def. When two or more forces act upon a body, or upon a material point, at the same time, the aggregate effect produced by their simultaneous action, or a force of a certain magnitude and direction equivalent to their joint energy, is called the *Resultant*, or the *Equivalent*; and the several forces of which this is compounded are called *Components* or *Composants*.

30. Def. The problem of the *Composition of Forces*, is that in which it is proposed to find the *resultant* of any given system of forces: and the *Resolution of Forces*, which is the converse problem, proposes to trace out the *composants*, which by their united energy might produce, or did produce, a given resultant.

Thus, supposing two forces C, C′, whose magnitudes and directions are represented (17.) by CP, C′P, to act at the same time upon a material point P (Pl. I. fig. 2.); by the first of these problems we determine the magnitude and direction of the *equivalent* RP to these two forces; and by the latter we enquire the two forces CP, C′P, or various pairs of forces, of which the resultant is RP. It may be proper to remark here that the Composition and Resolution of *forces*, and the similar Composition and Resolution of *motions*, are completely distinct objects of enquiry: the former is entirely a physical question, the latter a problem purely mathematical. Some authors have inferred from their demonstrations of the latter problem, the truth of the former: but this cannot well be admissible, because wherever statical equilibrium obtains there can be no motion, and of course the principle on which the inference is grounded is foreign to the nature of the thing to be proved. It seems most consistent with scientific precision, to establish subjects so very distinct, each on its own proper basis: and though the attempting to accomplish this may sometimes give an *appearance* of tautology, it is still thought better than to aim at conciseness by the sacrifice of accuracy.

31. DEF. *The moment of a force* or power, is a term by which we shall denote the product of its magnitude into the nearest distance of its direction (17.) from any fixed point.

Thus, in fig. 2. Pl. I. if the directions of the two powers C, C′, are at the respective perpendicular distances, *d*, *d′*, from a point S taken arbitrarily, then will the products C*d*, C′*d′* represent what we call the *moments* of those powers. They express the energies of those powers, and differ only in the manner of application from momenta in Dynamics.

32. In every system of an invariable form we may take for points of application of powers, any points whatever in the lines of their respective directions.

For the distances of the several points of the system remaining constantly the same, no one of the points in the direction of any power can move without drawing into its motion all the points in the same line, nor, of consequence, without causing such motion as would be produced if the power were applied immediately at that individual point.

This proposition may be illustrated by a simple and obvious example. If the body AB (fig. 5. Pl. II.) be acted upon by a force *pushing* against it by means of an inflexible bar OQ, the effect upon the body will be just the same whether the force be exerted on the lever at O, g, or Q: and if, on the contrary, the body were acted upon by a force *drawing* it by means of an inextensible cord P E C, the tendency to move the body would be just the same, whether the cord were pulled at P, E, or C. Here we exclude the consideration of gravity from the bar and the cord.

33. Hence, if an obstacle be employed to destroy any force, it is sufficient if such obstacle be applied at any point whatever in the direction of this force, provided the point be one of those in a system of unchanging form: so that we may in all such cases, without danger of error, conceive that the power is immediately applied to the obstacle.

34. 1. Two forces acting in the same right line, and directed the same way, are equivalent to one force equal to their sum.

2. Two equal forces acting in the same right line, but in opposite directions, destroy each other.

3. Several forces acting in opposite directions in the same right line are equivalent to one only, namely, to the excess of

the sum of the forces acting in one direction above the sum of the forces acting in the contrary direction.

These are obvious truths necessarily resulting from the definition of force (17.), and Newton's third axiom (21.).

35. Since, then, a system is in a state of equilibrium when it is subject to the joint action of two equal forces exerted in opposite directions; a system subjected to the united energy of various forces is in equilibrio, when any one of the forces is equal and opposite to the *resultant* of all the rest: and, of consequence, the general problem of the equilibrium between various powers, is reduced to that of the *Composition of Forces* (30.). We shall, therefore, now proceed to the solution of this problem, and, for the sake of perspicuity, shall consider it under four separate heads, according to those divisions which the order of nature seems to present, viz.

1. Of forces disposed in one plane, and concurring in the same point.
2. Of forces directed to one point, but not confined to one plane.
3. Of forces situated in one plane, but applied to different points of a body.
4. Of forces not confined to one plane, directed to various points of a body.

I. Of Forces disposed in one Plane, and concurring in the same Point.

36. Prop. *The equivalent of several forces situated in one plane is in the same plane.*

For if we suppose the equivalent to be out of the plane of the forces, on either side, we may always find a line on the other side of the plane situated in a perfectly similar manner; and since there can be no reason why the resultant should be in one of these directions rather than in the other; it is therefore in neither of them, unless we admit the absurd consequence that it is in both, that is, unless we admit that the same forces, acting in like manner, can produce two distinct effects.

Cor. The resultant of two equal forces must be in their plane; and it must be in a line which bisects the angle of their direction, since there is no reason why it should tend more to one side than to another.

37. Prop. *If to a material point already kept in equilibrio by a system of forces, another system is applied also in equilibrio, this will not destroy the pre-existing equilibrium.* This is manifest.

COR. Hence, if the three forces C, C′, O (Pl. I. fig. 3.), are in a state of equilibrium, and if each of the forces were doubled, or tripled, or quadrupled, or if they were halved, quartered, &c. or changed in any proportion, the equilibrium would remain, so long as they continued to act in the same directions, CP, C′P, OP.

COR. 2. Hence also, since the resultant R is always equal and opposite to one of the forces (as O), it follows that when the magnitudes of equilibrated forces concurring in a point are made to vary in any ratio, the resultant retains its position, but changes its magnitude in the same ratio.

38. PROP. *If three equal forces are inclined to one another in angles each of 120 degrees, any one of them will balance the joint action of the other two.*

This is likewise incontrovertible; for neither of the forces can prevail.

39. PROP. *Two equal forces inclined in an angle of 120 degrees, have for their equivalent a third which has the direction and proportion of the diagonal of the rhombus constructed on the lines which represent the forces.*

For, if C, C′, are the forces (fig. 3.) acting on the point P, the force O whose measure is $OP = CP = C'P$, and is situated so that the angles CPO and C′PO are each equal to CPC′, will (38.) ensure the equilibrium. But RP, the measure of the equivalent R, is equal and opposite to OP (35.): therefore $CP = PR = C'P$; and because angle $CPR = 60° = C'PR$, $CR = CP$ and $C'R = C'P$. Consequently CPC′R is a rhombus, and RP the representative of the equivalent of the forces C, C′, is its diagonal. Q. E. D.

COR. If half the angle CPC′ be denoted by a, we shall have $PD = PC \cos a = C \cos a$, whence the equivalent $RP = 2C \cos a$.

40. PROP. *Any two equal forces have for their equivalent the diagonal of the rhombus constructed on the right lines which represent them in magnitude and direction.*

For, 1. If this proposition be true with regard to any two equal forces C, C′, acting in the directions CP, C′P (Pl. I. fig. 4.), and forming with their resultant R, the angles CPR, C′PR, each for example equal to a, it is true likewise for two other equal forces c, c', acting according to the directions cP, c'P, which bisect those angles. In this case c may be considered (36 *cor.*) as the resultant of two equal forces x and y, acting in the directions CP, RP; and in like manner c' may be considered as the resultant of two other equal forces x' and y', acting in the directions C′P, RP: so that, in lieu of the two equal forces c, c', we may consider four equal but unknown forces x, x', y, y', acting in the directions just assigned them. The two first of these, x, x', acting in the directions CP, C'P, have, by hypothesis, the diagonal of the rhombus for their resultant; that is, they are equivalent to a force expressed by $2x \cos a$ acting in RP: therefore the resultant z of c and c' will be equal to $2y + 2x \cos a$. But $x = y$, therefore $z = 2x(1 + \cos a)$. Now, the angles CPR, CPC', being each equal to half CPC', are equal to each other; and z being the resultant of two equal components acting in CP and CP, while c is the resultant of two others acting in CP and RP, we have (37, cor. 2.) $z : c :: c : x = \frac{c^2}{z}$. Substituting this value of x for it in the preceding equation, we obtain $z = \frac{2c^2}{z}(1 + \cos a) = c\sqrt{2}\sqrt{1 + \cos a} = c\sqrt{2 \times (1 + \cos a.)}$ But it is known that $\cos \frac{1}{2} a = \sqrt{\frac{1 + \cos a}{2}}$. (See *Hutton's Dictionary*, art. SINE, or *Carnot de Géométrie de Position*, p. 154.); whence, by reduction $z = 2c \cos \frac{1}{2} a$. Consequently the proposition, if true for a, is true for $\frac{1}{2}a$.

2. In exactly the same manner may the proposition be proved true with respect to the half of $\frac{1}{2}a$, or $\frac{1}{4}a$, and in succession for $\frac{1}{8}a$, $\frac{1}{16}a$, $\frac{1}{32}a$, &c. That is, since it is true (38.) when the angle CPC' is measured by $\frac{1}{3}$ of the circumference, it is likewise true when the angle between the equal components is measured by $\frac{1}{6}$, $\frac{1}{12}$, $\frac{1}{24}$, $\frac{1}{48}$, &c. of the circumference, where the series may be continued, *sine limite*.

3. If the proposition be demonstrated for the three angles a, b, and $a - b$, it will be true for the angle $a + b$; that is, if we take two equal components c and c' making with their resultant x, angles $= a + b$, we shall have $x = 2c \cos (a + b.)$ Thus, if in Pl. I. fig. 5. the angles CPR, C'PR, are each equal to a, and CPC, CPd, C'PC', C'Pd', each equal to b: conceiving two forces dP, d'P, each equal to c, their resultant will, by hyp. be $= 2c \cos (a - b)$, because dPR $= a - b$; and this quantity subtracted from the resultant of c, c', d, d', will give x. But c and d have their resultant C, acting in CP, and $= 2c \cos b$; the same thing holding with respect to c' and d', we have two forces equal to C, and equivalent to one which is $2C \cos a$ or $4c \cos a. \cos b$: whence $x = 4c \cos a \cos b - 2c \cos (a - b)$. But $\cos a \cos b = \frac{1}{2} \cos (a + b) + \frac{1}{2} \cos (a - b)$. (See *Crakelt's Mauduitt*, p. 50. *Pantologia*, art. TRIGONOMETRY, or HUTTON'S *Course*, vol. iii. ch. 3.) Which value of $\cos a. \cos b$, substituted for it in the preceding equation gives $x = 2c. \cos (a + b)$. So that the proposition when true for a, b, and $a - b$, is true for $a + b$.

4. Let b be taken as small as we please in the series $\frac{1}{6}$, $\frac{1}{12}$, $\frac{1}{24}$, $\frac{1}{48}$, &c. and let a be the preceding term in the series, then,

a, b, $a - b$, $a + b$, are $2b$, b, b, and $3b$ respectively, in each of which the proposition holds: again, if $a = 3b$, $a + b = 4b$; if $a = 4b$, $a + b = 5b$, &c. So that the theorem is demonstrated for all angles in the series b, $2b$, $3b$, $4b$, $5b$, &c. in which b may be taken of a magnitude less than any one which can be assigned. Consequently the theorem is true with respect to any rhombus whatever: for let any rhombus be proposed which it is affirmed is an exception to this proof; we can, it is obvious, by choosing b lower than any assigned number, and taking a suitable multiple of it, approach nearer the excepted angle than by any assignable difference, that is, we show that our theorem is applicable to the angle itself. Q. E. D.

41. PROP. *Any two forces having the ratio of the sides of a rectangle, and whose directions coincide with those sides, have for their equivalent the diagonal of that rectangle.*

Let the two forces C, C′ (pl. I. fig. 6), act in the directions CP, CP′, which comprise the right angle P: complete the parallelogram CPC′R, and draw its diagonals; parallel to CC′, draw *cc*′ terminated by C*c*, C′*c*′, which are drawn parallel to the resulting diagonal. Conceive *c* and *c*′ to be two equal forces acting in the equal lines *c*P, *c*′P, opposite to each other, and consequently annihilating each other's effects; then *c*PDC and *c*′PDC being rhombi, the force CP is the equivalent of *c*P, DP, and C′P that of *c*′P, DP, by the preceding prop. Therefore the components CP, C′P, are the same in effect as the opposite ones *c*P, *c*′P, together with DP, DP; that is, the equivalent sought is 2DP or RP, the diagonal of the parallelogram. Q. E. D.

COR. Since RP : rad : : CP : cos CPR : : C′P : cos C′PR, we have the resultant equal to either component divided by the cosine of the angle which it makes with the resultant.

42. PROP. *Any two forces whatever have their equivalent expressed in magnitude and direction by the diagonal* RP *of the parallelogram constructed on the lines* CP, C′P, *which represent these forces.*

Having completed the parallelogram CPC′R (fig. 7. pl. I.) on the given sides, draw *cc*′ perpendicular, and C*c*, C′*c*′, parallel to the diagonal, demit also CD, C′D′, perpendicular to the diagonal: then will C*c*PD, C′*c*′PD′, be rectangles, and the triangles CRD C′PD′, equal in all respects; consequently C*c* = DP, RD = D′P, and *c*P = *c*′P. The addition of the equal forces *c*,*c*′, acting in the opposite directions *c*P, *c*′P, will make no difference as to the state of the system: and since the components DP, *c*P, have CP for their resultant, and the components D′P, *c*′P, the resultant C′P (by the preceding prop.), we may instead of the original forces CP, C′P, substitute the forces CP, C′P, DP, D′P, of which the two former destroy each other's effects, and the latter DP, D′P, are manifestly equal to RP; that is, the resultant of the two forces CP, C′P, is equal to the diagonal RP of the parallelogram. Q. E. D.

43. Thus have we, by a series of connected propositions, demonstrated that which is justly reckoned the most important in the theory of Statics, and which is now commonly spoken of under the title of the *Parallelogram of Forces.* The demonstration here given is commenced upon the same principle (38.) as that proposed by *D'Alembert* in the Memoirs of the French Academy for 1769: it was somewhat simplified by *Francœur* in his Mechanics; but what is here offered, at the same time that it is more concise than the demonstration of Francœur, is freed, it is hoped, from some objectionable positions into which that author has certainly fallen. This demonstration would manifestly be the same in every step, supposing the forces to act in the directions PC, PC′: so that the proof is general for the composition of any systems of forces whether pressing, or impulsive, or attracting, or repelling, &c. Professor *Robison*, by ingeniously blending the Demonstrations of Bernouilli, D'Alembert, and Frisi, deduces the same conclusion by a series of propositions commencing with the case of two equal forces acting at right angles. See art. DYNAMICS. Supp. Ency. Britan. A demonstration passing from parallel forces to oblique may be seen in *Poinsot's Statics.* The demonstrations by Laplace and Lagrange, though confessedly curious, depend upon principles too abstruse to be admitted, without impropriety, into this work. Laplace's demonstration has been somewhat simplified by Mr. Knight, in Leybourn's Mathematical Repository, No. 9, N. S. In order to facilitate the application of the Parallelogram of Forces, it will be proper to specify the most useful propositions and corollaries which immediately flow from it.

44. COR. 1. *If two component forces are represented by the two sides of a triangle, the resultant will be represented by the double of the line drawn from the point where the forces act, to the middle of the third side of the triangle.*

Thus, in figs. 3, 6, the resultant $PR = 2PD$; because the diagonals of the parallelogram OPC′R bisect each other.

45. COR. 2. *If the angle at which two given forces act be diminished, their resultant will be increased, and* vice versa.

For in the triangle CPC′, in which the line PD bisects the base, it is known that $CP^2 + C'P^2 = 2DP^2 + 2CD^2$ (Hutton's Geom. th. 38.). Whence $2DP^2 = CP^2 + C'P^2 - 2CD^2$, or $4DP^2 = RP^2 = 2CP^2 + 2C'P^2 - C'C^2$. But CP, and C′P remaining constant, the side CC′ will be greater or less as the angle CPC′ is; and since the square of CC′ is subtractive in the above value of RP^2, the latter will increase as the former decreases, and *vice versâ.*

46. COR. 3. *Two given forces produce the greatest effect when they act in the same direction, and the least when they act in opposite directions.*

For, in the former case the diagonal of the parallelogram becomes equal to the sum of its sides, and in the latter equal to their difference. The same conclusion may also be deduced immediately from art. 34. The conception of the student may be assisted by referring to figs. 8 and 9.

46. PROP. *If a material point be kept in equilibrio by the simultaneous action of three forces, those forces will be represented in magnitude and direction by the three sides of a triangle respectively parallel to the directions of the forces.*

For, when a system of forces sustains a point in equilibrio, any one of the forces, as that represented by OP (fig. 10. pl. I.) is equal and opposite to RP the resultant of the others (35). But the resultant of the two forces represented by CP and C′P, is represented by the diagonal of the parallelogram constituted upon CP, C′P, (42.) and consequently if PR be equal to OP, CR will be parallel to PC′. Whence, it is obvious that the three sides of the triangle PCR are respectively equal to the three forces O, C, C′, and parallel to the directions in which they act.

47. COR. 1. *If a body be kept in equilibrio by three forces, and two of them be represented in magnitude and direction by two sides of a triangle, the third side, taken in order, will represent the magnitude and direction of the other force.*

48. COR. 2. Since the sides of triangles are as the sines of the opposite angles, it follows that *when three forces keep a point in equilibrio, each force is proportional to the sine of the angle made by the directions of the other two.* Thus, in the forces represented in fig. 10.

$$O : C : C' :: \sin CPC' : \sin OPC' : \sin OPC :: \sin PCR : \sin CRP : \sin CPR.$$

49. COR. 3. *If a material point be kept at rest by three forces, and lines be drawn at right angles to the directions in which they act, and produced till they form a triangle, the sides of this triangle will be proportional to the respective forces: or, if lines be drawn each making the same given angle with the directions of the forces, the sides of the triangle formed by these lines will likewise be proportional to the forces.*

For it may easily be shewn that each of these triangles will be similar to the triangle formed by the directions of the forces.

50. PROP. *If, when three given forces* O, C, C′, *acting upon a point* P *keep it in equilibrio, right lines* PO, PC, PC′, *be set off from* P *proportional to the several forces and in their re-*

spective directions, the point P *will be the centre of gravity of the triangle* OCC′.

Produce OP to R (fig. 11. pl. I.) making PR = OP, then will PR be equal to the equivalent of the components C, C′, and consequently it will be the diagonal of the parallelogram CPC′R: wherefore PD, its half, is likewise equal to half OP, and falls upon the middle of the side CC′ of the triangle OCC′. But when OP = 2PD, CD being = C′D, P is the centre of gravity of the triangle. (See Art. 114.) Q. E. D.

51. PROP. *To find the resultant of any number of forces acting upon a point, their directions all lying in the same plane.*

This is done with much facility by an obvious application of the parallelogram of forces: for since the united action of any two forces upon a point puts it in the same state as if their equivalent had acted upon it, we may compound any two of the forces, and substitute their resultant for them; this compounded with a third force will give a new resultant; and this blended with a fourth force, another; and so on as far as necessary. Thus suppose the resultant of the five forces represented in magnitude and direction by PC, PC′, Pc, Pc′, and PC″ (fig. 12. pl. 1.), were required. First complete the parallelogram CPC′R, and instead of the component forces PC, PC′, make use of the resultant PR: then take PR, Pc, as components, and complete the parallelogram RPcR′, so will PR′ be the resultant of the two forces PR, Pc, or of the three PC, PC′, Pc: again, take PR′, Pc′, as components, and complete the parallelogram PRc′r: next, take the new resultant Pr, and the fifth force PC′ as components, and complete the parallelogram Prr′C″; so will the new resultant Pr′, be the resultant of the five original forces; and of consequence (35.) a force equal and opposite to Pr′ will keep the equilibrium in the whole system.

52. COR. 1. Since CR is both equal and parallel to PC′, RR′ equal and parallel to Pc, R′r to Pc′, &c. the resultant when the forces are all in one plane may be readily determined, by drawing consecutive lines, each from the end of the preceding, and having the directions and proportions of the several forces. Thus draw from C (fig. 12.) CR equal and parallel to PC′, RR′ to Pc, R′r to Pc′, rr′ to PC″, and Pr′ will be the resultant, as before.

53. COR. 2. *If any number of forces represented in magnitude and direction by the sides of a polygon taken in order, act simultaneously upon the same point of a body at rest, they will keep it at rest.*

Let PC, CR, RR′, R′r, rr′, r′P (fig. 12.), represent the forces; then, since the first five of these have Pr′ for their equivalent (52.), and r′P the fifth force is equal and opposite to Pr′, it will preserve the equilibrium of the system (35.).

54. PROP. *A single force may be resolved into any number of forces.*

Since any single force as PR (fig. 10.) is the equivalent of the two forces PC, PC′, which are two contiguous sides of a parallelogram whose diagonal is PR; and since it will likewise be the resultant of any two forces proportional to, and in the direction of any parallelogram which has PR for a diagonal; it follows that the single force PR may be decomposed into as many pairs of forces as there can be described parallelograms having the same common diagonal, that is, into an indefinite number of such pairs. And since either, or both, of these component forces may again be decomposed, in like manner, into two, and either of these again into two; it is manifest that the original force may in this way be conceived to be resolved into any number of others, *ad libitum;* and these either in the same plane, or any others.

55. SCHOLIUM. The preceding proposition is useful in what is called the *Reduction of forces,* or estimating their effects in any given direction. Thus, if it were required to estimate the effect of the force PC (fig. 7. pl. I.) in the direction of the given line PR: construct the rectangular parallelogram PC CD, and PC will manifestly be the equivalent of the two forces PC, and PD, of which the former is perpendicular to the given direction, and therefore neither promotes nor obstructs the effect of the force in that direction, so that PD is the aggregate of force in the direction PR. Of this species of reduction many examples will occur hereafter.

56. PROP. *The effects of forces, when estimated in given directions, are not altered by composition or resolution.*

Let the forces PC, PC′, and their resultant PR, be estimated in the proposed directions PX, PY (fig. 14. pl. I.). Draw the various lines respectively parallel to PX, PY, as in the figure. Then, the force PC, when referred to the given directions, will furnish the components PD, P*d*; the force $CR = PC$, the components *ce*, CE; and the force PR gives the components PF, P*f*. But the forces PD, *ce*, being in opposite directions will have opposite signs ($+$ and $-$), and by reason of the parallels DE, FR, P*f*, and PD, *dc*, *f*E, it will be $PD - ce = PD - DF = PF$, the effect of the two forces PC, PC′, estimated upon PY; also $Pd + CE = Pd + df = Pf$, the effect of the same two forces estimated upon PX; and these, it is evident, are likewise the effects of the resultant PR, when estimated in the same directions. In a similar manner the truth of the proposition may be shewn, for any number of forces, by compounding each pair, and tracing their simultaneous effects, in the proposed directions. Q. E. D.

Cor. If a system of equilibrated forces is reduced to any one direction, the reduced forces will be in equilibrio.

57. Prop. *To find the resultant of a system of forces applied to one point and acting in the same plane, by the method of rectangular co-ordinates.*

Let the three forces C, C′, C″, (fig. 13. pl. I.), solicit the point P, with energies proportional to PC, PC′, PC″, and in those directions: through any point A draw two lines XX′, YY′, in any directions respectively perpendicular to each other; through P draw Pd'' parallel to XX′ and PD parallel to YY′, and resolve each of the forces into components respectively parallel to these co-ordinates; for instance, PC into PD, Pd; PC′ into PD′, Pd'; and PC″ into PD″, Pd'': set off AY and AX respectively equal to PD + PD′ + PD″, and Pd + Pd' + Pd''; then complete the rectangle ARXY, and its diagonal AR will represent the magnitude of the resultant of the system, and will be parallel to its direction. The truth of which is manifest, from the foregoing proposition and scholium.

58. Cor. In the case of an equilibrium AR being opposed by an equal and contrary force, the resultant will be nothing, or zero: or when the several forces are reduced to the rectangular co-ordinates, the sum of the forces upon AY′ which may be considered as negative, must be equal to the sum falling upon AY, and the sum upon AX′, in like manner considered negative, equal to those upon AX.

59. Scholium. In cases where accuracy is required, the student will find it necessary to aim at something further than a graphical solution; to assist his progress, we, therefore, add a few algebraical formulæ which may often be advantageously applied.

Let the angles which the directions of the component forces C, C′, C″, make respectively with the axis AX (fig. 13.), be a, a', and a''; and let the angle which R the resultant makes with AX be denoted by r. Then when each of the forces is reduced to rectangular co-ordinates, we shall have (41. *cor.*) $C = \frac{Pd}{\cos a}$, $C' = \frac{Pd'}{\cos a'}$, $C'' = \frac{Pd''}{\cos a''}$, whence $C \cos a$, $C' \cos a'$, $C'' \cos a''$, are equal to Pd, Pd', Pd'', the various components of these forces parallel to AX; and, in a similar manner may be obtained $C \sin a$, $C' \sin a'$, $C'' \sin a''$, the components of the same forces parallel to AY: the former of these acting in the same line, are equivalent to a single force expressed by their sum [using the word *sum* in the sense of the algebraists], that is, $C \cos a + C' \cos a' + C'' \cos a''$, + &c.: and the latter are likewise equivalent to a single force expressed by their sum, or

$c \sin a + c' \sin a' + c'' \sin a'' + \&c.$ The two components of the resultant R referred to the same axis, are evidently $R \cos r$ and $R \sin r$: consequently (56.) the following equations obtain:

$$\text{(I.)} \quad \left\{ \begin{array}{l} c \cos a + c' \cos a' + c'' \cos a'' + \&c. = R \cos r \\ c \sin a + c' \sin a' + c'' \sin a'' + \&c. = R \sin r \end{array} \right.$$

If we denote the sum of the components in the direction of the axis AX by x, and the sum of those falling upon AY by y, the preceding will be represented in an abridged form, thus:

$$\text{(II.)} \ \ldots\ldots \ R \cos r = x, \ \ldots\ldots \ R \sin r = y.$$

When the system is in a state of equilibrium, the resultant is zero (58): thus our first equations give for the conditions of equilibrium:

$$\text{(III.)} \left\{ \begin{array}{l} x = 0 \\ y = 0 \end{array} \right\} \text{or} \left\{ \begin{array}{l} c \cos a + c' \cos a' + c'' \cos a'' + \&c. = 0. \\ c \sin a + c' \sin a' + c'' \sin a'' + \&c. = 0. \end{array} \right.$$

In those cases where the equilibrium has not place, it is still easy to find the magnitude and direction of the resultant: for, adding together the squares of the equations (II.) we shall have $R^2 (\sin^2 r + \cos^2 r) = x^2 + y^2$, wherefore, because $\sin^2 r + \cos^2 r = \text{rad}^2 = 1$, we shall have

$$\text{(IV.)} \ \ldots\ldots \ R = \sqrt{x^2 + y^2}$$

Which is, in fact, the well-known property of the right-angled triangle.

Our second equations likewise give, by a simple division,

$$\text{(V.)} \ \ldots \ \cos r = \frac{x}{R} \ \ldots \ \sin r = \frac{y}{R}.$$

Or, if we divide the one of our second equations by the other, another equation will be obtained, by which we may determine the direction of the resultant: for, since sin ÷ cos = tan, we have

$$\text{(VI.)} \ \ldots\ldots \ \tan r = \frac{y}{x}.$$

When only one of the equations (III.) obtains, there will not be an equilibrium, but the resultant will be parallel to one of the axes: thus, if we have only $x = 0$, this will give $R \cos r = 0$, of course $\cos r = 0$, or $r = 180°$: that is, the resultant is parallel to AY or perpendicular to AX. If only $y = 0$, the resultant will, in like manner, be parallel to AX.

60. We may deduce from the equations (I.) in the preceding article, two or three curious consequences; which may likewise be found of utility in some subsequent investigations.

Let us take in the plane of the forces c, c'c'', any point s at pleasure, and let the capital S denote its distance from P the point acted upon by the system, while the small s represents the angle formed by SP and XA (fig. 15. pl. I.). Now multiplying the first of the equations just referred to, by S sin s, the second by S cos s, and subtracting, we have

$$cS(\cos a \sin s \sim \sin a \cos s) + c'S(\cos a' \sin s \sim \sin a' \cos s)$$

$+ \mathrm{C}''\mathrm{S}\ (\cos a'' \sin s \sim \sin a'' \cos s) + \&c. = \mathrm{RS}\ (\cos r \sin s \sim \sin r \cos s)$.

But it is known (See *Crakelt's Mauduitt*, pa. 48. *Hutton's Course*, vol. iii. ch. 3. *Pantologia*, art. TRIGONOMETRY,) that $\cos a \sin s \sim \sin a \cos s = \sin (a \sim s)$, and the like of all the other expressions between the parentheses; whence the equation becomes $\mathrm{CS}\ \sin (a \sim s) + \mathrm{C}'\mathrm{S}\ \sin (a' \sim s) + \mathrm{C}''\mathrm{S}\ (\sin a'' \sim s) + \&c. = \mathrm{RS} \sin (r \sim s)$.

Here the angle $a \sim s = \mathrm{CPS}$, and $r \sim s = \mathrm{RPS}$, RP being the resultant: wherefore, if upon PC the perpendicular Ss be demitted, we shall have in the right-angled triangle PSs, $\mathrm{S}s = \mathrm{S} \sin (a \sim s)$; and in like manner $\mathrm{S}s' = \mathrm{S} \sin (a' \sim s)$ the perpendicular let fall upon PC′, and so on. If, therefore, we call the perpendiculars from S upon the several directions of C, C′, C″, &c. p, p', p'', &c. and the perpendicular upon the resultant ρ, the preceding equation will be transformed to this:

(VII.) . . . . $\mathrm{C}p + \mathrm{C}'p' + \mathrm{C}''p'' + \&c. = \mathrm{R}\rho$.

Hence, if we use the term *moment* in the sense of art. 31. this equation will furnish the following theorem.

The moment of the resultant of a system of forces, disposed in the same plane, and directed to one point, is equal to the sum of the moments of the components.

By the *sum* of the moments is here meant their aggregate when incorporated according to their signs, using the affirmative sign for the moments of those powers which lie on one side of the point S, the negative sign for those which are found on the other side. Or if the point S be considered as fixed, and the lines Ss, &c. as inflexible rods, the action of each of the forces upon the point M will tend to make it turn about S: viewing the matter thus, the *positive* moments will be those of such forces as tend to produce a rotation in one sense, and the *negative* moments, of those which tend to produce motion in the opposite sense. The previous deduction may, therefore, be stated thus:

When several forces in the same plane are applied to a particle of matter, the moment of the resultant is equal to the excess of the sum of the moments of those forces which tend to produce a rotation in one direction, over that of the moments of such as tend to produce rotation in the contrary direction.

It must be observed, however, that the idea of *rotation* introduced here, is merely called in for the purpose of assisting the student in determining the signs; but is no essential part of the principle.

61. The equation (VII.) will in two cases assume this form:

(VII. 2.) . . . $\mathrm{C}p + \mathrm{C}'p' + \mathrm{C}''p'' + \&c. = 0$.

That is, 1. When $\mathrm{R} = 0$; or when the system is in equilibrio. 2. When $\rho = 0$; that is, when the point S is taken on the line of direction of the resultant. Hence it appears, that,

The sum of the moments of forces which tend to produce rotation in one direction, is equal to the sum of the moments of forces which tend to produce rotation in a contrary direction, 1. *when the forces are in equilibrio;* 2. *when the moments are taken with relation to some point in the direction of the resultant.*

62. To illustrate the use of the formulæ for finding the resultant of a system of forces, we will now propose an instance:

Let the forces C, C′, C″ (*fig.* 13. *pl.* I.) *be represented by the numbers* 30, 25, *and* 20, *and* P *being the point acted upon, let the angles made by their directions, be* CPC′ = 30°, *and* C′PC″ = 28°; *it is proposed to find the magnitude and direction of the equivalent of these forces.*

If we suppose the direction C″P to make an angle of 20° with the assumed axis AX, then will the angles which the several forces make with that axis, be 78°, 48° and 20°. Wherefore, using a table of natural sines, &c. we shall have $C \cos a + C' \cos a'' + C'' \cos a'' = 30 \sin 12° + 25 \sin 42° + 20 \sin 70° = (30 \times \cdot 2079117) + (25 \times \cdot 6691306) + (20 \times \cdot 9396926) = 41 \cdot 759468 = R \cos r = x$; and $C \sin a + C' \sin a' + C'' \sin a'' = 30 \sin 78° + 25 \sin 48° + 20 \sin 20° = (30 \times \cdot 9781476) + (25 \times \cdot 7431448) + (20 \times \cdot 3420201) = 54 \cdot 76345 = R \sin r = y$. Then $R = \sqrt{x^2 + y^2} = 68 \cdot 86857$ the magnitude of the resultant; and $\frac{y}{x} = 1 \cdot 31140203 = \tan 52° \, 40' \, \frac{30}{79}$ the angle which the resultant makes with AX; this lessened by 48°, leaves $4° \, 40' \, \frac{30}{79}$ = the angle C′PR between C′P and CP. Or when the direction of the resultant is determined, its magnitude may be found without the extraction of roots: for, (V.) since $\sin r = \frac{y}{R}$ we have $R = \frac{y}{\sin r} = 68 \cdot 86857$, as before.

To shew that the same result may be obtained although the position of the co-ordinates AX, AY, be changed, suppose AX to coincide with C″P: then $C \cos a + C' \cos a' + C'' \cos a'' = 30 \sin 32° + 25 \sin 62° + 20 \text{ rad} = 57 \cdot 971269 = x$, and $C \sin a + C' \sin a' + C'' \sin a'' = 30 \sin 58° + 25 \sin 28° + 20 \sin 0° = 37 \cdot 178233 = y$. Hence $\frac{y}{x} = \cdot 6413332 = \tan 32° \, 40' \, \frac{30}{79}$, which lessened by C″PC′ gives $4° \, 40' \, \frac{30}{79}$ = C′PR, and $y \div \sin 32° \, 40' \, \frac{30}{79} = 68 \cdot 86857$, the resultant: agreeing with the former.

63. To be convinced of the simplicity of this method of finding the magnitude and direction of the resultant, in cases where the forces meet in a point and are disposed in one plane, it is only necessary to compare it with the common method.

Now, if the resultant of forces were determined in the usual way, the process would be this: Suppose PC, PC′ (fig. 16. pl. I.), to represent the magnitude and direction of two of the forces: then letting fall CE and RF perpendicularly upon PC, we should have rad : CP :: sin P : CE :: cos P : PE; and PF (= PC′ + PE) : rad :: RF (= CE) : tan RPF; also rad : PF :: sec RPF : PR. Then taking PR, and PC″, for the two forces of given magnitude and directions, four more proportions would determine their resultant: and in like manner four others would be necessary for a fourth force, and so on: whereas, by adopting the method just explained there is but little more labour necessary to determine the resultant of five or six forces than of two.

II. *Of Forces directed to one Point, but not confined to one Plane.*

64. Prop. *If three forces are represented in magnitude and direction by the three edges contiguous to the same angle of a parallelopiped, their equivalent will be represented in magnitude and direction by the diagonal drawn from that angle of the solid.*

Let the three components soliciting the particle P be represented by PC, PC′, PC″ (fig. 17. pl. I.). and let the parallelopiped be completed. The equivalent of the two forces C and C′ will be P*r*; we may, therefore, substitute P*r* for those two forces: but since the plane in which are the parallels C″P, R*r*, cuts the two parallel planes BD, CC′, the lines C″R, P*r*, in which it intersects them will be parallel (Euc. XI. 16.); and consequently C″P*r*R is a parallelogram: wherefore, compounding the two forces PC″, P*r*, we have PR the diagonal of the parallelogram C″*r*, or of the parallelopiped BD*r*, for the equivalent of the three forces C, C′, C″. Q. E. D.

65. Cor. *Hence, if four forces act upon a particle in different planes, and keep it in equilibrio, they are to each other in magnitude and direction as the three edges and the diagonal of a parallelopiped, constructed upon lines respectively parallel to the directions of the forces.*

For a parallelopiped may be constructed so that its three edges contiguous to any one angle, shall have magnitudes and directions analogous to any three of the four forces; the diagonal of this solid will, by the prop. be the equivalent of those three forces: and since in the case of an equilibrium the fourth force (35.) must be equal and opposite to the equivalent of the other three, it may be represented by the diagonal of the solid.

66. Cor. 2. Hence, likewise, *any number of forces whatever,*

acting upon a particle in different planes, may be compounded by pairs till there remain either two forces in the same plane, or three in different planes, and of these the resultant may be found either by art. 42. or by this proposition.

67. COR. 3. And hence, conversely, a single force may be resolved into three forces, in different planes, and each of these into other forces in the same or other planes, each of these again into others; the resolution admitting of being carried on, *ad infinitum.*

68. PROP. *If three forces in any one plane are in equilibrio, and these are reduced to any other plane, the reduced forces will likewise be in equilibrio.*

Let the three forces O, C, C′ (fig. 1. pl. II.), represented by OP, CP, C′P, in the plane OCC′ be in equilibrio, and let them be reduced by perpendiculars to the plane LM, then shall the reduced forces *op*, *cp*, *c′p*, be in equilibrio. For the lines O*o*, P*p*, C*c*, C′*c′*, R*r*, being all perpendicular to the same plane are parallel to each other; and because CP, RC′, are equally inclined to the plane LM, they are equally inclined to the lines *cp*, *rc′*, in that plane; hence CP : *cp* : : RC′ : *rc′*, and by permutation CP : RC′ : : *cp* : *rc′* : : 1 : 1; in like manner it may be shewn that *cr* is to *pc′* in a ratio of equality; and consequently *pcrc′* is a parallelogram: thus also, since OP = PR, *op* = *pr*: whence *op* being equal and opposite to the diagonal of the parallelogram of forces *cpc′r*, the reduced system is in equilibrio. Q. E. D.

69. PROP. *To find the resultant of a system of forces applied to one point, but disposed in different planes; by the method of rectangular co-ordinates.*

Here the process will be similar to that described in art. 57, and may be explained by a reference to the same figure (fig. 13. pl. I.): in which let the three forces, C, C′, C″, directed to the same point but not in the same plane, be represented by PC, PC′, PC″, supposed not in the same plane: through an assumed point A draw any two lines XX′, YY′, perpendicular to each other, and suppose another line AZ drawn through A perpendicular to the plane of the figure; these will be the co-ordinates to which the forces are to be referred. Through P the point of application of the forces, let lines PD, P*d*, P*δ*, be drawn parallel to the respective co-ordinates: to each of these lines refer the component forces, by perpendiculars CD, C*d*, C*δ*, C′D′, C′*d′*, C′*δ′*, &c. and the sums PD + PD′ + PD″, P*d* + P*d′* + P*d″*, P*δ* + P*δ′* + P*δ″*, will denote the equivalents of the three original forces, upon the edges of the rectangular parallelopiped; whence the diagonal of the parallelopiped whose sides are expressed by these sums represents the magnitude and direction of the resultant sought.

70. Schol. In this case of many forces in different planes, the algebraic method of solution will often be peculiarly useful; and it will be easily understood, being an obvious extension of the formulæ applied to forces in one plane. Thus, let

the component forces which we call . . . $\mathrm{C}, \mathrm{C}', \mathrm{C}''$, &c.
form with the axis AX (x) the angles . . . a, a', a'', &c.
with the axis AY (y) . . . . b, b', b'', &c.
with the axis AZ (z) . . . . c, c', c'', &c.

Each of these forces being decomposed into three others, of which the directions are parallel to each axis, it has been shewn (59.) that for components parallel to these axes, we have

Of x . . . . $\mathrm{C}\cos a$, $\mathrm{C}'\cos a'$, $\mathrm{C}''\cos a''$, &c.
y . . . . $\mathrm{C}\cos b$, $\mathrm{C}'\cos b'$, $\mathrm{C}''\cos b''$, &c.
z . . . . $\mathrm{C}\cos c$, $\mathrm{C}'\cos c'$, $\mathrm{C}''\cos c''$, &c.

These groups of forces are each equivalent to a single force, represented by their sum, acting in the several lines to which the original forces are referred; so that we have for the components of the resultant parallel to each axis, as below:

$$x = \mathrm{C}\cos a + \mathrm{C}'\cos a' + \mathrm{C}''\cos a'' + \&c.$$
$$y = \mathrm{C}\cos b + \mathrm{C}'\cos b' + \mathrm{C}''\cos b'' + \&c.$$
$$z = \mathrm{C}\cos c + \mathrm{C}'\cos c' + \mathrm{C}''\cos c'' + \&c.$$

Now denote by r, r', r'', &c. the unknown angles formed by the direction of the resultant and each of the co-ordinates, and $\mathrm{R}\cos r$, $\mathrm{R}\cos r'$, $\mathrm{R}\cos r''$, will represent the equivalents of this resultant in the several directions of the axes: hence, we have

$$\text{(ii.)} \quad . \; . \; . \; . \quad \begin{cases} \mathrm{R}\cos r = x. \\ \mathrm{R}\cos r' = y. \\ \mathrm{R}\cos r' = z. \end{cases}$$

When the system is in a state of equilibrium, we have $\mathrm{R} = 0$, so that the equations expressing the equilibrium are

$$\text{(iii.)} \quad . \; . \quad \begin{cases} x = \mathrm{C}\cos a + \mathrm{C}'\cos a' + \mathrm{C}''\cos a'' + \&c. = 0. \\ y = \mathrm{C}\cos b + \mathrm{C}'\cos b' + \mathrm{C}''\cos b'' + \&c. = 0. \\ z = \mathrm{C}\cos c + \mathrm{C}'\cos c' + \mathrm{C}''\cos c'' + \&c. = 0. \end{cases}$$

If an equilibrium does not obtain in the system, the magnitude and direction of the resultant may be deduced from the three equations (ii.), for by adding together their squares, we have $\mathrm{R}^2(\cos r + \cos r' + \cos r'') = x^2 + y^2 + z^2$. But when r, r', & r'', are angles made by any line, and three rectangular co-ordinates, it is known that the sum of their squares is equal to unity: therefore,

$$\text{(iv.)} \quad . \; . \; . \quad \mathrm{R} = \sqrt{x^2 + y^2 + z^2}.$$

Or, referring to fig. 17. pl. I. and conceiving the parallelopiped rectangular, and $PC = x$, $PC' = y$, $PC'' = z$, we have (Euc. I. 47.) $Pr^2 = PC^2 + Cr^2 = x^2 + y^2$, and $PR^2 = Pr^2 + rR^2 = x^2 + y^2 + z^2$, whence R is found, as above. And with respect to the position of the resultant, that is easily determined; for from the equations (ii.) we deduce

$$\text{(v.)} \quad . \quad . \quad \cos r = \frac{x}{R} \quad . \quad . \quad \cos r' = \frac{y}{R} \quad . \quad . \quad \cos r'' = \frac{z}{R}.$$

When either of the quantities in equation (ii) become $= 0$, the consequences are similar to those we traced with respect to forces disposed in one plane: thus, if the equations take at once this form,—

$$x = R \cos r = 0, y = R \cos r', z = R \cos r'':$$

it is then manifest that $\cos r = 0$, or that r is a right angle; or that the resultant is situated in a plane perpendicular to the axis AX. If we have, at one and the same time,

$$x = \cos r = 0, y = \cos r' = 0, z = \cos r'',$$

it will be obvious that the direction of the resultant is perpendicular to AX, AY, and parallel to AZ.

III. *Of Forces situated in one Plane, but applied to different Points of a Body.*

71. PROP. *If two parallel forces act perpendicularly upon a right line, in the same direction, their resultant is parallel to them, equal to their sum, acts in the same direction, and divides the line of application into two parts which are reciprocally proportional to the components.*

Let the two forces C, C' acting in the directions CP, C'P' (fig. 2. pl. II.), perpendicular to PP', be those whose equivalent is sought. Conceive any two forces c, c', equal to each other, to act in the opposite directions cP, c'P', opposite to each other; and it is obvious they will cause no change in the state of the system: therefore, if r the resultant of C and c act in the direction rP, and r' the resultant of C', c', in the direction r'P', the lines rP, r'P, when produced, will intersect in a point A, through which Rp, the direction of the resultant of the components C, C', must likewise pass. This granted, through A, the point of concourse of rP, r'P, draw BD, Ap, respectively perpendicular and parallel to the directions of the original forces, and decompose each of the forces r, r', into two others acting in the directions BA, pA, and DA, pA. Now, since the circumstances of the resolution of r, and r', are the same in A as in P, P', the force r acting upon A will be decomposed into the two, c acting in BA, and C in pA; and the force r' acting upon A will be resolved into c' acting in DA and C' acting in pA. But the two forces c, c', being equal and opposite are annihilated, therefore R the equivalent of C and C′ acts in a parallel direction RA and is equal to their sum.

Again to determine the point p in PP through which the resultant passes: since r is the equivalent of c, and C, it follows (59. VI.) that the angle $r\mathrm{PC}$ has $\frac{c}{\mathrm{C}}$ for its tangent; while in the triangle $\mathrm{P}p\mathrm{A}$, we have $\tan \mathrm{PA}p = \tan r\mathrm{PC} = \frac{\mathrm{P}p}{\mathrm{A}p}$: thus $\frac{c}{\mathrm{C}} = \frac{\mathrm{P}p}{\mathrm{A}p}$, and $c = \frac{\mathrm{C}\,.\,\mathrm{P}p}{\mathrm{A}p}$. Proceeding in a similar manner, we have $\frac{c'}{\mathrm{C}} = \frac{\mathrm{P}'p}{\mathrm{A}p}$, and $c' = \frac{\mathrm{C}'.\,\mathrm{P}'p.}{\mathrm{A}p}$. But, by hypothesis $c = c'$; consequently $\mathrm{C}\,.\,\mathrm{P}p = \mathrm{C}'.\,\mathrm{P}'p$, and $\mathrm{P}p : \mathrm{P}'p :: \mathrm{C} : \mathrm{C}$. Q. E. D.

72. COR. 1. *When three parallel forces acting perpendicularly upon a right line keep it in equilibrio, one of them will act in a direction opposite to the other two, and it will be equal to their sum; and any two of them will be to each other inversely as their distances from the point to which the third force is applied.*

For the third force A must be equal and opposite to the resultant R of C, C′, and must be applied at the same point p. And since $\mathrm{C} \propto p\mathrm{P}'$, and $\mathrm{C}' \propto \mathrm{P}p$, $\mathrm{A} = \mathrm{C} + \mathrm{C}'$ will $\propto \mathrm{P}p + p\mathrm{P}' = \mathrm{PP}'$. That is, $\mathrm{A} : \mathrm{C} : \mathrm{C}' :: \mathrm{PP}' : \mathrm{P}'p : \mathrm{P}p$, as affirmed in the corollary.

73. COR. 2. *When two component forces act in contrary directions, their resultant is equal to their difference, and is applied at the same point as the power which establishes the equilibrium.*

Thus the resultant of the forces A and C (fig. 2. pl. II.) acts at the point C′ and is equal and opposite to C′P′.

74. COR. 3. *The equivalent of any number of parallel forces acting perpendicularly upon a line will be equal to their sum when they act the same way; or to the excess of the sum of those which act in one direction above those which act in the contrary direction, when they act contrarily:* and to find their point of application compound them, two by two, as in the proposition.

75. COR. 4. *If a right line be kept in equilibrio by any number of forces acting perpendicularly either at the same, or at different points, the sum of all the forces acting on one side, will be equal to the sum of all those acting on the other.*

76. PROP. *The moment of the resultant of two parallel forces taken with relation to any point whatever in the same plane is equal to the sum of the moments of the components.*

1. Let C, and C′ (fig. 3. pl. II.), be two forces acting in the parallel directions CP, C′P, and S any point taken in their plane: perpendicular to CP, C′P′, draw the line SPP′, and consider P, P′, as the points of application of the forces. The resultant R being directed towards Q, we have (71.) $\mathrm{C} \times \mathrm{PQ}$

$= C' \times P'Q$. Now $R \times SQ = (C + C') \times SQ = (C \times SQ)$ $(C' \times SQ)$ and $SQ = PQ + SP = SP' - P'Q$: therefore, by substitution, $R \times SQ = C \cdot PQ + C \cdot SP + C' \cdot SP' - C' \cdot P'Q = C \cdot SP + C' \cdot SP'$, because $C \cdot PQ - C' \cdot P'Q = 0$.

2. If instead of taking S out of the space included by CP, $C'P'$, we take a point S' between those lines, the same thing will hold, using the word *sum* in the extensive algebraic sense (60.). For, in that case, we have again, $R \times S'Q = (C + C') \times S'Q = (C \times S'Q) + (C' \times SQ.)$ But, as we have $S'Q = S'P' - P'Q = PQ - P'S'$, we obtain by substitution, $R \times S'Q = C \times (PQ - PS') + C' \times (S'P' - P'Q)$, and because $C \cdot PQ - C' \cdot P'Q = 0$, we have $R \cdot S'Q = C' \cdot S'P' - C \cdot S'P$.

77. If we denote as in art. 60. the perpendiculars from S upon the directions of C, C', and R, by p, p', ρ, we shall have the equation $R\rho = Cp + C'p'$, corresponding with the equation (VII) in that article. So that the consequence stated there, has equally place here, and the observations relative to the positive and negative signs, are equally useful in both cases.

78. Since a force O equal and opposite to the resultant R, acting at the point Q will sustain the system in a state of equilibrium, we have $O = -R$, and the preceding equation assumes this form:

$$R\rho + Cc + C'c' = 0.$$

Consequently, *when three parallel forces are in equilibrio, the sum of their moments with respect to any point in the same plane, is equal to zero.*

79. Drawing from the point S (fig. 3. pl. II.) any right line whatever spp', and supposing the three forces C, C', O, which are in equilibrio, applied at the points p, p', q, we have, by reason of the parallels, $SQ : SP : SP' :: sq : sp : sp'$. But the equation of the moments (78.) is $O \cdot SQ + C \cdot SP + C' \cdot SP' = 0$: in which, substituting the consequents of the proportion for the antecedents, there results $O \cdot sq + C \cdot sp + C \cdot sp' = 0$; whence it follows, *generally, that the resultant of two parallel forces divides any right line to the extremities of which they are applied into parts reciprocally as the forces:* and moreover that the results in arts. 71 .... 75. are applicable to all parallel forces acting upon one line, without regarding the angle that line makes with the direction of the forces.

80. PROP. *When a given power acts upon a certain point in a line, to determine its effects upon any other two points in the same line.*

Let the force R act upon the point q, it is required to determine the effort exercised upon the points p, p' (fig. 3. pl. II.). This is nothing else than to resolve the force R into the two parallel components C, C', acting at the proposed points. Consequently we must have $\mathrm{c} + \mathrm{c}' = \mathrm{R}$, and $\mathrm{c} : \mathrm{c}' :: qp' : qp$; whence, $\mathrm{c} + \mathrm{c}' : \mathrm{c} :: qp' + qp \; (= pp') : qp'$, and $\mathrm{c} = \frac{\mathrm{R} \cdot qp'}{pp'}$. In like manner we find $\mathrm{c}' = \frac{\mathrm{R} \cdot qp}{pp'}$.

81. **Scholium.** We may now deduce from what is gone before, a few theorems, which will be of utility in enquiries respecting parallel forces in one plane. Thus let c, c', c'', &c. constitute a system of parallel forces acting in one plane, R their resultant, p, p', p'', ϱ, &c. the perpendiculars demitted from any point in the same plane, upon the directions of the several forces. Then with relation to two forces c, c', we have $\mathrm{R} = \mathrm{c} + \mathrm{c}'$, and $\mathrm{R}\varrho = \mathrm{c}p + \mathrm{c}'p'$, for the equations by which we may determine the magnitude and position of R. Substituting for the two forces c, c', their equivalent R, and compounding this with the third force, we find the new resultant, by means of the equations $\mathrm{R}' = \mathrm{R} + \mathrm{c}''$ and $\mathrm{R}'\varrho' = \mathrm{R}\varrho + \mathrm{c}''p''$, or,

(I.) $\mathrm{R}' = \mathrm{c} + \mathrm{c}' + \mathrm{c}'' + \&\mathrm{c}.$ and $\mathrm{R}'\varrho' = \mathrm{c}p + \mathrm{c}'p' + \mathrm{c}''p'' + \&\mathrm{c}.$

And thus may we proceed with other forces.

When the forces are in a state of equilibrium, the equations become

(II.) … $\mathrm{c} + \mathrm{c}' + \mathrm{c}'' + \&\mathrm{c}. = 0.$ .. $\mathrm{c}p + \mathrm{c}'p' + \mathrm{c}''p'' + \&\mathrm{c}. = 0.$

When the equilibrium does not obtain, the first equation (1) determines the magnitude of the resultant: its direction will evidently be parallel to those of the components, and its position will be determined from this equation:

$$\varrho = \frac{\mathrm{c}p + \mathrm{c}'p' + \mathrm{c}''p'' + \&\mathrm{c}.}{\mathrm{c} + \mathrm{c}' + \mathrm{c}'' + \&\mathrm{c}.} = \frac{\mathrm{c}p + \mathrm{c}'p' + \mathrm{c}''p'' + \&\mathrm{c}.}{\mathrm{R}}$$

In these equations we consider as negative the forces which act in a contrary direction to those which we reckon positive: and if R come out negative, the resultant of the system will be a force acting in a contrary direction to the forces we account affirmative: if ϱ be negative, the resultant must be disposed, with regard to the origin of the moments, on the side opposite to that on which the forces are whose distances from this point we considered as positive.

81 a. **Prop.** *To find the resultant of any number of forces acting upon different points of a body, their directions being all in the same plane.*

This is performed very readily by a graphic process similar to that described in art. 51. Thus, let any two of the forces be taken, and, prolonging their directions, conceive them to be applied at their point of concourse (32.) and find their resultant by means of the parallelogram of forces (42.). Compound this, in like manner with any one of the remaining forces; and so on: the magnitude and direction of the diagonal of the last parallelogram, will shew the magnitude and direction of the force equivalent to the several components. For an example, let it be proposed to find the equivalent of the three forces c, c′, c″, acting upon the body AB (fig. 4. pl. II.) in the directions CP, C′P′, C″P″, and with intensities proportional to those lines. Produce CP, C′P′, till they meet in D, and on those directions set off DE, DF, respectively equal to CP, C′P′: complete the parallelogram EF, and its diagonal will represent in magnitude and direction the equivalent of the two forces c,c′. Then produce GD till it meet C″P″ produced in *d*: make *de* = DG, and *df* = P″C″, and complete the parallelogram *ef*. Lastly, produce the diagonal *dg*, till QR is equal to it; so will RQ shew the magnitude, position, and direction of the force equivalent to the three components c, c′, and c″.

82. Prop. *When three forces in the same plane, acting upon different points of a body (considered as void of gravity) keep it in equilibrio, they are such as would balance if applied to one point; their directions continuing parallel.*

If the forces c, c′, o, (fig. 5. pl. II.) which act upon the body AB, at the points P, P′, Q, in the directions CP, C′P′, OQ, with energies proportional to those lines, keep it in equilibrio, they would balance if applied to one point. For, producing two of the directions *e. g.* CP, C′P′, till they meet at a point D, making ED, FD, equal to CP, C′P′, respectively, and completing the parallelogram EF, its diagonal GD will represent the magnitude and direction of the equivalent of the components c, c′. And since no single force but one that is equal and opposite to the resultant of c, c′, can keep them in equilibrio, the direction of the third force o must pass through D, and its magnitude OQ must be equal to GD, or DG. Q. E. D.

83. Cor. 1. *Any two of these forces are inversely proportional to the perpendiculars demitted upon their respective directions, from the point of application of the third force, or from any point in its direction.*

For (48.) o : c : c′ :: sin CDC′ : sin ODC′ : sin ODC (fig. 6. pl. II.) And if we produce the directions of any two of the forces as CP, C′P′, and upon them let fall from Q, the point of application of the third force, the perpendiculars QK, QI, or from any other point *q* in the direction of that force, the perpendiculars *qk*, *qi*, these perpendiculars will be to each other as the sines of the angles QDK, QDI, or as the sines of the supplemental angles ODC, ODC′. Wherefore c : c′ :: QI : QK :: *qi* : *qk*.

84. Cor. 2. *When four forces are in equilibrio by the intervention of a solid body, they are such as would balance each other if applied to one point.*

Thus (fig. 5. pl. II.) the four forces c, c′, c″, c‴, which when the parallelograms of forces are constituted on the respective directions, give the equal and opposite resultants GD, Dg, are in equilibrio whether the points of application be $P, P' P'' P'''$, or whether they are blended in D.

85. Cor. 3. *When any number of forces are in equilibrio by the intervention of a solid body, they would balance each other if applied at one point.* For either the lines of directions of all the forces will meet in one point when produced, or they will meet in different points forming so many systems in equilibrio; and all these systems when applied to one point, are in equilibrio, by art. 37.

86. Prop. *If a body* AB *(fig. 7, pl. II.) be sustained in equilibrio by the simultaneous action of several forces* C, C', C'', *&c. in one plane, their quantities and directions being represented by* CP, $C'P'$, $C''P''$, *&c. cutting any line* AX *drawn through the body, in the points* P, P', P'', *&c. and if lines* CS, $C'S'$, $C''S''$, *&c. be drawn parallel to each other from the points,* C, C', C'', *till they intersect the line* AX*: then,* 1. *the sums of the portions* PS, PS, $P'S'$, *and* PS', $P''S''$, $p''s''$, *estimated in contrary directions must be equal.* 2. *The sums of the parallel forces on each side* AX, *i. e.* $CS, + C'S' + C''S''$, *and* $cs + c's' + c''s''$, *must be equal.* 3. *The sums of the moments on each side from any point* A, *i. e.* $AP \cdot C + AP' \cdot C' + AP'' \cdot C''$ *and* $AP \cdot c + AP \cdot c' + Ap'' \cdot c''$, *must be equal.*

Conceive the various parallelograms of forces to be constructed, as in the figure, then will the force represented by CP, be the equivalent of DP, SP, the force $C'P'$, the equivalent of $D'P'$, $S'P'$, and so of the others. And it is evident that the forces PS, $P'S'$, &c. must make up equal sums in the contrary directions, otherwise, instead of the system being at rest, it would move either in the direction of AX or of XA. And with respect to the parallel forces DP, $D'P'$, &c. or their equals CS, $C'S'$, &c. they fall under the deductions in arts. 75 .... 79. Whence the truth of the proposition is manifest.

87. Cor. If we consider the forces reduced to AX to have either positive or negative signs according as they act towards or from A; and the parallel forces on different sides of AX to have contrary signs: then will the sum of the forces reduced to AX, the sum of the parallel forces CS, $C'S'$, &c. and the sum of the moments of the forces with respect to any point on AX, be each equal to nothing.

88. Schol. Since the preceding proposition is true, whatever direction the parallel lines CS, $C'S'$, &c. make with the assumed line AX, we may readily deduce from hence the formulæ for several forces acting at different points in various directions in the same plane, according to the method of rectangular co-ordinates. To this end, let CS, $C'S'$, &c. (fig. 7.)

be perpendicular to AX, and let the ordinate AY be drawn parallel to CS. Then if the angles which the directions of the forces C, C′, C″, &c. make with AX, be denoted by a, a', a'', &c. and the angle made by R the resultant with the same be r; also if the ⊥s from A upon CP, C′P′, C″P″, &c. be represented by p, p', p'', &c. we shall have SP = C cos a, S′P′ = C′ cos a', &c. and CS = C sin a, C′S′ = C′ sin a', &c. And in the case of an equilibrium, we shall have the following equations, the resultant being absolutely nothing.

$$\text{(I.)} \quad . \quad \begin{cases} c \cos a + c' \cos a' + c'' \cos a'' + \&c. = x = 0. \\ c \sin a + c' \sin a' + c'' \sin a'' + \&c. = y = 0. \\ cp + c'p' + c''p'' + \&c. = \pi = 0. \end{cases}$$

These equations are (like those in arts. 59, 70, 81,) each composed of as many terms as there are forces, unless one of the forces should have its direction coinciding either with AX, or AY, or parallel to either of them, or passing through A their assumed origin, — when one term will manifestly vanish, in the corresponding equation.

89. If the system of forces is not in equilibrio, the subsequent equations will assist in finding the resultant, its magnitude, and its point of application. Besides the above characters, let ρ denote the perpendicular An from the origin of the axes upon the direction of the resultant: then, since an equal force to the resultant applied in a contrary direction restores the equilibrium, making it to subsist between C, C′, &c. and $-$ R, by introducing the expressions $-$ R cos r, $-$ R sin r, and $-$ Rρ, into the equations, we shall get,

$$\text{(II.)} \quad . \quad . \quad R \cos r = x, \quad . \quad . \quad R \sin r = y, \quad . \quad . \quad R\rho = \pi.$$

Adding together the squares of the two first of these, there will be found

$$\text{(III.)} \quad . \quad . \quad . \quad . \quad R = \sqrt{x^2 + y^2}.$$

And from the equations (II) we deduce, by division,

$$\text{(IV.)} \quad . \quad \cos r = \frac{x}{R} \; . \, . \; \sin r = \frac{y}{R} \; . \, . \; \tan r = \frac{y}{x} \; . \, . \; \rho = \frac{\pi}{R}. \; \}$$

Of these equations III determines the magnitude; either of the first three of IV ascertains its direction, and the last gives the perpendicular distance of its direction from the origin of the ordinates. Thus, drawing a line An = ρ, making with AY an angle = r, and drawing nq perpendicular to An, the line nqR will be the position of the resultant; any point in this line may be considered as the point of application: the signs of r and of ρ, will determine on which side of AY and of AX, the line An must be drawn.

90. When the three equations (88. I.) all obtain at the same time, they denote that a system of forces disposed in the same plane, but not concurring in the same point, is perfectly in

equilibrio: but it will not be entirely uninteresting to consider what will be the consequence when all three have not place at once. When only the first, or the second exists, we may apply the remarks made at the end of art. 59, so that it only remains for us to examine here, what will occur when the third only exists. Thus, if instead of the system being in absolute equilibrium, we have the three following equations:

$$\text{R} \cos r = x, \qquad \text{R} \sin r = y,$$
$$\text{C}p + \text{C}'p' + \text{C}''p'' + \&\text{c.} = \text{R}\rho = 0.$$

Here $\text{R}\rho = 0$, gives $\rho = 0$, that is to say, the resultant passes through the origin A of the co-ordinates: it is obvious that if that point in the system is fixed, this resultant ought to be annihilated, and that there is an equilibrium whatever x and y may be. The equation $\text{C}p + \text{C}'p' + \text{C}''p'' + \&\text{c.} = 0$, therefore, suffices solely to indicate that there is an equilibrium, on the supposition that the origin is a fixed point: and hence

In order that a system of forces disposed in the same plane be in equilibrio about any fixed point, it will be sufficient if the sum of the moments of the forces, with respect to this point, be equal to zero.

Supposing that a body subjected to the action of several powers is retained by a fixed point, it follows necessarily that it can only have a motion of rotation about this point: if, therefore, the equilibrium does not exist, neither can the resultant pass by this point, nor can we have $\text{C}p + \text{C}'p' + \text{C}''p'' + \&\text{c.} = 0$. Hence, when $\pi = 0$, there can be no rotatory motion, and when both $x = 0$, and $y = 0$, there can be no rectilinear motion.

IV. *Of Forces not confined to one Plane, and directed to various Points of a Body.*

91. Def. *Any point in a system of parallel forces, through which the resultant passes, and which retains the same place, although all the forces change their directions, provided they continue respectively parallel, may be called the Centre of Parallel Forces.*

92. Prop. *To find, by a graphic process, the resultant of any number of parallel forces, however disposed.*

Conceive any plane to be intersected by the directions of the several forces, and the points of intersection to be joined by right lines falling on the plane. Then, by the method already given (71, 79.) find the resultant of any two of the forces and its point of application at the imaginary plane: then take this resultant and its point of application, and in like manner compound with a third force, and ascertain their resultant and its point of application: let this new resultant be compounded with a fourth force, and so on through the whole. Thus for example, let the five forces represented by C, C', C'', C''', c, have P, P', P'', &c. for their points of application in the plane $PP'P''$ (fig. 8, pl. II.) In the right line PP' make $Pq : qP' :: C' : C$, and q will be the point of application of $R = C + C'$, the resultant of C and C'. Join the points q, P', in the same plane, and make $qq' : q'P'' :: C'' : R$, then will q' be the point of application of the new resultant $R' = R + C'' = C + C' + C''$. Again, for the fourth force C''', join $q'P'''$, and make $q'q'' : q''P''' :: C''' : R'$, then q'' will be the point of application of the third resultant $R'' = R' + C''' = C + C' + C'' + C'''$. Lastly, join $q''p$, and make $q''q''' : q'''p :: c : R''$, then will q''' be the point of application of the resultant $R''' = R'' + c = C + C' + C'' + C''' + c$ Had any one or more of these powers been exerted in an opposite direction, as for instance C'' and c, the point of application of the resultant R''' would still have been the same; but its magnitude would then have been $C + C' - C'' + C''' - c$: as is evident from art. 73.

93. COR. 1. If the components C, C', C'', &c. all change their directions in such a manner as to remain parallel to each other, their resultant will still be applied to the same point in the plane $PP'P''$; or, if all the powers change their magnitudes in one and the same ratio, although the resultant will undergo a corresponding mutation in its magnitude, its point of application will still be the same; and will, therefore, be the *centre of parallel forces* in the system whose variations are thus regulated.

94. COR. 2. Hence it will be easy to put any system of parallel forces into an equilibrated state: for it requires nothing more than to find, by this proposition, the magnitude and point of application of the resultant; and then to apply an equal force to the same point in an opposite direction.

95. SCHOLIUM. Here also it may be proper to deduce a few general theorems which may be useful in future enquiries relative to parallel forces however disposed. For which purpose let us begin with three parallel forces C, C', C'', in a state of equilibrium, which, it is obvious, must necessarily be in one plane: let YAX (fig. 9. pl. II.) be the plane of the rectangular co-ordinates to which we mean to refer; this plane must be cut (Euc. xi. 3.) by the plane in which are the directions of the forces, in a right line, suppose $ACC'C''$ in the figure; in this line let C, C', C'', be the points where those three forces are applied in the plane of the axes. From any point A in the line AC'' draw any rectangular co-ordinates AX, AY, and on these let fall from C, C', C'', the perpendiculars Cd, $C'd'$, $C''d''$, and CD, $C'D'$, $C''D''$: let $Ad = d$, $Ad' = d'$, $Ad'' = d''$, $AD =$

D, $AD' = D'$, $AD'' = D''$. Then, since what has been shewn in art. 81, is applicable here, we have, in addition to the condition that the points C, C', C'', are in one right line, these equations:

$$C\cdot AC + C'\cdot AC' + C''\cdot AC'' = 0, \text{ and } C + C' + C'' = 0.$$

But by reason of the parallel lines, we have $AC : AC' : AC'' :: Ad : Ad' : Ad'' :: AD : AD' : AD''$, and substituting the consequents of these for their respective antecedents in the first preceding equation, we obtain the following equations of condition for three parallel forces in equilibrio:

$$(\text{I.})\ \ldots\ldots \left\{\begin{aligned} Cd + C'd' + C''d'' &= 0 \\ CD + C'D' + C''D'' &= 0 \\ C + C' + C'' &= 0 \end{aligned}\right.$$

Some persons, from a slight consideration, might conclude, that since the quantities in the two first of these three equations are proportional, they need not both be taken at the same time: but it should be recollected, that when any one of them is taken together with the third, this does not ensure the essential condition of the three points C, C', C'', being in one right line; for while d, d', d'', remained the same, the points might not be found in AC'', but in any unlimited positions upon Cd, $C'd'$, $C''d''$; whereas the contemporaneous existence of all the three equations establishes the proper magnitudes of the forces, and confines their points of application to the intersections of DC, dC, &c. upon AC''.

96. A force $-R$ which is equal and applied in an opposite direction to the resultant R of two forces C, C', will manifestly establish the equilibrium. Therefore if x and y be put for the distances at which perpendiculars from the point of application of the resultant will cut the axes AX, and AY; and $-R$, x, and y, being substituted for C'', d'' and D'', in the equations I, they will be transformed to

$$\begin{aligned} Rx &= Cd + C'd' \\ Ry &= CD + C'D' \\ R &= C + C' \end{aligned}$$

And these equations will evidently serve to determine the magnitude and position of the resultant of two forces.

Having four forces C, C', C'', C''', if we compound two of them, as C, C', into one force R, we may establish the equilibrium between this and the other two by causing them to satisfy the equations (I): we shall have, therefore,

$$\begin{aligned} Rx + C''d'' + C'''d''' &= 0 \\ Ry + C''D'' + C'''D''' &= 0 \\ R + C'' + C''' &= 0. \end{aligned}$$

But the first terms of each of these equations being already known, we may substitute for them their values, and we shall thence obtain the equations for four forces in equilibrium:

$$\text{viz} \ldots\ldots \left\{\begin{array}{l} \mathrm{C}d + \mathrm{C}'d' + \mathrm{C}''d'' + \mathrm{C}'''d''' = 0 \\ \mathrm{CD} + \mathrm{C}'\mathrm{D}' + \mathrm{C}''\mathrm{D}'' + \mathrm{C}'''\mathrm{D}''' = 0 \\ \mathrm{C} + \mathrm{C}' + \mathrm{C}'' + \mathrm{C}''' = 0 \end{array}\right.$$

Here, however, it is not necessary that the directions of all the four forces should be in one plane: it is merely requisite that the three forces R, C, C', should act in one plane, and R, C'', C''', in one plane; or that the plane which contains the directions of C, C', should intersect that which contains the directions of C'', C''', in a right line which will be the resultant of both these pairs of forces. Now, on comparing the equations last exhibited with the equations (95. I.) it will appear, that they are exactly the same, excepting that each of the latter comprises one term more than the former: and, if a similar process were adopted with respect to a fifth force, similar equations would be obtained with another additional term in each. Hence, we may generalise without hesitation, and for as many parallel forces as we please may lay down the following equations, as conditions of equilibrium.

$$\text{(II.)} \ldots\ldots \left\{\begin{array}{l} \mathrm{C}d + \mathrm{C}'d' + \mathrm{C}''d'' + \&\text{c.} = 0 \\ \mathrm{CD} + \mathrm{C}'\mathrm{D}' + \mathrm{C}''\mathrm{D}'' + \&\text{c.} = 0 \\ \mathrm{C} + \mathrm{C}' + \mathrm{C}'' + \&\text{c.} = 0 \end{array}\right.$$

Each of these equations will comprise as many terms as there are forces, unless one of these forces have its direction passing through the origin A of the axes, in which case one of the terms will vanish from each of the first two equations: as to the signs, they will be positive or negative according as the forces to which they are attached act in the same or a contrary direction, or as their points of application fall on the same or different sides of either of the co-ordinates.

97. When the equilibrium does not obtain, we shall by proceeding as in the former part of art. 96. have these equations:

$$\text{(III.)} \ldots \left\{\begin{array}{l} \mathrm{R}x = \mathrm{C}d + \mathrm{C}'d' + \mathrm{C}''d'' + \&\text{c.} \\ \mathrm{R}y = \mathrm{CD} + \mathrm{C}'\mathrm{D}' + \mathrm{C}''\mathrm{D}'' + \&\text{c.} \\ \mathrm{R} = \mathrm{C} + \mathrm{C}' + \mathrm{C}'' + \&\text{c.} \end{array}\right.$$

If the plane XAY be perpendicular to the directions of the forces, a supposition which will not diminish the universality of the deductions; and if we conceive AX and AY to be two planes perpendicular to XAY, the rectangles $\mathrm{R}x$, and $\mathrm{R}y$, will then become *moments* (31.) of the resultant, taken with regard to each of those two planes; and $\mathrm{C}d$, CD, will in like manner be moments of C; and so of the others. Hence then, taking the word *sum* in the sense we have so often explained, we may deduce the following theorem for all parallel forces, however situated:

The resultant of any number of parallel forces whatever is parallel to them, equal to their sum, and has its moment (esti-

mated with respect to every parallel plane) equal to the sum of the moments of the components.

Thus the resultant is determined in magnitude and direction, and the point in which it intersects the plane XAY may be readily found by means of these equations:

$$\text{(IV.)} \ldots \begin{cases} x = \dfrac{\mathrm{C}d + \mathrm{C}'d' + \mathrm{C}''d'' + \&\mathrm{c.}}{\mathrm{R}} \\ y = \dfrac{\mathrm{CD} + \mathrm{C}'\mathrm{D}' + \mathrm{C}''\mathrm{D}'' + \&\mathrm{c.}}{\mathrm{R}} \end{cases}$$

98. The equilibrium will be absolute when all the three equations (II.) obtain at once; but it will not be useless if we enquire what will be the consequence when the equations of condition are taken partly from equa. II, and partly from equa. III. If, for example, we have the following equations:

$$\mathrm{R} = \mathrm{C} + \mathrm{C}' + \mathrm{C}''$$

$$\mathrm{R}x = \mathrm{C}d + \mathrm{C}'d' + \mathrm{C}''d'' + \&\mathrm{c.} = 0$$

$$\mathrm{R}y = \mathrm{CD} + \mathrm{C}'\mathrm{D}' + \mathrm{C}''\mathrm{D}'' + \&\mathrm{c.}$$

Here $\mathrm{R}\,x$ being $= 0$, and R an absolute quantity, we conclude that $x = 0$; in this case the direction of the resultant will intersect the plane XAY somewhere in AY. If therefore this axis is fixed in the system, whatever R and y may be, the only necessary condition of equilibrium is indicated by the equation

$$\mathrm{C}d + \mathrm{C}'d' + \mathrm{C}''d'' = 0$$

In like manner the equation

$$\mathrm{CD} + \mathrm{C}'\mathrm{D}' + \mathrm{C}''\mathrm{D}'' = 0$$

has place when the direction of the resultant passes through some one of the points in AX; and when this axis is fixed in the system, this equation is alone necessary to indicate an existing equilibrium.

If we suppose that the plane XAY is perpendicular to the directions of the forces, the terms of these two equations denote the moments of these forces with relation to planes passing through AX and AY perpendicular to their plane. Hence it follows that,

In order to have a system of parallel forces in equilibrio about any fixed axis, it is necessary that the sum of the moments of these forces with respect to a plane parallel to their forces, and passing through that axis, shall be equal to zero.

If the two equations $\mathrm{R}x = 0$, and $\mathrm{R}y = 0$, exist together, there will result $x = 0$, $y = 0$; that is, the resultant will pass both through some point in AX and some point in AY; that is, it will pass through A the origin of the axes: if, therefore, this point be fixed, these two equations denote an equilibrium.

Again, if we have only this equation

$$\mathrm{R} = \mathrm{C} + \mathrm{C}' + \mathrm{C}'' + \&\mathrm{c.} = 0,$$

we are not to conclude that because $\mathrm{R} = 0$, the equilibrium obtains: for this isolated equation merely denotes that the sum of the forces which act in one direction, is equal to the sum of those which are exerted in the contrary direction; and this may take place independently of the positions of the forces: but the equilibrium exists only when the equal and opposite forces are applied at the same point. In this case, however, we may always establish the equilibrium by means of one additional force; or we may restore it in an endless variety of ways by means of two additional forces.

99. PROP. *To find the resultant, or resultants, of a system of forces applied to different parts of a body, and acting in various directions in different planes.*

The best method of performing this will be by the method of rectangular co-ordinates, as follows. Suppose the directions of the several forces prolonged till they meet the plane YAX (fig. 9. pl. II. or fig. 18. pl. I.), and conceive them all applied at their points of intersection with that plane: then each of these forces may be resolved into two others, the one perpendicular to the plane YAX, the other situated in the plane. The perpendicular forces will have for their resultant a force in like manner perpendicular to the plane, and the forces situated in the plane will obviously have their resultant lying in the same plane. If the directions of these resultants meet in a point, they may be compounded by art. 41. and the system will have *one resultant:* but in many, indeed, in most cases, these directions will *not* meet; and then, as the effects of perpendicular forces not meeting in a point are independent, the system will have *two distinct resultants.*

COR. In the first of the above cases a single additional force may restore the equilibrium; in the latter, two at least will be requisite.

100. PROP. *If a body be kept in equilibrio by several forces acting at different points and in various directions not in one plane, the forces are such as would be in equilibrio if applied to one point, and in directions respectively parallel to the former.*

For in any assumed plane the forces parallel and perpendicular to any line, will be the same whether applied at one point or many: and when the directions of any of the forces are out of this plane, such extraneous forces may be reduced to others, one set acting in the plane, the other perpendicular to it; and both these will be equal in quantity in each case; therefore if the equilibrium obtains in the one case, it must in the other, both with respect to the perpendicular and parallel forces, all which will be sufficiently obvious after recollecting what was shewn with regard to several forces in one plane. (arts. 82...86.)

If the weight of the body be taken into the consideration it may be regarded as a single force applied vertically at the centre of gravity: the reason of which will appear from the discussions in the next chapter.

SCHOLIUM.

101. We might now proceed to deduce the equations of equilibrium, &c. for forces acting at different points and in various planes: but as the process would be complex and intricate, and after all but of trifling utility in elementary mechanics, except in the case of parallel forces already treated in arts. 95 ...98. it is thought best to omit them. We add here what follows at once from a comparison of the preceding proposition with art. 70. namely that

When a system of forces acting upon different points of a body in various planes keeps it in equilibrio, the sum of the moments taken relatively to each of three rectangular co-ordinates, of the components estimated in a plane perpendicular to that ordinate, is equal to zero.

CHAPTER III.

Centre of Gravity.

102. DEF. *THE Centre of Gravity of any body or system of bodies is that point about which the body or system, acted upon only by the force of gravity, will balance itself in all positions: or it is a point which when supported, the body or system will be supported, however it may be situated in other respects.*

The centre of gravity of a body is not always *within* the body itself: thus the centre of gravity of a ring is not in the substance of the ring, but in the axis of its circumscribing cylinder; and the centre of gravity of a hollow staff, or of a bone, is not in the matter of which it is constituted, but somewhere in its imaginary axis. Every body, however, has a centre of gravity, and so has every system of bodies, as will soon be made evident: but it will be proper to premise a few brief remarks with respect to gravity itself, and its effect upon bodies subjected to its operation.

103. It is a fact established by general observation in all ages and all countries, that whenever bodies are unsupported or left to themselves, they begin to move downwards in vertical lines, and continue thus to move until they meet with something which interrupts their motion or prevents their further descent. This is observed to take place not only with respect to large and very ponderous bodies, but to smaller ones, and even to the most minute particles into which they can be separated, provided they are not so small as to elude the observation of our senses. And if certain substances, such as smoke, and vapours, &c. seem to contradict this universal fact; it is because they are only in *appearance* left to themselves, while in reality they are supported, and put into an ascending motion, by the action of the fluids, &c. that compose the atmosphere which surrounds the earth. All bodies, and their most intimate particles, tend towards a point which is either accurately or very nearly the centre of the terraqueous globe; yet this tendency is certainly not essential to matter, it is an effort which matter of itself is not able to make, being indifferent to either motion or rest (18, 26.): we are authorised, then, to conclude that this tendency to motion is caused by a power not existing in the matter on which our observations are made, but in something exterior; and this

force, without attempting to explain its nature and essence, we designate by the term *Gravity:* the general fact or event of bodies falling is denoted by the verbal noun *Gravitation;* and it is a part or consequence of a more universal property, not here entered upon,—that of the mutual *Attraction* of the different bodies in the universe towards each other.

104. Since gravity impresses, or has a tendency to impress, on every particle of bodies, in an instant, a certain velocity with which they would begin to fall, if they were not supported; and since, abstracting the influence of the air, this velocity would be the same for each of the moleculæ of bodies, whatever be their substance, it will not be difficult to attach a just and scientific meaning to that which is commonly called *weight:* it is the effort necessary to prevent a body from falling. But bodies fall in consequence of the action of the force of gravity upon each of their particles, and they can be prevented from falling by a force equal and opposite to the resultant or equivalent of all these actions. Hence, we may readily distinguish between the effect of gravity and that of weight, by adopting the language of *Condorcet,* when he says "the former is the power of transmitting, "or a tendency to transmit into every particle of matter a certain "velocity which is absolutely independent on the number of "material particles; and the second is the effort which must be "exercised to prevent a given mass from obeying the law of "gravity. *Weight,* accordingly, *depends on the mass, but gravity has no dependance at all upon it.*"

A verbal distinction is also made between *weight* and *heaviness.* Thus *heaviness* is that quality of a body which we feel and distinguish by itself: *weight* is the measure and degree of that quality, which we ascertain by comparison. Absolutely and in an undetermined sense, we say that a thing is *heavy;* but relatively and in a manner determined, that it is of such a *weight,* as of 2, 3, 4 pounds, &c. In illustration we may add that many circumstances prove the *heaviness* of atmospheric air; but the mercury in a barometer determines its exact *weight.*

105. Every particle of which bodies are composed receiving from gravity equal solicitations towards the centre of the earth, it follows that if the supports of bodies, whether large or minute, were taken away, and they were permitted to fall from equal altitudes, they would arrive at the surface of the earth after equal portions of time: and this is confirmed by experience; for under the exhausted receiver of the Air-pump (where the resistance of the air is removed) the heaviest metals and the lightest feathers, or down, fall in the same time. If, therefore, a body is divided into ever so many parts, each of them left to itself would arrive

at the surface of the earth, after the same time as would have been employed by the whole body in descending. All bodies being more or less porous, and possessing different degrees of density (10.) they will contain a greater or less number of equal moleculæ in the same volume or bulk; hence all bodies of equal bulk are not equal in weight. But since the weight is equal to the sum of all the efforts exercised by gravity upon the constituent moleculæ of a body, it is proportional to its *density* or to its *mass*. If p, p', p'', &c. be the several particles of which a body is composed, and M its mass, then will $M = p + p' + p'' +$ &c, and if g represent the force of gravity soliciting each particle, we shall have the weight $= gM = gp + gp' + gp'' +$ &c.

106. When bodies are composed of moleculæ, which are of the same size and substance, and similarly posited throughout, they are said to be *homogeneous:* such are the bodies which we shall consider in this chapter; and in which the mass will manifestly be proportional to the extension or the magnitude, so that the one may be substituted for the other in our investigations. The vertical lines which would be described by bodies if subjected to the free action of gravity, are frequently called *lines of direction*. Since they would, if produced, meet at the centre of the earth, they cannot, strictly speaking, be parallel: but, with respect to any body or any system of bodies connected for mechanical purposes, the whole space occupied by all their particles must be so very minute compared with the magnitude of the earth, that their several lines of direction may be considered as parallel without any danger of sensible error; just as we speak of a moderate portion of the earth's surface as a plane, although it is, in fact, nearly spherical. Hence, then, the actions of gravity upon a body, or system, may be considered as those of parallel forces applied to their various particles; and, of consequence, the conclusions and theorems which were deduced, arts. 92....98. with regard to such forces, may be adopted in our present investigations relating to the *centre of gravity*. This being admitted, the ensuing particulars are without difficulty inferred.

I. By the definition of the centre of gravity, when *it* is supported the body is in equilibrio; and from the nature of equilibrium it can only be produced singly by the exercise of a force equal and opposite to the resultant of all the other forces acting upon the several particles of the body, that is, since in this case the forces are parallel, by a force (104.) equal to the weight of the body applied at the centre of parallel forces (91.): consequently the centre of gravity coincides with the centre of parallel forces.

II. Varying the position of the body, will not cause any change in the centre of gravity; since any such mutation will be nothing more than changing the directions of the forces, without their ceasing to be parallel; and if the forces do not continue the same, in consequence of the body being supposed at different distances from the earth, still the forces upon all the moleculæ vary proportionally, and their centre remains unchanged (93.).

III. Let any system be conceived in which no other forces than weights are applied; and let it be imagined of any form or construction whatever, but without any motion. In this case, whatever be the disposition of the bodies of the system, it is clear that if there be an equilibrium, the sum of the resistances of the fixed points or obstacles, estimated in the vertical direction, will be equal to the total weight of the system. But if any motion arises, a part of the force of gravity will be employed in producing it, so that it is only with the surplus that the fixed points can be charged. Therefore, in this case, the sum of the vertical resistances of the fixed points will be less at the first instant of motion than the entire weight of the system: consequently, from those two forces combined, there will result a single force equal to their difference, which will solicit the system downwards. Hence the centre of gravity will necessarily descend with the velocity due to that difference: and hence it follows that *to assure ourselves that several weights applied to any system or machine whatever are in equilibrium, it suffices to prove that if the system be left to itself, its centre of gravity will not descend.*

The immediate and universal consequence of this principle is, that if the centre of gravity of any system is at the lowest point possible, there will necessarily be equilibrium: for, if not, the centre of gravity must descend; yet, how can it *descend* if it be already at the *lowest* point?

It would not, however, be correct to say, reciprocally, that always when equilibrium obtains the centre of gravity is at the lowest point possible: for that point might be (and in many cases of unstable equilibrium often *is*) in the *highest* possible position; or it might be found at neither the highest nor the lowest point. The exceptions occurring with sufficient frequency in the usual theory of *maxima* and *minima*. But the *principle*, as above stated, has no exception.

IV. When a heavy body is suspended by any other point than its centre of gravity, it will not rest unless that centre is in the same vertical line with the point of suspension: for in all other positions the force which is intended to ensure the equilibrium, will not be directly opposite to the resultant of the parallel forces of gravity upon the several particles of the body, and of course the equilibrium will not be obtained.

V. If a heavy body be sustained by two or more forces, their directions must meet either at the centre of gravity of that body, or in the vertical line which passes through it. Thus, let B (fig. 10. pl. II.) be a heavy body whose centre of gravity is G, it will be at rest if the string *a*B by which it is suspended hang vertically, whether the string itself be fixed at *a*, or be attached to two other strings whose lengths are *ba*, *ca*, and fixed at *b* and *c*. If the string *a*B were either longer or shorter, the point *a* continuing fixed and the positions of *b* and *c* being unchanged, the body would still hang at rest, and the strings *ba*, *ca*, under the same tension. If the body were removed vertically to the dotted situation in the diagram, it would be sustained in that position by the strings *bd* and *ce*, which would suffer the same tension as when they were united in the point *a*. If instead of the strings we applied props *fg*, *hi*, in the same directions on the opposite sides of the body, their feet *f*, *h*, being fixed; or, if other props *mn*, *op*, were applied, either parallel to the former, or having their directions meeting in the vertical line *q*G, the body would still be supported: and if the directions of the props and of the strings were parallel, the compression in the one case would be equal to the tension in the other. Either the compression or the tension may in every case be readily estimated by means of the parallelogram of Forces, &c. (46.) the weight of the body being known.

VI. When a body stands upon a plane, if a vertical line passing through the centre of gravity fall within the base on which the body stands, it will not fall over; but if that vertical line passes without the base, the body will fall, unless it be prevented by a prop or a cord. When the vertical line falls upon the extremity of the base, the body *may* stand, but the equilibrium may be disturbed by a very trifling force; and the nearer this line passes to any edge of the base the more easily may the body be thrown over; the nearer it falls to the middle of the base, the more firmly the body stands.

VII. The various motions of animals, if attentively considered, will appear to be regulated consistently with the principles just stated.

Thus, when a man endeavours to rise from his seat he thrusts forward his body, and draws his feet backward till the vertical line from the centre of gravity falls just before his feet; this enables or indeed compels him to rise; and to prevent falling forwards he advances one of his feet, till the vertical line of direction is brought between his feet, in consequence of which he may stand firmly. In walking, he first extends his hindmost leg and foot almost to a right line, and at the same time bends the knee of his fore leg a little; by this means his body is thrust forward, and the line of direction from its centre of gravity falls beyond the fore foot, on which account he is ready to fall, but prevents it by immediately taking up the other foot, and putting it forward beyond the line of direction. After the same manner, he thrusts himself forward by the leg which is now the hindmost, till the line of direction from the centre of gravity be beyond his fore foot, when he again sets his hind foot forward: and thus he continues the motion of walking at pleasure. While walking, a man always sets down one foot before the other is taken up; so that at each step he has both feet upon the ground. But in running he takes one up before he sets the other down; so that his feet touch the ground alternately for moments of time, and in the intermediate portions he does not touch it at all.

In walking up hill a man bends his body more forward than in walking on a horizontal road, that the line of direction may be thrown before his feet: in walking down hill he rather leans backwards to prevent the line of direction from being too forward, which would occasion his fall. In carrying a burthen a man always leans the contrary way to that in which the burthen lies, in order that the common centre of gravity of both may have its line of direction between his feet.

When a quadruped, as a horse, moves, he leans forward, all at once lifting up one of his fore feet and one of his hind feet; when the right leg before is pushed forward the left leg behind is moved on at the same time; and this motion being made, the left leg before takes its turn conjointly with the right leg behind, and so on: as the body when standing is supported by four props which form a rectangle, the most commodious mode of moving is to change the positions of two feet at a time diagonally, and thus to cause the centre of gravity of the animal's body to make but a small movement, and to remain always very nearly in the direction of the two points of support. This rule of motion is always observed, but with these differences: in the pace there are four times in the complete movement; if the fore right leg be moved first, the left leg behind follows the instant after; then the left fore leg has its turn, which is followed the next instant by the right foot behind. Thus the fore right foot comes first to the earth, the left foot behind next, the left fore foot third, and the right hind foot last: so that there are four motions, and three intervals, of which the first and last are shorter than the middle one. In trotting, he takes up two feet together, and sets down two together, diagonally opposite. In galloping he takes up his feet one by one, and sets them down one by one; but he strikes with the two fore feet nearly at once, and the two hind feet nearly at once: all the time the line from the centre of gravity lies forward, so that the animal requires the fore feet to come to the ground at short intervals to prevent a fall. If the various motions of other animals be considered, they will all be found conformable to the same principles. But it would be tedious to enter further into the detail in this place.

VIII. To find the centre of gravity mechanically, it is only requisite to dispose the body successively in two positions of equilibrium, by the aid of two forces in vertical directions, applied in succession to two different points of the body; the point of intersection of these two directions will shew the centre required.

This may be exemplified by particularising a few methods. If the body have plane sides, as a piece of board, hang it up by any point, then a plumb-line suspended from the same point will pass through the centre of gravity; therefore mark that line upon it: and after suspending the body by another point, apply the plummet to find another such line, then will their intersection shew the centre of gravity.

Or thus: hang the body by two strings from the same point fixed to different parts of the body; then a plummet hung from the same tack will fall on the centre of gravity.

Another method: Lay the body on the edge of a triangular prism, or such like, moving it to and fro till the parts on both sides are in equilibrio, and mark a line upon it close by the edge of the prism: balance it again in another position, and mark the fresh line by the edge of the prism; the vertical line passing through the intersection of these lines, will likewise pass through the centre of gravity. The same thing may be effected by laying the body on a table, till it is just ready to fall off, and then marking a line upon it by the edge of the table: this done in two positions of the body, will in like manner point out the centre of gravity.

107. When a plane or a line can be so drawn as to divide a solid or a plane into two parts equal and similar, or so that its moleculæ shall be disposed two by two, in the same manner, with respect to such plane or such line, we may call the body symmetrical with regard to that plane or axis. And in all such bodies, it is obvious that the sum of the moments of its several moleculæ with relation to such plane or axis, will be nothing: for, if we take two particles disposed in the same manner but on different sides, their moments will be equal but with contrary signs; and, consequently, their sum will be equal to *zero:* and

the same may be shewn of every other pair of moleculæ, similarly situated: whence, as (by hyp.) there are none but what are similarly situated, the resultant of the system will be in such plane, or line, (97.) and, of consequence, its centre of gravity will be there also. The same reasoning may be extended to the centre of figure or of magnitude, that is, the point with respect to which a whole body shall be symmetrical. Hence we conclude that the centre of gravity of a right line, or of a parallelogram, prism, or cylinder, is in its middle point; as is also that of a circle, or of its circumference, or of a sphere, or of a regular polygon; that the centre of gravity of a triangle is somewhere in a line drawn from any angle to the middle of the opposite side; that of an ellipse, a parabola, a cone, a conoid, a spheroid, &c. somewhere in its axis. And the same of all symmetrical figures whatever.

108. Prop. *To deduce some general theorems which may be useful in finding the centre of gravity of any proposed body.*

The determination of the centre of gravity, being reduced to that of the centre of parallel forces, we may here adopt the theorem for the moments laid down at the end of art. 97. From which it will follow, that if p, p', p'', &c. (fig. 11. pl. II.) be equal material particles, and g the point through which the resultant R of the gravitating forces upon these particles always passes; and ABCD be a vertical plane, on which perpendiculars from p, p', p'', and g are let fall, then will the sum of the products of the forces upon p, p', p'', into their respective distances from ABCD, be equal to the product of the resultant R into its distance, where the force R would be equal to those upon $p+p'+p''$. The same would likewise obviously hold with respect to perpendiculars upon the other plane ABCG: and since the same will also obtain with relation to any vertical plane, although the position of p, p', and p'' be changed, provided they retain their *relative* situations, it will of course obtain when the position of the system is so varied that AEBF becomes a vertical plane: consequently the equality of the products may be affirmed with regard to all the three planes at the same time, and if the distances from the several planes be referred to the rectangular co-ordinates AX, AY, AZ, we may readily appropriate the equations (97. III.) to our present purpose. Denote, as before, the force of gravity by g, the distances referred to AX by d, d', d'', &c. the distances referred to AY, by D, D$'$, D$''$, &c. and those referred to Z by δ, δ', δ'', &c. the distances from the centre of parallel forces to the same axis being denoted by X, Y, and Z: then we shall have

$$\text{R}\,\text{X} = gpd + gp'd' + gp''d'' + \&\text{c}.$$

$$\mathrm{RY} = gp\,\mathrm{D} + gp'\mathrm{D}' + gp''\mathrm{D}'' + \&c.$$
$$\mathrm{RZ} = gp\delta + gp'\delta' + gp''\delta'' + \&c.$$

But $\mathrm{R} = gp + gp' + gp'' + \&c.$ and $\mathrm{M} = p + p' + p'' + \&c.$ whence

$$\text{(I.)} \ldots \begin{cases} \mathrm{X} = \dfrac{pd + p'd' + p''d'' + \&c.}{\mathrm{M}} \\ \mathrm{Y} = \dfrac{p\mathrm{D} + p'\mathrm{D}' + p''\mathrm{D}'' + \&c.}{\mathrm{M}} \\ \mathrm{Z} = \dfrac{p\delta + p'\delta' + p''\delta'' + \&c.}{\mathrm{M}} \end{cases}$$

Here, if we adopt the language of fluxions, and put x, y, and z, for the variable distances from A upon AX, AY, and AZ, respectively, we may convert these equations into the following form, which will render it more useful in many investigations.

$$\text{(II.)} \ldots \begin{cases} \mathrm{X} = \dfrac{\text{fluent of } x\,\dot{\mathrm{M}}}{\text{fluent of } \dot{\mathrm{M}}} = \dfrac{\text{fluent } x\,\dot{\mathrm{M}}}{\mathrm{M}}. \\ \mathrm{Y} = \dfrac{\text{fluent of } y\,\dot{\mathrm{M}}}{\text{fluent of } \dot{\mathrm{M}}} = \dfrac{\text{fluent } y\,\dot{\mathrm{M}}}{\mathrm{M}}. \\ \mathrm{Z} = \dfrac{\text{fluent of } z\,\dot{\mathrm{M}}}{\text{fluent of } \dot{\mathrm{M}}} = \dfrac{\text{fluent of } z\,\dot{\mathrm{M}}}{\mathrm{M}}. \end{cases}$$

As these values together determine only one point, we see that a body has but one centre of gravity; of which the three equations determine the three co-ordinates, and of consequence the distances of the centre from three planes respectively perpendicular to each other.

These results being entirely independent of g, that is, of the force of gravity, some philosophers have preferred the term *centre of inertia* to that of *centre of gravity:* other philosophers have, on account of other properties, preferred different terms, which will be mentioned as we proceed.

When it is required to find the centre of gravity of any line whatever, it is considered as composed of a series of material heavy particles contiguous to each other, and connected by a law which is expressed by the equation of the curve, with respect to any two rectangular co-ordinates x and y. In this case the centre of gravity will manifestly be in the same plane as the proposed line, so that the plane YAX may contain the centre of gravity, whence $\mathrm{Z} = 0$, and the value of y being deduced, from the equation of the curve in terms of y, the centre of gravity may be determined by these two equations:

$$\text{(III.)} \ldots \mathrm{X} = \frac{\text{flu. } x\,\dot{\mathrm{M}}}{\mathrm{M}} \ldots \mathrm{Y} = \frac{\text{flu. } y\,\dot{\mathrm{M}}}{\mathrm{M}}$$

If the curve have two legs symmetrical with relation to any axe, then we may reckon the vertex of that axe the origin of the co-ordinates, and y being $= 0$, we shall only require $\mathrm{X} =$

$\frac{\text{flu. } x \dot{\mathrm{M}}}{\mathrm{M}}$; but in this case, the fluxion of M the curve being $= \sqrt{\dot{x}^2+\dot{y}^2}$, we have also,

$$\text{(IV.)} \ldots \mathrm{X} = \frac{\text{flu. } x\sqrt{\dot{x}^2+\dot{y}^2}}{\mathrm{M}}$$

If the figure is a plane, its centre of gravity will be in the same plane, and of course we may take $\mathrm{Z} = 0$: and because $\dot{\mathrm{M}} = y\,\dot{x}$, our equations become,

$$\text{(V.)} \ldots \begin{cases} \mathrm{X} = \dfrac{\text{flu. } x\,y\,\dot{x}}{\text{flu. } y\,\dot{x}} = \dfrac{\text{flu. } x\,y\,\dot{x}}{\mathrm{M}} \\ \mathrm{Y} = \dfrac{\text{flu. } y^2\,\dot{x}}{\text{flu. } y\,\dot{x}} = \dfrac{\text{flu. } y^2\,\dot{x}}{\mathrm{M}} \end{cases}$$

Here again, if the plane be symmetrical with respect to the axe, the equation for X will alone be wanted.

When the figure is the superficies of a body generated by the rotation of any line about an axis, then will $\mathrm{Y} = 0$, and $\mathrm{Z} = 0$: and putting $\pi = 3{\cdot}14159$, &c. $2\,\pi\,y$ will denote the circumference of the generating circle, and $2\,\pi\,y\,\dot{\mathrm{M}}$ the fluxion of the surface, wherefore

$$\text{(VI.)} \ldots\ldots \mathrm{X} = \frac{\text{flu. } 2\pi\,y\,x\,\dot{\mathrm{M}}}{\text{flu. } 2\,\pi\,y\,\dot{\mathrm{M}}} = \frac{\text{flu. } y\,x\,\dot{\mathrm{M}}}{\text{flu. } y\,\dot{\mathrm{M}}}$$

When the figure is a solid of revolution, the centre of gravity being upon its axis, we have $\mathrm{Y} = 0$, $\mathrm{Z} = 0$: and $\pi\,y^2$ denoting the area of the circle whose radius is y, and $\pi\,y^2\,\dot{x} = \dot{\mathrm{M}}$ the fluxion of the solid, we readily find,

$$\text{(VII.)} \ldots \mathrm{X} = \frac{\text{flu. } \pi\,y^2 x\,\dot{x}}{\text{flu. } \pi\,y^2\dot{x}} = \frac{\text{flu. } y^2\,x\,\dot{x}}{\text{flu. } y^2\dot{x}}.$$

Cor. When $\mathrm{X} = 0$, $\mathrm{Y} = 0$, $\mathrm{Z} = 0$; that is, when the centre of gravity is at the origin of the co-ordinates, the equations (I.) will give $p\,d + p'\,d' + p''\,d'' + \&c. = 0$, or $d + d' + d'' + \&c. = 0$; in like manner, $\mathrm{D} + \mathrm{D}' + \mathrm{D}'' + \&c. = 0$, and $\delta + \delta' + \delta'' + \&c. = 0$: and the same will hold with respect to any other co-ordinates whose origin is the centre of gravity; that is, the sum of perpendiculars from all the particles affected with contrary signs as they lie on different sides of either axis is then equal to zero: and consequently, *if on any plane passing through the centre of gravity of a body, perpendiculars be let fall from each of its moleculæ, the sum of all the perpendicular distances on one side of the plane will be equal to the sum of all those on the other side.*

109. Prop. *The position, distance, and motion of the centre of gravity of any body is a medium of the positions, distances, and motions of all the particles in the body.*

First, the *distance* of the centre of gravity of any body from a given plane, is an average of the distances of each of its constituent particles from the same plane: For, let AB (fig. 1. pl. III.) be a body whose centre of gravity is G, and CD any plane, from which the distances are to be estimated: then is the distance of any particle as p, beyond the plane AB (drawn through the body parallel to CD), equal to $p\mathrm{E} + \mathrm{EH} = p\mathrm{E} + \mathrm{GK}$; and if n be the number of particles p on the same side of AB, then will the sum of all the distances $p\mathrm{H}$, be equal to the sum of all the perpendiculars $p\mathrm{E}$, added to all the distances EH or GK, that is, $n \cdot p\mathrm{H} = (n \cdot \mathrm{GK}) + (n \cdot p\mathrm{E})$. Again, let p' be a particle of the body between the parallel planes AB, CD, its distance from the plane CD will be equal to $\mathrm{FI} - \mathrm{F}p' = \mathrm{GK} - \mathrm{F}p'$; and, if n' be the number of all the particles p' of the body between AB and CD, we shall have the sum of all their distances, that is, $n' \cdot p'\mathrm{I} = (n' \cdot \mathrm{GK}) - (n' \cdot \mathrm{F}p')$: Hence $(n \cdot p\mathrm{H}) + (n' \cdot p'\mathrm{I}) = (n + n') \cdot \mathrm{GK} + (n \cdot p\mathrm{E}) - (n \cdot \mathrm{F}p')$. But by the corollary to the preceding proposition, the sum of all the perpendiculars, on one side of the plane AB, is equal to the sum of all those on the other: consequently $n \cdot p\mathrm{E} - n' \cdot \mathrm{F}p' = 0$, and $n \cdot p\mathrm{H} + n' \cdot p'\mathrm{I} = (n + n')\ \mathrm{GK}$; that is, GK is the mean of all the distances from every particle of the body to the plane CD. And the like may be shewn of any other plane, or, of the body in any other position.

Secondly, the *motion* of the centre of gravity is an average of the motions of the several particles of which the body is composed: For, suppose the body to have moved towards the plane CD, its centre of gravity having passed from G to *g*, along the right line GK, and the body itself now situated as in *ab*. Here, it might be shewn, in the same manner as in the first case, that $(n + n')\ g\mathrm{K}$, is the sum of the perpendiculars from all the particles in the body to the plane CD: consequently, $(n + n')\ \mathrm{GK} - (n + n')\ g\mathrm{K} = (n + n')\ \mathrm{G}g$, is the sum of the approaches of all the particles towards CD, and $\mathrm{G}g$ being the $\frac{1}{n+n'}$ of this sum is evidently their mean. And if the motion of G were along a curvilinear path, the same conclusion would be deduced, if we conceive the curve to be separated into its infinitely small elements, and the motions with respect to each determined. The conclusion will likewise be the same although the body may have turned round some centre or axis.

This property of the centre of gravity has occasioned it to be called, by some authors, *the centre of position*; by others, the *centre of mean distances*. The celebrated French mathematician *Carnot*, in his *Geometrie de Position*, makes use of the latter term, and has endeavoured to show "that the theory of "this centre belongs to geometry, and that it would be very ad

" vantageous for the progress of that science, to re-establish in " this respect the natural order of ideas."

110. Prop. *The common centre of gravity, or of position, of two bodies divides the right line drawn between the respective centres of the two bodies in the inverse ratio of their masses.*

Let A and B (fig. 12. pl. II.) be two bodies whose centres of gravity are united by the inflexible line AB, then, if $AG : GB :: B : A$, G will be the common centre of gravity of those two bodies, that is, if G be supported, those two bodies actuated by the force of gravity will be in equilibrio in any position: for, through G let the vertical line CD be drawn, on which let fall the perpendiculars AF, BE, from the centres of gravity of the two bodies, then, because of the similar triangles BEG, AFG, we have $AG : GB :: AF : BE :: B : A$, whence $A \cdot AF = B \cdot BE$. But $A \cdot AF$, is equal to the sum of the products of all the particles in A into their respective distances from CD, by the last prop. and, in like manner $B \cdot BE$, is equal to the sum of the products of the particles in B into their respective distances; therefore $g \cdot A \cdot AF = g \cdot B \cdot BE$; that is, the sums of the moments of the forces of gravity upon A and B, with respect to CD are equal. If A and B be removed to any other position as a, b, the point G remaining fixed, it will appear in like manner that $g \cdot a \cdot af = g \cdot b \cdot be$: so that G is the centre of the forces of gravity with respect to A and B, that is, it is their common centre of gravity. In a manner but very little different, G may be shewn to be their common centre of position: and the two bodies, if considered as united by their centres of position at G, will then, as well as when their centres are separated by AB, have the sum of the perpendiculars from the several particles on one side of any plane passing through G, equal to the sum of all the perpendiculars on the other side of it.

Cor. The centre of gravity of three or more bodies may, hence, be found, by considering the first and second as a single body equal to their sum and placed in their common centre of gravity, determining the centre of gravity of this imaginary body, and a third. These three again being conceived united at their common centre, we may proceed, in like manner, to a fourth; and so on, *ad libitum.*

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111. It may not be altogether useless to shew that the centre of gravity of three or more bodies, as determined by this proposition and corollary, will be the same by whatever steps the process is conducted. Let, therefore, a, b, c, denote the masses of the three bodies placed at A, B, C (fig. 2. pl. III.); and let

G be the centre of gravity of A and B: make EG : EC :: c : $a+b$, and draw AEF; then F will be the centre of gravity of B and C, and AE : EF :: $b+c$: a. Draw GD parallel to BC, and FH parallel to BA; then will AG : AB :: GD : BF $= \text{GD} \cdot \frac{\text{AB}}{\text{AG}} = \text{GD} \cdot \frac{a+b}{b}$; and GE : EC :: GD : CF $= \text{GD} \cdot \frac{\text{EC}}{\text{GE}} = \text{GD} \cdot \frac{a+b}{c}$: wherefore, BF : CF :: $\frac{a+b}{b} : \frac{a+b}{c}$ :: c : b, and F is the centre of gravity of B and C. Again, CB : CF :: BG : FH $= \text{BG} \cdot \frac{\text{CF}}{\text{CB}} = \text{BG} \cdot \frac{b}{b+c}$; but AG $= \text{BG} \frac{b}{a}$: and FH : GA :: $\frac{b}{b+c} : \frac{b}{a}$ :: a : $b+c$ :: EF : AE, whence E is the same point as was determined from A and E. From this the same may be shewn true in cases where the number of bodies is greater, following the changes, step by step. Thus, in 4 bodies *a, b, c, d,* the order *a, b, c, d,* will give the same result as *c, a, d, b;* since *(a, b, c,) d,* is shewn to give the same as *(c, a, b,) d;* and *(c, a,) b, d,* the same as *(c, a,) d, b,* or as *c, a, d, b.*

112. *If the particles or bodies of any system be moving uniformly and rectilineally, with any velocities and directions whatever, the centre of gravity is either at rest, or moves uniformly in a right line.* For, let one of the bodies as C (fig. 3. pl. III.) move uniformly from C to D: then, *g* being the centre of gravity of the remaining bodies, join D*g*, and take *g*E to ED, as the mass D, to the sum of the other masses; then, is GE obviously parallel to CD, and CD : GE :: A + B &c. : C, GE being in this case the path of the common centre. And thus may the motion of the centre of gravity be found, which would be produced by the uniform rectilinear motion of each body in the system. Then, because each corresponding motion of the centre of gravity is uniform and rectilinear, the result of the whole will be either a uniform rectilinear motion; or no motion at all.

Hence, if a rotatory motion be communicated to a body and it be then left to move freely, the axis of rotation will pass through the centre of gravity: for, that centre, either remaining at rest or moving uniformly forward in a right line, has no rotation.

Here too it may be remarked, that *a force applied at the centre of gravity of a body, cannot produce a rotatory motion.* For every particle resists, by its inertia, the communication of motion, and in a direction opposite to that in which the force applied tends to communicate the motion; the resisting forces, therefore, act in parallel lines, in the same manner as the gravitating forces: consequently, since the latter balance each other on the centre of gravity, the former will do so likewise.

The truths thus briefly shewn in this article, are often discussed more at large; but a bare statement of them must suffice in this place, since they fall naturally under the province of Dynamics.

113. Prop. *To find the centre of gravity of the perimeter of any right-lined figure.*

If the figure be a regular polygon, the centre of gravity of its perimeter will be the centre of its circumscribing or inscribed circle. But if it be irregular, we conceive the particles of each line to be all placed at their respective centres of gravity, that is, at the middle of each line (107.), and proceed thus. Join the middle points of any two of the sides, as FB, BC, (fig. 4. pl. III.) by the line fb: make $f\mathrm{G} : \mathrm{G}b :: \mathrm{BC} : \mathrm{BF}$, or $f\mathrm{G} : fb :: \mathrm{BC} : \mathrm{BC} + \mathrm{BF}$, and G will be the common centre of gravity of the two sides BC, BF. Then join Gc, c being the middle point of a third side CD; make $\mathrm{GG}' : c\mathrm{G}' :: \mathrm{CD} : \mathrm{BC} + \mathrm{BF}$, and G' will be the centre of gravity of the three sides, FB, BC, CD. In like manner, join G' and the middle point d of a fourth line, and find the new centre of gravity G'': and so on, for all the sides of the figure.

Or, drawing through any point A in the same plane, the rectangular co-ordinates AX, AY, and denoting the lines FB, BC, CD, &c. by p, p', p'', &c. the distances of their middle points from AX, by d, d', d'', &c. and the distances of the said middle points from AY, by $\mathrm{D}, \mathrm{D}', \mathrm{D}''$, &c. we shall have (108. I.)

$$\mathrm{X} = \mathrm{AI} = \frac{pd + p'd' + p''d'' + \&c.}{p + p' + p'' + \&c.} \cdot \quad \mathrm{Y} = \mathrm{AH} = \frac{p\mathrm{D} + p'\mathrm{D}' + p''\mathrm{D}'' + \&c.}{p + p' + p'' + \&c.} \cdot$$

Then, drawing HG''', IG''', parallel to AX and AY, their intersection G''' will be the centre of gravity required.

114. Prop. *To find the centre of gravity of a plane triangle.*

Let ABC (fig. 5. pl. III.) be any triangle: draw AE from one of its angles to the middle of its opposite side, then will AE divide every line which can be drawn in the triangle parallel to BC into two equal parts; consequently the surface of the triangle is symmetrically disposed with respect to AE, and the centre of gravity will be found in that line (107.). For a like reason, if from any other angle, as C, we draw CD to the middle of its opposite side, the centre of gravity of the triangle will be somewhere upon that line: it will, therefore, be at G the intersection of those lines.

Now, since the points D, and E, divide the sides BA, BC, of the triangle proportionally, the line DE which joins them must be parallel to the third side AC: hence the triangles BDE, BAC are similar, and so are the triangles GDE, GCA. Consequently, GD : GC :: GE : GA :: DE : AC :: BD : BA :: 1 : 2.

Therefore $\mathrm{AG} = \frac{2}{3}\mathrm{AE}$, and $\mathrm{CG} = \frac{2}{3}\mathrm{CD}$, also $\mathrm{BG} = \frac{2}{3}\mathrm{BF}$.

Cor. If AB, BC, CA, be denoted by a, b, and c, and AG, BG, CG, by m, n, and r, we have, by a well-known theorem in geometry, the three following equations:

$$\begin{cases} AB^2 + AC^2 = 2\,BE^2 + 2\,AE^2, \text{ i. e. } & a^2 + c^2 = \frac{1}{2}b^2 + {}^{9}m^2 \\ AB^2 + BC^2 = 2\,CF^2 + 2\,BF^2, & a^2 + b^2 = \frac{1}{2}c^2 + {}_{2}n^2 \\ AC^2 + BC^2 = 2\,AD^2 + 2\,CD^2, & c^2 + b^2 = \frac{1}{2}a^2 + \frac{9}{2}r^2 \end{cases}$$

Adding together these equations and clearing them of fractions, there results $a^2 + b^2 + c^2 = 3m^2 + 3n^2 + 3r^2$; which gives this curious theorem: *In any plane triangle the sum of the squares of the three sides, is equal to thrice the sum of the squares of the distances of each of its angles from the centre of gravity.*

Cor. 2. From the three equations in the preceding corollary, we readily find $m = \frac{1}{3}\sqrt{2a^2 + 2c^2 - b^2}$, $n = \frac{1}{3}\sqrt{2a^2 + 2b^2 - c^2}$, and $r = \frac{1}{3}\sqrt{2b^2 + 2c^2 - a^2}$: by means of which the distance of the centre of gravity from either angle of any given triangle may be soon found.

115. Prop. *To find the centre of gravity of a trapezium.*

Let ABCD (fig. 6. pl. III.) be any trapezium. Divide it into two triangles by the diagonal AC: find their centres of gravity H and I, by the last proposition; then join IH, and make IG, to GH, as the triangle ABC, to the triangle ADC, and G will be the centre of gravity of the trapezium.

Or, divide the trapezium into two other triangles by the diagonal BD; find their centres of gravity E, and F, and draw EF. Then the centre of gravity of the trapezium must be in the line HI; and it must likewise be in the line EF; consequently it must be in G their point of intersection.

When two sides of the quadrilateral, as AD and BC, (fig. 7. pl. III.) are parallel, the centre of gravity is somewhere upon the line KL joining their middle points: and it is somewhere upon the line EF joining the centres of gravity of the two component triangles ABD, CBD: it is, therefore, at G the point of intersection of EF and KL. And in this case $KG = \frac{1}{3}KL, \frac{BK + AD}{BK + AL}$.

For, drawing the lines Ee, Ff, parallel to AD, BC; since $EL = \frac{1}{3}BL$, and $KF = \frac{1}{3}KD$, we have $Ee = \frac{1}{3}BK = \frac{1}{6}BC$, and $Ff = \frac{1}{3}LD = \frac{1}{6}AD$. Also $Le = \frac{1}{3}KL$, and $Kf = \frac{1}{3}KL$. Whence $ef = \frac{1}{3}KL$. Now, the similar triangles GeE, GfF give $Ee : Ge :: Ff : Gf$, therefore $Ee + Ff : Ge + Gf :: Ff : Gf$; that is, $\frac{1}{6}BC + \frac{1}{6}AD : \frac{1}{3}KL :: \frac{1}{6}AD : Gf = \frac{\frac{1}{3}KL \cdot AD}{BC + AD}$. Hence then, $KG = Kf + fG = \frac{1}{3}KL + \frac{\frac{1}{3}KL \cdot AD}{BC + AD} = \frac{1}{3}KL \cdot \frac{BC + 2AD}{BC + AD} = \frac{1}{3}KL \cdot \frac{BK + AD}{BK + AL}$.

116. Thus also, to find the centre of gravity of the area of any rectilinear figure, divide it into triangles, and find their

respective centres of gravity, by art. 114. then conceiving each of the triangles collected into their respective centres of gravity, their common centre of gravity may be found, either by the method described in art. 111. or by the theory of moments (equa. I. 108.)

117. **Prop.** *To find the centre of gravity of any triangular pyramid.*

Let ABCD (fig. 8. pl. III.) be a triangular pyramid: on one of its faces, as BCD, draw DI from the angle D to the middle of its opposite side BC; set off $I\gamma = \frac{1}{3}ID$, and γ will (114.) be the centre of gravity of that face. From A the vertex of the pyramid, draw $A\gamma$, and it will pass through the centres of gravity of every section of the pyramid parallel to BCD: consequently, it will pass through the centre of gravity of the pyramid. Again, on the face ABC, draw AI from A to the middle of the opposite side; set off $Ig = \frac{1}{3}IA$, and join gD: the centre of gravity of the pyramid will, it is obvious, be found upon this line also. But the two lines $A\gamma$, Dg, being both in the plane of the triangle ADI, must intersect each other, and G their point of intersection must necessarily be the centre of gravity of the pyramid.

Now, if we conceive $g\gamma$ drawn, it will be parallel to AD, since $Ig : IA :: I\gamma : ID$: hence the triangles $Gg\gamma$, GDA, are similar, and,

$$Gg : GD :: G\gamma : GA :: g\gamma : AD :: 1 : 3.$$

Therefore, $Gg = \frac{1}{3}GD = \frac{1}{4}gD$; $G\gamma = \frac{1}{3}AG = \frac{1}{4}A\gamma$, &c.

Cor. 1. If AB be put $= a$, AC $= b$, AD $= c$, BC $= d$, BD $= e$, CD $= f$; and the distances from the angles to the centre of gravity, AG $= m$, BG $= n$, CG $= r$, DG $= s$: then, by a well known theorem, $AI^2 = \frac{AB^2 + AC^2 - 2BI^2}{2} = \frac{a^2 + b^2 - \frac{1}{2}d^2}{2}$; and, by the same, $DI^2 = \frac{e^2 + f^2 - \frac{1}{2}d^2}{2}$: again, since $AG = \frac{3}{4}A\gamma$, we have $AG^2 = \frac{9}{16}A\gamma^2 = \frac{9}{16}\left(\frac{AI^2 . D\gamma + AD^2 . I\gamma - ID . I\gamma . \gamma D}{ID}\right)$. (See the Lemma at p. 128. Simpson's Select Exercises.) Substituting in this expression the literal values of AI, $D\gamma$, &c. and reducing, we at length obtain $AG^2 = \frac{1}{16}(\overline{3a^2 + 3b^2 + 3c^2} - \overline{d^2 + e^2 + f^2})$. A like equation being in the same manner deducible for BG^2, CG^2, DG^2, we have the following general theorem: *In any triangular pyramid the distance of any one of the angles of the pyramid, from the centre of gravity, is equal to one-fourth of the square root of the difference of thrice the sum of the squares of the three edges meeting at that angle, and the sum of the squares of the other three edges.*

Cor. 2. Adding together the four equations found by the preceding corollary, viz.

$$AG^2 = \tfrac{1}{16}(3AB^2 + 3AC^2 + 3AD^2) - \tfrac{1}{16}(BC^2 + BD^2 + CD^2)$$
$$BG^2 = \tfrac{1}{16}(3BA^2 + 3BC^2 + 3BD^2) - \tfrac{1}{16}(AC^2 + AD^2 + CD^2)$$
$$CG^2 = \tfrac{1}{16}(3CA^2 + 3CB^2 + 3CD^2) - \tfrac{1}{16}(AB^2 + AD^2 + BD^2)$$
$$DG^2 = \tfrac{1}{16}(3AD^2 + 3DC^2 + 3BD^2) - \tfrac{1}{16}(AB^2 + AC^2 + BC^2)$$

there arises the following:

$$AG^2 + BG^2 + CG^2 + DG^2 = \tfrac{1}{4}(AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2).$$

That is, *the sum of the squares of the distances from the angles of any triangular pyramid to its centre of gravity, is equal to a fourth of the sum of the squares of the six edges of the pyramid.*

These two corollaries are manifestly analogous to those of art. 114.

COR. 3. If the sides of the base are equal, as $BD = BC = CD$, we have $AG^2 = \frac{3}{16}(AB^2 + AC^2 + AD^2 - BC^2)$.

COR. 4. If, moreover, $AB = AC = AD$, then $AG^2 = \frac{3}{16}(3AB^2 - BC^2)$.

COR. 5. If all the edges of the pyramid are equal, we have for the regular tetraedron, $AG^2 = BG^2 = CG^2 = DG^2 = \frac{3}{8}AB^2$, whence $AG = \frac{1}{4}AB\sqrt{6}$.

COR. 6. A pyramid whose base is any polygon, will have its centre of gravity upon the line drawn from the vertex to the centre of gravity of the base, and at the distance of $\frac{3}{4}$ of its length from its vertex. Which will be sufficiently obvious if it be considered, that any such pyramid is composed of triangular pyramids, whose centres of gravity all lie in one plane parallel to the base; consequently their common centre of gravity must be in the same plane, and must likewise be in the line drawn from the vertex to the common centre of gravity of all the triangles which constitute the base.

118. **PROP.** *To find the centre of gravity of a circular arc.*

From the middle point of the proposed arc MAM' (fig. 9. pl. III.) conceive the line AC drawn through C the centre of the circle, on which let AP be denoted by x, the variable ordinate PM by y, and the radius AC of the circle by r, the half arc being denoted by z. Then accounting A the origin of the coordinates, since the curve is symmetrical with respect to AC, we need only make use of the equation (108. IV.) which, substituting z for M will become, $\mathrm{x} = \dfrac{\int x\sqrt{\dot{x}^2 + \dot{y}^2}}{z}$[*]. Now the equation expressing the relation between the rectangular co-ordinates of a circle is $y^2 = 2rx - x^2$; by means of which we have

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[*] Here, as well as in the subsequent parts of this work, we make the character $\int$ denote the fluent of the expression which stands after it.

$\text{x} = \frac{r}{z}\int\frac{x\dot{x}}{\sqrt{(2ax - x^2)}}.$ The fluent of this is $\text{x} = \frac{r}{z}(z - y) = r - \frac{ry}{z}$: which needs no correction, because when $y = 0$, $z = 0$. Hence then, G being the centre of gravity, we have $\text{AG} = r - \frac{ry}{z} = \text{AC} - \frac{ry}{z}$. Consequently $\text{CG} = \text{AC} - \text{AG} = \frac{ry}{z}$: that is, *the distance of the centre of gravity of a circular arc from its centre, is a fourth proportional to the arc, the radius, and the chord of the arc.*

Cor. 1. When the arc is a semicircle, the chord is double the radius, and $\text{CG} = \frac{2r}{3.141593} = \frac{r}{1.57079} = \cdot 63662\, r$.

Cor. 2. When $x = 2r$, $y = 0$, and consequently $\text{CG} = \frac{ry}{z} = 0$; that is, the centre of gravity coincides with the centre of the circle; as is sufficiently obvious independent of the fluxional process.

119. Prop. *To find the centre of gravity of a circular segment.*

Let MAM′P in the figure last referred to be the segment proposed, and let the parts be denoted as before. Here we take the first of the equations (108. V.), that is $\text{x} = \frac{\text{flu } xy\dot{x}}{\text{M}}$, where M denotes the area of APM. Now, since $y = \sqrt{2rx - x^2}$, we have $\text{xM} = \int xy\dot{x} = \int x\dot{x}\sqrt{2rx - x^2} = -\frac{(2rx - x^2)^{\frac{3}{2}}}{3} + r\text{M}$: this divided by M gives $\text{x} = \text{AG} = r - \frac{\text{PM}^3}{3\text{M}} = \text{CA} - \frac{\text{PM}^3}{3 \text{ area APM}}$. Consequently $\text{CG} = \frac{\text{PM}^3}{3 \text{ area APM}} = \frac{\text{MM}^3}{12 \text{ area seg.}}$.

Cor. When the segment becomes a semicircle, we have $\text{M}'\text{M} = 2r$, and $\text{CG} = \frac{r}{2.356194} = \cdot 42441\, r$.

120. Prop. *To find the centre of gravity of any parabola.*

Here the general equation is $y = x^n \div a^{n-1}$. And by substitution $\frac{\text{flu. } xy\dot{x}}{\text{flu. } y\dot{x}} = \frac{\int x^{n+1}\dot{x}}{\int x^n\dot{x}} = \frac{n+1}{n+2} \times x = \text{x}$.

Cor. 1. If $n = \frac{1}{2}$, which is the case in the common or Apollonian parabola, $\frac{n+1}{n+2}x = \frac{3}{5}x$. That is, $\text{AG} = \frac{3}{5}\text{AC}$. (fig. 10. pl. III.)

Cor. 2. If $n = 1$, the figure becomes a triangle, and then $\text{AG} = \frac{2}{3}x$: which agrees with art. 114.

121. Prop. *To find the centre of gravity of any semiparabola.*

In this case the distance on the absciss, or the value of X is determined by the foregoing problem; we have now to find Gg (fig. 10. pl. III.): in order to which we take the second equation (108. V.) $\text{Y} = \frac{\text{flu. } y^2\dot{x}}{\text{M}}$. By substituting for y^2 in the numerator of this expression, its value in the equation of the curve, we have $\text{YM} = \int\frac{x^{2n}\,\dot{x}}{a^{2n-2}}$, which, by the known rules for the determination of fluents is $\doteq y^2 \times \left(\frac{h}{2} - \frac{nx}{2n+1}\right)$, where $h = \text{AC}$ the height of the parabola. And this when $x = h$, becomes $y^2 \times \frac{h}{4n+2}$: wherefore, dividing by $\text{M} = \frac{h \times \text{CD}}{n+1,}$ we obtain $\text{Y} = \frac{(n+1)\text{CD}}{4n+2} = \text{G}g$.

COR. 1. If $n = \frac{1}{2}$, as in the common parabola, $\text{G}g = \frac{3}{8}\text{CD}$.

COR. 2. In the triangle, where $n = 1$, $\text{G}g = \frac{1}{3}\text{CD}$.

122. **PROP.** *To find the centre of gravity of the segment of a spheroid.*

Here the figure may be divided into two parts symmetrical with respect to the axis: so that if we account the vertex A the origin of the co-ordinates, we shall need the equation for X (108. VII.). Let the fixed axis of the spheroid be called a, the revolving axis c, then the equation of the curve is $y^2 = \frac{c^2}{a^2} \times (ax - x^2)$: consequently $\frac{\text{flu. } y^2 x\dot{x}}{\text{flu. } y^2\dot{x}} = \frac{\int(ax - xx)\cdot x\dot{x}}{\int(ax - xx)\cdot\dot{x}}$ $= \frac{\frac{1}{3}ax^3 - \frac{1}{4}x^4}{\frac{1}{2}ax^2 - \frac{1}{3}x^3} = \frac{4a-3x}{6a-4x}\,x = \text{AG}$.

COR. 1. When the segment becomes a hemispheroid, $x = \frac{1}{2}a$, and $\frac{4a-3x}{6a-4x}\,x = \frac{5}{8}x$ for the distance of the centre of gravity from the vertex, therefore $\frac{3}{8}x$ is its distance from the centre of the base.

COR. 2. When $c = a$, the spheroid becomes a sphere, and as the theorem is independent of c, it is alike applicable to both solids, and to their corresponding segments.

COR. 3. Since the equation to the hyperbola is $y^2 = \frac{c^2}{a^2}(ax + x^2)$ which differs from that to the ellipsis only in the sign of the second term, we get by a similar process $\frac{4a+3x}{6a+4x}\,x$, for the distance of the centre of gravity from the vertex of a hyperboloid.

123. **PROP.** *To find the centre of gravity of a sector of a sphere.*

A spheric sector may be considered as a solid of rotation; for it may be generated by the rotation of a circular sector: thus, in fig. 9. pl. III. if the circular sector CMA make a complete revolution upon AC as an axis, it will describe the spheric sector of which CMAM′ will be a section. This solid is composed of a spheric segment and a cone: the centre of gravity of the first of these we suppose at G, that of the second at g. If AP be denoted by x, and AC by r, we shall have AG $= \frac{8r-3x}{12r-4x}\,x$, and, the cone being a kind of pyramid, C$g = \frac{3}{4}$CP $= \frac{3}{4}(r-x)$: consequently Ag = AC − Cg $= \frac{3x+r}{4}$. Let π denote, as before, the number 3·141593, and by well known theorems the solid contents of the respective figures will be thus expressed:

The content of the spheric sector $= \frac{2}{3}\pi r^2 x$.
the segment . $= \frac{1}{3}\pi x^2 (3r-x.)$
the cone . . $= \frac{1}{3}\pi (2rx-x^2).(r-x)$.

Now, if we conceive each solid to be condensed into its centre of gravity, we must first take the sum of the moments with respect to A, which are,

For the segment, $\frac{1}{3}\pi x^2(3r-2x)\frac{8r-3x}{12r-4x}\,x = \frac{1}{3}\pi x^3.\,\frac{8r-3x}{4}$

the cone . $\frac{1}{3}\pi(2rx-x^2).\,(r-x)\frac{3x+r}{4}$.

Then, dividing the sum of these moments by the sum of the masses, or $\frac{2}{3}\pi r^2 x$, the content of the sector, we shall have $\text{X} = \frac{\frac{1}{3}x^2(8r-3x)+\frac{1}{4}(2r-x)\,.\,(r-x)\,.\,(3x+r)}{2r^2} = \frac{1}{8}(2r+3x)$; an equation expressing the distance of the centre of gravity of the spheric sector from A the centre of its base.

SCHOLIUM.

124. Since the knowledge of the position of the centre of gravity in any body or system of bodies is of very great importance in almost every part of Mechanics, we have thought it right to be tolerably copious in the discussion of this subject; and have, therefore, given instances of various methods of ascertaining the situation of the centre of gravity, one or other of which may be applied in most cases which can arise. To save, however, the trouble of a distinct investigation in some instances which often occur, we may now state in addition to the preceding propositions, a few known results.

1. In a circular sector, the distance from the centre of the circle is $\frac{2cr}{3a}$; where a denotes the arc, c its chord, and r the radius.

2. The centres of gravity of the surface of a cylinder, of a cone, and of a conic frustum, are respectively at the same di-

stances from the origin as are the centres of gravity, of the parallelogram, triangle, and trapezoid, which are vertical sections of the respective solids.

3. The centre of gravity of the surface of a spheric segment, is at the middle of its versed sine or height.

4. The centre of gravity of the convex surface of a spherical zone, is in the middle of that portion of the axis of the sphere which is intercepted by the two bases of the zone.

5. In a cone, as well as any other pyramid, the distance from the vertex is $\frac{3}{4}$ of the axis.

6. In a conic frustum, the distance on the axis from the centre of the less end, is $\frac{1}{4}h \cdot \frac{3R^2+2Rr+r^2}{R^2+Rr+r^2}$: where h denotes the height, and R and r, the radii of the greater and less ends.

7. The same theorem will serve for the frustum of any regular pyramid, taking R and r, for the sides of the two ends.

8. In the paraboloid, the distance from the vertex is $\frac{2}{3}$ of the axis.

9. In the frustum of the paraboloid, the distance on the axis from the centre of the less end, is $\frac{1}{3}h \cdot \frac{2R^2+r^2}{R^2+r^2}$: where h denotes the height, R and r the radii of the greater and less ends.

Of the Centrobaryc Method.

Among the several uses to which the doctrine of the centre of gravity may be appropriated, one, which for its elegance and simplicity deserves being mentioned here, is that which is called the Centrobaryc method, and by which the magnitudes of surfaces and solids may often be determined with great facility. The relation between the centre of gravity and the figure generated by the revolution of any line or plane, which is the foundation of this method, was first distinctly stated by *Pappus* in the preface to his 7th book: but it was not completely discussed till the time of father *Guldin;* who, in the 2d and 3d books of his Treatise on the Centre of Gravity, treated this method very fully, and exhibited its utility in a variety of examples. The doctrine is comprised in the following proposition, and the corollaries which naturally flow from it.

125. **PROP.** *If any line, right or curved, or any plane figure, whether it be bounded by right lines or curves, re-*

volve about an axis in the plane of the figure, the surface or solid generated will be respectively equal to the surface or solid whose base is the given line or figure, and its height equal to the arc described by the centre of gravity of the said generating line or figure.

Let the line or plane BCD (fig. 11. pl. III.) be that by whose motion a certain surface or solid is generated, and let QAR be the axis about which the rotation is performed: through G the centre of gravity of BCD draw ABGD perpendicular to QR, and either suppose the various particles in BCD to be referred, by perpendiculars, to QR, or, which will come to the same thing, estimated upon ABD by lines drawn from the several moleculæ parallel to QR; and let B and D be two of those reduced points. Now, if ABD after performing a rotation, or part of a rotation, be brought to the position AEH, BDHE will represent the figure generated, and GF the path of the centre of gravity; the generated figure being equal to all the BE, GF, DH, &c. But the arcs BE, GF, DH, &c. are similar (being equal parts of concentric circles), and therefore their sums are as all the AB, AG, AD, &c. and, by art. 109. these sums are as so many times AG: consequently the sum of all the BE, GF, DH, &c. is equal to as many times FG, or equal to BCD × FG; that is, the figure BEHD is equal to the base drawn into the path described by its centre of gravity. Q. E. D.

The same thing may be shewn very concisely by means of the two values of Y in formulæ III. and V. art. 108. Annexing the co-efficient 2π to bot hnumerator and denominator will not change the value of the fractions: they may, therefore, be expressed thus:

$$Y=\frac{\int 2\pi y\dot{M}}{2\pi M}. \quad . \quad . \quad . \quad . \quad . \quad Y=\frac{\int 2\pi y^2\dot{x}}{2\pi M}.$$

The first of these equations expresses the distance from the axe AX of the centre of gravity of any line: a simple transformation gives us $2\pi YM=\int 2\pi y\dot{M}$, where $2\pi Y$ is the circumference of which Y is the radius, and is that which is described by the centre of gravity of M if that curve be made to turn about AX as an axis: moreover $\int 2\pi y\dot{M}$ is the expression for the area of the surface which is generated by the arc M during this rotation. Consequently, the surface generated by the rotation of any given curve about an axis is equal to the product of the generating arc into the circumference described by its centre of gravity.

From the second of these formulæ we have $2\pi YM=\int 2\pi y^2\dot{x}$;

where if we suppose a revolution upon the axis AX, the area of which the algebraic expression is M or $y\dot{x}$, generates a solid denoted by $\int 2\pi y^2\dot{x}$, and the centre of gravity will describe the circumference $2\pi Y$. Therefore the solid generated by the rotation of any plane figure about an axis, has for its capacity *the product of the generating area into the circumference described by its centre of gravity.*

If instead of 2π we introduced any fraction, $\frac{1}{n}\pi$, into the numerators and denominators of the same formulæ, it would thence be equally obvious that the same property was applicable to the curve surfaces or the capacities of figures generated by a partial revolution about a fixed axis. And if many curves comprised in the same plane generate at once as many surfaces and solids, we may apply the same proposition, by taking the common centre of gravity of the system. From the whole, then, we may deduce a few corollaries.

COR. 1. If the distance of the centre of gravity of any line or surface from the axis of rotation, and the magnitude of the line or surface be given, the value of the surface or solid generated by either a total or a partial revolution may be found, and consequently any two of the three being given, the third may be easily determined.

Thus, if the rectangle whose sides are m and n revolve upon m as an axis, it will generate a cylinder whose altitude is m and radius of its base n: now the centre of gravity of the rectangle will be at the distance of $\frac{1}{2}n$ from m, and will therefore describe in a complete rotation the circumference $\frac{1}{2}n \times 2\pi = \pi n$: consequently, by the proposition, $mn \times \pi n, = \pi mn^2$, is the capacity of the solid; and this is equal to $\pi n^2 \times m$, the base of the cylinder into its altitude, as it ought to be.

Again, putting b the base and a the axis of a semiparabola, which by its rotation generates a paraboloid; we have $\frac{3}{8}b$ for the distance of the centre of gravity of the semiparabola from the axis, and $\frac{3}{8}b \times 2\pi = \frac{3}{4}b\pi$, the circum. described by the centre of gravity: and, since the area of the parabola is $\frac{2}{3}ab$, we have $\frac{3}{4}b\pi \times \frac{2}{3}ab = \frac{1}{2}ab^2\pi$, for the content of the paraboloid; which is manifestly equal to half the circumscribing cylinder, as it ought to be.

For another example, let us take a sphere whose radius is r, and capacity $\frac{4}{3}\pi r^3$, being generated by the rotation of the semicircle whose area is $\frac{1}{2}\pi r^2$. Here $\frac{4}{3}\pi r^3 \div \frac{1}{2}\pi r^2 = \frac{8}{3}r$, the circumference described by the centre of gravity; and this divided by 2π, gives $\frac{4r}{3\pi} = \cdot 42441\, r$, for the distance of the centre of gravity of a semicircle from its centre: agreeing with the result in art. 119. By a similar process the centre of gravity may be found in several figures, more easily than by the usual direct method.

COR. 2. If the generating or revolving lines or surfaces are equal, but the distances of their centres of gravity unequal, then the generated surfaces or solids will be directly as the distances of their centres of gravity from the axis of motion.

Thus, if two solid rings of iron have both the same thickness of metal, but have different diameters, the capacities of the rings, or their weights, will be as the arithmetical means between their respective inner and outer diameters.

COR. 3. If the distances of the centres of gravity of the generating or revolving lines or surfaces from the axis of motion be equal, but the generating lines or surfaces unequal, then the generated surfaces and solids will be as the generating lines and surfaces.

COR. 4. If neither the generating lines or surfaces nor the distances of their centres of gravity from the axis of motion be equal, the generated surfaces or solids will be to each other in a ratio compounded of the ratio of the generating lines or surfaces, and the ratio of the distances of their centres of gravity from the axis of motion.

Thus, for example, the triangle is to its circumscribed parallelogram as 1 to 2, and the distance of the centre of gravity of the triangle from its vertex is $\frac{2}{3}$ of the axis; but the distance of the centre of gravity of the parallelogram from its side is $\frac{1}{2}$ of the other side, or of the axis, therefore the distance of the centre of gravity of the triangle from the axis of rotation is to the distance of the centre of gravity of the parallelogram, as 4 to 3. So that, if both the spaces are supposed to revolve about a line touching the vertex of the triangle, and parallel to its base, the solid generated by the triangular space will be to that generated by the parallelogram as 4 to 6, or as 2 to 3.

Again, the common parabola is to its circumscribing parallelogram as 2 to 3; and the distance of the centre of gravity of the parabola is to the distance of that of the parallelogram from the same point of suspension, as 6 to 5; wherefore the solid generated by the parabola is to that generated by the parallelogram, each revolving about the same common axis passing through the vertex of the parabola, and parallel to its base, as 12 to 15, or as 4 to 5: and consequently, the solid generated by the triangle is to that generated by the parabola in a ratio compounded of the ratios of 5 to 4, and of 4 to 6; that is, of 20 to 24, or 5 to 6.

After a similar manner may the value of any surface or solid be found, and those of all kinds of surfaces and solids be compared together; for various other examples of which we may refer to *Dr. Wallis's* Treatise *de Calculo Centri Gravitatis, Dr. Hutton's Mensuration*, and *Stone's Mathematical Dictionary*, art. CENTRE *of Gravity.*

CHAPTER IV.

On the Mechanical Powers.

126. Def. A *machine* is any thing that serves to augment or to regulate moving forces or powers; or it is any body destined to produce motion so as to save either time or force; or, we may in general apply the term *machines* to the material agents by the aid of which forces operate one upon another, and which when employed to second the efforts of certain powers enable them to overcome others which may be more considerable.

127. Machines are distinguished into *simple* and *compound*. The only machines which we think can be strictly called simple, are the *lever*, the *inclined plane*, and *cords:* it has been customary, however, to consider as simple machines those of which all compound machines are found to be constituted, and into which when their combined powers are estimated they must be divided in the investigation; these are in number six, viz. the *lever*, the *wheel and axle*, the *pulley*, the *inclined plane*, the *screw*, and the *wedge*. These are most commonly known by the name of the *mechanical powers*. In all these the mechanical advantage arises from distributing the power to be overcome among the different parts of the machine, so that the part sustained by the contrary power shall bear but a small ratio to the whole: thus a power incapable of communicating motion to a body, or of supporting its pressure, without mechanical assistance, may effect its designed purpose by transferring a part of the weight upon a fulcrum, distributing it among a system of pulleys, placing it upon an inclined plane, screw, &c.

128. Forces of different kinds may, it is obvious, be made to operate so as to counteract each other's effects, by the intervention of machines: as for example, the muscular force of men and animals may be applied to machines so as to oppose and counteract the force of gravity; but in any such cases, since the just and adequate measures of these forces are their simultaneous effects in similar situations, the theoretic processes must be conducted in the same manner whether the forces exerted are unlike or the same, as when the force of gravity soliciting two bodies

causes them to act as moving powers by means of their weights. Hence then, whatever is shewn to obtain with relation to any two forces whose effects are known, while their nature is undefined, may be safely applied to all which are respectively equal and operate in a similar manner, whether they are dead weights, living powers, or forces existing naturally.

129. In the theory of simple machines, and indeed of machines in general, they are first considered as in a state of balanced rest: and when the ratio of the forces is ascertained which by the intervention of any given machine produces such equilibrium, it is manifest that a slight change in the magnitude of either of the forces, or in their points of application, by causing a preponderance on one side or other, will produce motion. In actual practice the change must always be made in favour of the power which is to produce the motion: and this may easily be accomplished when the conditions of the equilibrium are once established. When forces act upon each other by means of machines, they meet with various obstacles on account of the roughness of surfaces, the stiffness of cords, &c. which if introduced at the commencement of our investigations would render them very embarrassing and intricate: we therefore, at first, suppose surfaces to be perfectly smooth, cords to be perfectly flexible, &c. and afterwards blend with the deductions from the theory those which have resulted from experiment with regard to these obstacles. As to what relates to the disposition of powers upon machines so that when in motion their effects shall be greatest or least, it is a distinct branch of the theory, and falls under the head of *Dynamics*.

130. Writers on the subject of mechanics have often attempted to demonstrate the properties of the several simple machines, by means of a celebrated theorem, which is this: *When two heavy bodies counterpoise each other by means of any machine, and are then made to move together*, the products of each mass into its velocity, or, as it is technically expressed, *the quantities of motion with which one body descends and the other ascends perpendicularly, will be equal**. Since an *equilibrium* always accompanies this equality of motions; it bears

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* This theorem is due to *Galileo*, who is justly reckoned the father of the science of *Dynamics*. It is nearly analogous to the celebrated principle of the *virtual velocities*, from which M. La Grange has deduced many ingenious and elegant solutions of some difficult problems in his *Mécanique Analytique*. The principle is thus enunciated: " *Si un systême quelconque est sollicité par des puissances en equilibre, et qu'on donne à ce systême un petit mouvement quelconque, en vertu duquel chaque point parcoure un espace infiniment petit, la somme des puissances multipliées chacune par l'espace que le point où elle est appliquée parcourt suivant la direction de cette même puissance, sera toujours égale à zero : en regardant comme posi-*

such a resemblance to the case wherein two moving bodies stop each other when they meet together with equal quantities of motion, that many have thought that the cause of an equilibrium in the several machines might be immediately assigned by saying, that because one body always loses as much motion as it communicates to another, two heavy bodies counteracting each other must continue at rest when they are so circumstanced that one cannot descend without causing the other to ascend at the same time, and with velocities inversely proportional to their masses; for then, should one of them begin to descend, it must instantly lose its whole motion by communicating it to the other. But this argument, however plausible it may seem, is (as Dr. Hamilton remarked) by no means satisfactory; for when we say that one body communicates its motion to another, we must necessarily suppose the motion to exist first in the one, and then in the other; but in the present case, where the two bodies are so connected that one cannot possibly begin to move before the other, the descending body cannot be said to communicate its motion to the other, and thereby make it ascend: but whatever we should suppose causes one body to descend, must be also the immediate cause of the other's ascending, since, from the connection of the bodies, it must act upon them both together, as if they were really but one. And therefore, without contradicting the laws of motion, we might suppose the superior weight of the heavier body, which is in itself more than able to sustain the lighter, would overcome the lighter, and cause it to ascend with the same quantity of motion with which the heavier descends; especially as both their motions, taken together, may be less than what the difference of the weights, which is here supposed to be the moving force, would be able to produce in a body falling freely. For these reasons, and various others which might easily be assigned, we are of opinion that all proofs founded upon this theorem as a basis, are necessarily unsatisfactory: we have, nevertheless, thought it right to notice it; and, as it may serve as a good *index* of an *equilibrium* in many machines, and admits in some instances of a useful application,

tifs les petits espaces parcourus dans le sens des puissances, et comme négatifs les espaces parcourus dans un sens opposé." This principle, or rather, universal fact, La Grange expresses by a very simple formula, from which he deduces his whole system algebraically, without a single diagram: a mode of procedure which, though it displays astonishing analytical skill, is certainly far from the best way of *teaching* mechanical philosophy; since the whole may be gone over, without either a distinct or even an indistinct idea of the things represented by the symbols. M. Carnot in his *Geometrie de Position*, pa. 339, demonstrated a purely geometrical theorem which is very similar to this celebrated mechanical principle.

we may again refer to it in the practical part of this Treatise. Our present investigations will be made to depend upon the principles of equilibrium explained in the second and third chapters.

I. *Of the Lever.*

131. DEF. A *lever* is an inflexible bar or rod, which may be subjected to the action of two or more forces or powers, at different points, while it is supported on a fulcrum or prop, about which it may move freely.

Of levers there are two kinds which differ essentially: *i. e.* those in which the forces act on *contrary* sides of the centre of motion, or fulcrum, and those in which they act on the *same* side. They are, however, usually distinguished into *three* kinds, according to the respective dispositions of the prop, the power, and the resistance. In levers of the *first* kind, the fulcrum is between the power and the resistance: such are, the balance, steelyards, scissars, pokers, pincers, snuffers, &c. In levers of the *second* kind the resistance is between the fulcrum and the power: such are, cutting knives fixed at one end, the oars and rudders of boats, which may be considered as having their fulcra in the water, doors whose hinges serve as a fulcrum, &c. In levers of the *third* kind the power acts between the prop and the resistance; as in tongs, shears for sheep, a ladder while raising up, &c. This too is the kind of lever which is found most to obtain in the animal functions: for the muscles, by means of which the bones are turned upon their joints, are inserted much nearer to the centre of motion than the point in which is the centre of gravity of the weight to be raised. The mechanical contrivances in different parts of animal frames furnish a very curious and interesting topic: but as we cannot, without widely digressing, enlarge upon it here, we would beg to refer to Chapters 7 and 8 of Dr. Paley's Natural Theology, where it is discussed very admirably.

132. PROP. *Any two weights acting perpendicularly upon an horizontal lever (considered as void of gravity), and at contrary sides of the fulcrum, will balance each other if they be reciprocally proportional to the distances from the fulcrum at which they act.*

Conceive any cylinder or prism, as AB (fig. 12. pl. III.), hung by its ends to the inflexible line HI, by means of the cords AH, BI. Now if the whole be suspended by the cord C placed in the middle of HI, it will remain in equilibrio, from the nature of the centre of gravity. And if we suppose the prism divided into two unequal parts by a vertical plane passing through D, then if a cord, as ED, keep the parts AD, DB, in the same position with respect to HI, the equilibrium will not be affected.

Or, if instead of the three cords HA, ED, IB, the two pieces were attached to the line HI, by cords GL, FM, fixed to their middle points L, M, the equilibrium will still obtain, in like manner as at first. So that we may consider two heavy bodies AD, and DB, suspended from the terms G, and F of the beam, to be in equilibrio upon the fulcrum C. Here, since GE $= \frac{1}{2}$ EH, and EF $= \frac{1}{2}$ EI, we shall have GF $= \frac{1}{2}$ HI, and consequently $=$ CI: whence, taking away CF which is common, there remains GC $=$ FI $=$ FE. Therefore, adding CE, we have GE $=$ CF. Hence then, GE : EF :: FC : CG :: 2GE : 2EF :: AD : DB; wherefore, convertendo, GC : CF :: the weight BD : the weight AD. And if these two prisms AD, DB, were reduced to cubes, or spheres, or any other figures, the equilibrium would still remain, provided the points of suspension G and F continued the same: whence the proposition is manifest. Had the prism AB been supposed laid upon HI, the steps of the demonstration would still have been the same.

This demonstration, which, in our opinion, is far more simple and obvious than that of Archimedes for the straight lever, is given by *Galileo* in his Second Dialogue on the Resistance of Solids. It appears but little known: for *Dr. Matthew Young*, the late learned bishop of Clonfert, has, in his Analysis of the Principles of Natural Philosophy, deduced this property in a manner nearly similar, without seeming at all aware that the same thing had been long since effected by the illustrious Pisanian philosopher.

133. Prop. *When three forces act upon an inflexible lever, void of gravity, in any parallel directions in the same plane, and keep it in equilibrio, any two of them will be to each other inversely as their distances from the point to which the third force is applied.*

This proposition has been already demonstrated in Chap. II. For all that was shewn there (in arts. 71...81.) with respect to parallel forces in one and the same plane, will evidently apply to the case of the lever. So that, instead of repeating the demonstration, we refer to those articles; and shall here deduce from them a few evident corollaries more immediately applicable to what is before us.

Cor. 1. If two weights or parallel forces balance each other upon any straight lever in any position, they will balance each other in any other position of the lever while the same fulcrum is retained: for in this case the fulcrum corresponds with what is named the centre of parallel forces, (art. 91.)

Cor. 2. In the straight lever of the first order (fig. 1. pl. IV.) we have P × AF $=$ W × BF, and the pressure upon the fulcrum in the equilibrated state equal to the sum of the two weights, or of the power and the resistance P + W.

COR. 3. In the straight lever of the second kind (fig. 2.), we have likewise $\mathrm{P \cdot AF = W \cdot BF}$: but the pressure upon the fulcrum is $= \mathrm{W - P}$.

COR. 4. In the straight lever of the third kind (fig. 3.), $\mathrm{P \cdot AF}$ is again $= \mathrm{W \cdot BF}$: but the pressure upon the fulcrum is $= \mathrm{P - W}$.

COR. 5. If a given weight U be moved along either arm of a straight lever (as FA. fig. 1.) the weight W which it will balance at B on the other arm is proportional to FA, the distance from the fulcrum at which the given weight acts. For, in the case of equilibrium, $\mathrm{W \cdot BF = U \cdot FA}$; wherefore, since FB and U are invariable, $\mathrm{W} \propto \mathrm{FA}$.

COR. 6. If a weight W (fig. 5.) be placed upon a lever which is supported by two props A and B, in an horizontal position, the pressure on A : pressure on B :: BC : CA. Or pressure upon $\mathrm{A} = \frac{\mathrm{W \cdot BC}}{\mathrm{AB}}$. Pressure upon $\mathrm{B} = \frac{\mathrm{W \cdot AC}}{\mathrm{AB}}$. (art. 80.)

COR. 7. In the common balance (fig. 4. where the arms of the lever either are equal or ought to be so, the weights in the two scales must be equal also.

COR. 8. In the false balance, where arms of unequal lengths are made to balance each other when the scales are empty, *the true weight is a geometrical mean proportional between the apparent weights*.

For if X be the unknown weight, W the weight which puts it in equilibrio, x and w being the corresponding arms of the balance, we have $\mathrm{X}x = \mathrm{W}w$. If we put the weight X into the bason at the end of the arm w, some new weight as W' will restore the equilibrium, if put in the contrary bason, and we shall have $\mathrm{X}w = \mathrm{W}'x$. By multiplying these two equations together we obtain $\mathrm{X}^2 xw = \mathrm{WW}'xw$, whence $\mathrm{X} = \sqrt{\mathrm{WW}'}$.

In all cases where great accuracy is required, we may apply this rule advantageously, even though the balance be thought correct: if the difference between the apparent weights when tried in the opposite scales is but trifling, the arithmetical mean may suffice.

COR. 9. When several weights, as T, U, V, W, (fig. 6.) keep a lever in equilibrio upon a fulcrum F, the pressure upon that fulcrum is equal to $\mathrm{T + U + V + W}$, and the sums of the moments of the weights on different sides of the fulcrum are equal, *i. e.* $\mathrm{W \cdot AF + U \cdot CF = T \cdot BF + V \cdot DF}$.

COR. 10. If a straight lever be kept at rest by any number of parallel forces acting either at the same or different points, the sum of all the forces acting on one side will be equal to the sum of all the forces on the other.

SCHOLIUM.

134. The observation in the first corollary above, is equally

applicable to several forces as to two; that is, if they balance each other in any one position of a straight lever, they will likewise balance in any other position, provided the fulcrum is not changed. But it must be remarked that in any other position than that in which all the forces are perpendicular to the lever, and consequently their resultant perpendicular to it, the lever will have a tendency to slide along and change its fulcrum, unless prevented. For, since it is evident that a force F soliciting a material point, or body, in a direction perpendicular to any plane AB may be entirely destroyed by the resistance of the plane, and that a force acting in the same direction as the plane must have its entire effect for aught that depends upon that plane; it follows that, in order that when a single force acts upon a body it should remain immoveable, this force must be exerted in a direction perpendicular to the plane on which the body is supported. If any other force as F′ acting obliquely should be affirmed to keep the body at rest, we might resolve it into two, the one in the direction of the plane, the other perpendicular to it; of these the first producing its entire effect, the body must move in consequence of its action, which is contrary to the hypothesis.

135. Prop. *To establish the equilibrium on a straight lever, having regard to its weight.*

When we take the weight of the lever into our investigation, we may consider it as acting at its centre of gravity; or in levers of the first kind we may consider the weight of each arm of the lever as a new power acting at its centre of gravity; and in either case the equilibrium may be established as in *Cor.* 9. of the foregoing proposition. Let us consider each of the levers as prisms or cylinders; then will their centres of gravity reside in their middle points, and the centres of gravity of the arms in *their* middle points estimated from the fulcrum. Hence, in the lever of the first kind (fig. 1. pl. IV.), we have in the case of an equilibrium, $\mathrm{P}\cdot\mathrm{AF} + \text{weight } \mathrm{AF}\cdot\frac{1}{2}\mathrm{AF} = \mathrm{W}\cdot\mathrm{BF} + \text{weight } \mathrm{BF}\cdot\frac{1}{2}\mathrm{BF}$. In the second kind of lever (fig. 2.) $\mathrm{P}\cdot\mathrm{AF} = \mathrm{W}\cdot\mathrm{BF} + \text{weight } \mathrm{AF}\cdot\frac{1}{2}\mathrm{AF}$. And in the third kind of lever, (fig. 3.) $\mathrm{P}\cdot\mathrm{AF} = \mathrm{W}\cdot\mathrm{BF} + \text{weight } \mathrm{BF}\cdot\frac{1}{2}\mathrm{BF}$. Here we have supposed throughout, that the forces are all parallel to that of gravity.

SCHOLIUM.

136. As it is often requisite to pay a regard to the weights of levers, we shall here exhibit a few algebraic formulæ for each kind of straight lever, which may be found of utility in various calculations.

In all the figures (1, 2, 3, pl. IV.) let $\mathrm{FA} = p$, and $\mathrm{FB} = w$, and let the weight of the lever divided by its length be denoted by m; then will m represent the weight of a unit of AB, in the same terms as P and W are expressed in. And we shall have,

For a lever of the first kind: $\mathrm{P}p + \frac{1}{2}mp^2 = \mathrm{W}w + \frac{1}{2}mw^2$.

From which we get (I.) . . $\mathrm{P} = \frac{\mathrm{W}w + \frac{1}{2}mw^2 - \frac{1}{2}mp^2}{p}$.

$$\text{(II.)} \ldots \mathrm{W} = \frac{\mathrm{P}p + \frac{1}{2}m\,p^2 - \frac{1}{2}mw^2}{w}.$$

$$\text{(III.)} \ldots m = \frac{2\,\mathrm{P}p \backsim 2\,\mathrm{W}w}{w^2 \backsim p^2}.$$

$$\text{(IV.)} \ldots w = \tfrac{1}{m}\sqrt{(m^2p^2 + 2m\mathrm{P}p + \mathrm{W}^2)} - \tfrac{1}{m}\mathrm{W}.$$

$$\text{(V.)} \ldots p = \tfrac{1}{m}\sqrt{(m^2w^2 + 2m\mathrm{W}w + \mathrm{P}^2)} - \tfrac{1}{m}\mathrm{P}.$$

When $\mathrm{W} = 0$ . . (VI.) . . $p = \sqrt{\left(w^2 + \frac{\mathrm{P}^2}{m^2}\right)} - \frac{1}{m}\mathrm{P}$.

Or, when $\mathrm{P} = 0$ . . (VII.) . . $p = \sqrt{\left(w^2 + \frac{2}{m}\mathrm{W}w\right)}$.

The latter formula manifestly has place, when the longer arm of the lever balances the shorter arm together with the weight attached to it: so that P has no proper *minimum* here.

For a lever of the second kind: $\mathrm{P}p = \mathrm{W}w + \frac{1}{2}mp^2$.

Whence, ... (i.) . . . . $\mathrm{P} = \frac{\mathrm{W}w}{p} + \frac{mp}{2}$.

$$\text{(ii.)} \ldots \mathrm{W} = \frac{\mathrm{P}p - \frac{1}{2}mp^2}{w}.$$

$$\text{(iii.)} \ldots m = \frac{2\mathrm{P}p - 2\mathrm{W}w}{p^2}.$$

$$\text{(iv.)} \ldots w = \frac{\mathrm{P}p - \frac{1}{2}mp^2}{\mathrm{W}}.$$

$$\text{(v.)} \ldots p = \frac{1}{m}\mathrm{P} \pm \frac{1}{m}\sqrt{(\mathrm{P}^2 - 2m\mathrm{W}w)}.$$

When $\mathrm{W} = 0$, that is, when the power just sustains the lever, we have (vi.) . . . $p = \frac{2\mathrm{P}}{m}$.

In this case it is evident the equation for P is susceptible of a *minimum*: for, adopting the fluxionary process, since

$$\mathrm{P} = \frac{\mathrm{W}w}{p} + \frac{mp}{2} = \mathrm{W}w \times p^{-1} + \frac{m}{2}p, \text{ we have}$$

$\frac{m}{2}\dot{p} = \mathrm{W}wp^{-2}\dot{p}$, and consequently when P is a *minimum* we have

$$\text{(vii.)} \ldots p = \sqrt{\frac{2\mathrm{W}w}{m}}.$$

For a lever of the third kind: $\mathrm{P}p = \mathrm{W}w + \frac{1}{2}mw^2$.

Wherefore, . . . (1.) . . $\mathrm{P} = \frac{\mathrm{W}w + \frac{1}{2}mw^2}{p}$.

$$\text{(2.)} \ldots \mathrm{W} = \frac{\mathrm{P}p}{w} - \frac{mw}{2}.$$

$$(3.)\ldots m = \frac{2\mathrm{P}p - 2\mathrm{W}w}{w^2}.$$

$$(4.)\ldots w = \tfrac{1}{m}\sqrt{\mathrm{W}^2 + 2m\mathrm{P}p} - \tfrac{1}{m}\mathrm{W}.$$

When $\mathrm{W} = 0$, or the power just sustains the lever,

we have $$(5.)\ldots w = \tfrac{1}{m}\sqrt{(2m\mathrm{P}p)} = \sqrt{\frac{2\mathrm{P}p}{m}}.$$

$$(6.)\ldots p = \frac{\mathrm{W}w + \frac{1}{2}mw^2}{\mathrm{P}}.$$

Or, when the power just sustains the lever alone

$$(7.)\ldots p = \frac{mw^2}{2\mathrm{P}}.$$

137. Def. The Roman *statera*, or *steelyard*, is a lever of the first kind, having unequal arms; and is so contrived that one weight alone may serve to counterpoise a great variety of others, by sliding it to different distances from the point of suspension upon the longer arm of the lever.

138. Prop. *Upon the steelyard the distances at which the constant weight must be hung from some certain point will vary as the weights suspended from the shorter arm.*

Let $\mathrm{W}, \mathrm{W}', \mathrm{W}''$, &c. denote the weights which are suspended successively from B (fig. 7. pl. IV.) w the invariable distance FB, p, p', p'', &c. the variable distances FD, FD$'$, &c. at which the constant weight P must be hung from the fulcrum to keep the equilibrium: let also N, N$'$, denote the weights of the brachia BF, FA, respectively, n, n', the distances of their centres of gravity from F. Then, by the last proposition, we have these equations:

$$\mathrm{W}w + \mathrm{N}n = \mathrm{P}p + \mathrm{N}'n'. \qquad \mathrm{W}''w + \mathrm{N}n = \mathrm{P}p'' + \mathrm{N}'n'.$$
$$\mathrm{W}'w + \mathrm{N}n = \mathrm{P}p' + \mathrm{N}'n'. \qquad \mathrm{W}'''w + \mathrm{N}n = \mathrm{P}p''' + \mathrm{N}'n', \text{ \&c.}$$

Now subtracting in succession the first equation from the second, the second from the third, &c. we thence deduce,

$$p' - p = \frac{\mathrm{W}' - \mathrm{W}}{\mathrm{P}}w \quad \ldots \quad p'' - p' = \frac{\mathrm{W}'' - \mathrm{W}'}{\mathrm{P}}w.$$

$$p''' - p'' = \frac{\mathrm{W}''' - \mathrm{W}''}{\mathrm{P}}w. \text{ \&c.}$$

Here, if the weights are in arithmetical progression, we have $\mathrm{W}' - \mathrm{W} = \mathrm{W}'' - \mathrm{W}' = \mathrm{W}''' - \mathrm{W}''$, &c. consequently $p' - p = p'' - p' = p''' - p''$ &c. that is, the divisions on the graduated arm are then equal to each other, or the distances from a certain point on the steelyard will vary as the weights W, W$'$, &c.

The beam, with its appendages of scales or hooks, is either so contrived as to be in equilibrium upon the point F, without the weights P, W, &c. or so that the arm FA when both are unloaded shall preponderate. In the former case F is the point from which the equal divisions on the beam are to be estimated: in the latter, to find the point where the divisions are to commence, proceed thus. Find experimentally the point C, at which if a single weight equal to the constant weight P be suspended, it will keep the lever at rest; that point will be the one from which the graduations must be measured. For, when W and P are suspended upon the beam and the whole is in equilibrio, W sustains P together with a weight which would support P if placed at C: hence $Ww = Pp + P \cdot FC$, or $W \cdot BF = P \cdot DF + P \cdot FC = P \cdot DC$, where, because BF and P are invariable, W must vary as DC.

139. This instrument is very convenient, because it requires but one weight, is easily carried, and the pressure on the fulcrum is less than in scales, when the substance to be weighed is heavier than the constant weight. But when the constant weight is greater than that of the substance to be weighed, the pressure on the fulcrum is greater in the steelyard than in the common balance; for which reason, together with the following, it is not so accurate as scales in determining the quantity of small weights. 1st. Because, the length of the beam being given, the arms in the steelyard will be shorter than in the balance, whenever the substance to be weighed is equal to, or less than, the constant weight. 2dly, The balance admits of a more nice adjustment. 3dly, The subdivision of weights for the balance can be effected with greater precision than the subdivision of the arm of the steelyard.

140. PROP. *Any two forces acting upon a bent lever (considered as without weight) in different directions but in the same plane, are in equilibrio, when they are to each other inversely as the perpendiculars let fall from the fulcrum upon their directions.*

Let the two powers W, P, (fig. 8. pl. IV.) act upon the lever WFP, whose centre of motion is F, in the directions WM, PL, in the same plane; and let FM, FL, be the perpendiculars upon the directions of the powers: from F as a centre with radius FL equal to the longer perpendicular, describe a circular arc intersecting the direction of the force W in D: and, because the efficacy of forces is the same (32.) to whatever points of their directions they may be applied, we may conceive the two forces applied at D and L. Now, the forces being in equilibrio when acting at D and L, may be conceived the same in effect as forces acting at the periphery of a wheel DHL, perpendicular to its radii DF, LF, and preventing it from turning upon the centre F, and in this case the forces acting in the direction of the circumference at their points of application must manifestly be equal. Hence, if DE in the direction of W represent the magnitude of that force, and be resolved into the two, DG in the direction of radius FD, and GE perpendicular to it, the line GE will represent the part of W which tends to produce the motion on the centre, and is therefore equal to P. Consequently, by the nature of equilibrated forces and similar triangles W : P :: DE : EG :: DF (=FL) : FM. Q. E. D.

This demonstration is the same in principle as the one given by *Newton* in the *Principia*: we have adopted it, because, notwithstanding the various objections which have been urged against it, we think it more concise, and at the same time more satisfactory, and more universal in its application, than any other isolated demonstration extant. The two principles assumed are, 1st. That any force W will have the same energy in the direction EDM whether it be applied at D, W, or M; a truth which cannot well be denied (art. 32.) And, 2dly, That equal and contrary forces acting perpendicularly to any two radii of a wheel at equal distances, will prevent it from turning about its centre: and this presents itself to the mind as remarkably evident. With this view of the matter, we shall not here attempt to controvert the arguments of Dr. Hamilton and others; who do not seem to have attended very carefully to the nature of Newton's demonstration*.

Cor. 1. *If* DW *and* PL, *the directions of the forces (figs. 9. 10. pl.* IV.), *be produced till they meet at* S, *and from the fulcrum* F *the line* FI *be drawn parallel to the direction of one force till it meets that of the other, then will* SI, IF *represent the two forces, and* SF *the pressure upon the fulcrum.*

For, if SF be radius, FM and FL (perpendiculars to SD and SP respectively) are the sines of the angles FSM, FSL, or of the angles FSI, IFS: hence SI : IF :: sin SFI : sin ISF :: FL : FM; and therefore the forces W and P are, by the proposition, represented in magnitude and direction by SI, IF, and (art. 32.) may be supposed to be applied at S. But the third force by which the equilibrium is sustained is represented by SF, in the triangle SIF; and that third force is (by hyp.) no other than the re-action of the fulcrum, which is equal and opposite to the pressure upon it; consequently that pressure is denoted by SF.

Cor. 2. *The magnitudes of* W, P, *and the pressure, are each as the sine of the angle formed by the directions of the other two; or, any two of them are inversely as the perpendiculars let fall upon their direction from any point in the direction of the third.* So that these deductions coincide with what has already been demonstrated in arts. 82, 83. &c. Ch. II. as they certainly ought to do.

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* They, however, who are desirous of seeing this demonstration defended against the usual objections, and its application to the case of parallel forces acting on a straight lever fully shown, may turn to my paper on this subject in number 82 of the *Repertory of Arts*, &c., or to the article Lever, in the *Pantologia*.

COR. 3. Since the energy of the powers is the same at whatever point in their respective directions they may be applied, a bended or an angular lever may be reduced to a straight one, making an invariable angle with it, and the powers may therefore be always supposed to act upon different points of one straight line passing through the centre of motion.

COR. 4. If the directions of forces acting upon a straight lever are parallel, and keep it in equilibrio, they will be inversely as their distances from the fulcrum, either both measured along the lever, or both estimated perpendicular to the directions. This, therefore, agrees with what has previously been shown with respect to the straight lever.

COR. 5. If more than two forces act upon a lever, there will be an equilibrium when the sum of the products arising from multiplying each into the nearest distance of its direction from the fulcrum, on one side, is equal to the sum of the products on the other.

COR. 6. If the weight of the lever is taken into the account, that of each arm is to be regarded as a new force acting at its centre of gravity.

COR. 7. If any body AB (fig. 11. pl. IV.) is moveable about its centre of gravity C, and two forces P, P′, act upon it at those points in the figure in the directions PD, PD′, in the same plane: then joining CP, CP′, the body may be considered as a lever PCP′, and there will be an equilibrium when the powers are inversely as the perpendiculars upon their respective directions; that is, when P : P′ : : CM′ : CM.

COR. 8. If two weights are suspended from the ends of an angular lever, whose fulcrum is at its angular point, the whole will be at rest when the vertical line passing through the fulcrum divides the right line joining the extremities of the lever into two parts, which are inversely as the weights suspended from the contiguous arms of the lever.

For, let the lever WFP (fig. 1. pl. V.), having the weights W and P attached to its extremities, be in a state of equilibrium; the directions WL, PM of the forces being in this case parallel, the perpendiculars FL, FM, form one straight line, and (*Cor.* 3.) FL : FM : : P : W. But the vertical line FG is parallel to WL and PM: so that, if L′FM be drawn through F, parallel to WP, we shall have WG (= L′F) : GP (= FM′) : : LF : FM : : P : W.

COR. 9. If the arms of the angular lever be prisms or cylinders of uniform matter and thickness, the weight of the arm WF being denoted by *w*, and that of PF by *p*, then in the case of the equilibrium WG : GP : : P + ½*p* : W + ½*w*. For this is only to suppose ½ of the weight of either arm to act at double the distance of its centre of gravity from the fulcrum, which is manifestly the same in effect as the whole weight acting at the single distance.

141. **Prop.** *In any combination of straight levers* AB, $A'B'$, $A''B''$, &c. *whose centres of motion are* F, F', F'', &c. *the whole lying in one direction, the ratio of* P *to* W *acting in the same plane at their extremities, is that of* $B''F'' \cdot B'F' \cdot BF : AF \cdot A'F' \cdot A''F''$.

For, let the forces Q, Q', &c. (fig. 2. pl. V.) acting at the points B, B', &c. in directions parallel to those of P and W, be those which would keep each lever in equilibrio: then (art. 132.)

$$P : Q :: BF : AF.$$
$$Q : Q' :: B'F' : A'F'.$$
$$Q' : W :: B''F'' : A''F'', \text{ \&c.}$$

whence, by composition of ratios $P : W :: BF \cdot B'F' \cdot B''F'' : AF \cdot A'F' \cdot A''F''$. Q. E. D.

Cor. 1. The pressure upon the fulcrum $F = P + Q = \frac{P \times AB}{FB}$.

the pressure upon $F' = Q + Q' = \frac{P.AF}{BF} + \frac{W.B''F''}{A''F''}$.

the pressure upon $F'' = Q' + W = W + \frac{W.B''F''}{A''F''} + \frac{W.A''B''}{A''F''}$.

Cor. 2. If P and W act in different directions, as well as Q, Q', &c. and the whole be in equilibrio, we must substitute perpendiculars from the points F, F', &c. upon the several directions of the powers, for AF, FB, &c. in the original proportion.

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142. Before we close our discussions respecting the lever, it may not be amiss just to remark, that in every attempt to determine the advantage gained by this machine peculiar attention must be paid, not only to the directions in which the forces are exerted, but to the points on the lever to which their action is to be referred. Without a due regard to these particulars, the mechanist will often be involved in error, even in simple cases where there might be supposed but little probability of mistake. In fact, even the simple property of the straight lever that equal weights acting at equal distances from the fulcrum on opposite sides will be in equilibrio, while at unequal distances the one which acts most remotely from the fulcrum will preponderate, has more than once been a source of error in unskilful hands; and in particular, it has lain at the foundation of most of those ill-fated and useless contrivances which have been struck out by such as were in pursuit of the *perpetual motion.* In these contrivances the object of the projector has generally been to apply different weights to a rotatory machine in some such manner, that, at successive moments of time, first one and then another

should be brought to greater distances from the centre, and so, by being placed at the extremity of a longer lever, should produce a constant motion. To prevent, therefore, such waste of time and ingenuity, we shall here describe an apparatus invented by Dr. Desaguliers (See *Phil. Trans.* No. 419. Or, *New Abridg. vol.* VI. *pa.* 542.) in which two equal weights may be placed at any unequal distances whatever from the centre of motion, and still remain in equilibrio. In fig. 3. pl. V. AB represents a balance with equal arms, and EF another of equal dimensions: they turn freely upon the centres C, D, and their extremities are connected by equal inflexible bars AE, BF; the whole being permitted to move freely at the joints A, B, C, D, so as to assume the forms of varying parallelograms, in consequence of any motion upon the points C and D. Across the bars AE, BF, are fixed others as WU, PQ, from any points of which equal weights P, W, may be suspended. Now, on whatsoever part of the bar PQ the weight P is fixed, it is manifest that it will, on account of that bar being firmly connected with the vertical rod BF, *act* as though it were placed at F: and, in like manner, in whatever part of the bar WU the weight W be suspended, it will *act* as though it were placed at E: so that, however great may be the difference of the distances of the bodies P and W from CD, they will still, if equal in weight, balance each other in any position of the system. Nor is this in any respect incompatible with the principle of the equal products of weight and velocity, which we have mentioned (130.) as a useful indication of an equilibrium: for, suppose this compound balance to be brought by motion on its centres into the position *abcd*, the weights being then at *w* and *p*; those weights will have moved through the arcs W*w*, P*p*, while the extremities of the levers will have passed through the equal and respectively parallel arcs A*a*, E*e*, B*b*, F*f*; of consequence the velocities of the two weights will have been equal, as they ought to be, in conformity with that principle. Thus, then, it appears from this simple contrivance, that weights do not preponderate in machines merely on account of their different distances from the centre of motion; and consequently a mere increase of distance does not *universally* give a mechanical advantage.

II. *Of the Wheel and Axle.*

143. DEF. The *Wheel and Axle*, or, as it is often called, the *Axis in Peritrochio*, is a machine which consists of a cylinder and a wheel having the same axis, at the two extremities of which are pivots on which the whole may turn. The power is applied at the circumference of the wheel, generally in the di-

rection of a tangent, and a cord is wrapped about the cylinder, in order to overcome the resistance, or elevate the weight.

This machine is sometimes, without any impropriety, named the perpetual lever; it being, in reality, a lever on whose arms the power and weight may always act perpendicularly although the lever turns round its fulcrum. It is often constructed of the form represented fig. 4. pl. V. where CD is the cylinder, at the ends of which are the pivots FE, turning in the solid pieces of timber HF, AE; the weight W is raised by means of a rope coiled about the cylinder; the power being applied to the wheel SSB, either by the cord II, or by the handles S, S, S. Sometimes, instead of the wheel, we find this machine made up of levers, fixed into the cylinder, as spokes into the nave of a wheel; at others, a simple handle serves for the application of the power, as in fig. 5: but the effect is still the same, except that the rotation is less uniform. In some cases the cylinder is horizontal, as in the figures, and in some kinds of *cranes;* in others, it is vertical, as in the capstan, &c. But whether the cylinder be horizontal or vertical, this machine has a manifest advantage over the simple lever in point of convenience: for, by the continual rotation of the wheel the weight may be raised to any height, or from any depth; while it could be elevated only a little way by any lever.

144. Prop. *In the wheel and axle, if the power act perpendicularly to the radius of the former, and the weight perpendicularly to that of the latter, there will be an equilibrium when the weight and power are reciprocally as the radii of the circles at which they act.*

Conceive fig. 6. pl. V. to be a vertical section of the machine perpendicular to the axis: and, since the effort of the weight to turn the axle round is the same at whatever point of that axle it be applied, suppose both P the power, and W the weight or resistance, to be applied at A and B, in the same plane, perpendicular to the axis of rotation. Then, whether the power P act at A, or A′, since it acts perpendicularly to the radius CA, CA′, (by hyp.) while the weight acts perpendicularly to CB, it is manifest that either ACB, or A′CB may be considered as a lever, whose fulcrum is C; and consequently (140.) P : W : : CB : CA. Q. E. D.

Cor. 1. If the power act in any other direction than the tangential one, as in A′P″, for example; then, drawing from C the line CD perpendicular to the direction of the power, we have P : W : : CB : CD; whence, because CD is always less than CA, it is obvious that the *tangential is the most advantageous direction in which the power can be applied.*

Cor. 2. Since we have always P : W : : CB : CA, or CA·P =

CB·W, it follows, that when the power is variable, if the diameter of the wheel increase in the same proportion as the power diminishes, and *vice versa*, the force with which the wheel will continue to be turned, will always be of the same magnitude.—This principle is ingeniously applied in the action of the main spring on the fuzee of watches, and of the main spring on the tumbler of gun-locks.

Cor. 3. When the moving force is applied by means of handles, as at s, s, s, (fig. 4.) it often happens that many forces act simultaneously, one at each handle: then, if they all act in directions perpendicularly to the respective radii, there will be an equilibrium, when the sum of all the powers is to the weight, as the radius of the wheel, to the mean distance at which the powers act; or, when the sum of the moments (31.) of the powers, is equal to the moment of the weight.

Cor. 4. When the power and weight act by means of a rope, and it have a sensible thickness compared with that of the axle, there will be an equilibrium when the power is to the weight, as the sum of the radii of the axle and rope, to the sum of the radii of the wheel and rope.

For the action of both forces is transmitted by the axis of the rope, and consequently its radius ought to be added both to that of the wheel, and that of the axle.

Cor. 5. In the case of the last corollary the ratio of the power to the weight is greater than when the thickness of the rope is not taken into the account: if, therefore, the rope is so folded upon the axle as to cover the surface, and the weight acts by a second spire of rope, the power must be augmented to maintain the equilibrium; and so on continually for every increased course of rope upon the axle.

Cor. 6. The distance which the weight hangs from the axle will make no difference in the result, except that which may be occasioned by the weight of rope, or that which might be caused by a varying force of gravity at different distances from the earth's centre.

145. Prop. *To determine the pressure upon the pivots of the wheel and axle, when in equilibrio.*

Here we shall merely consider the most common and useful case, in which the power and the weight act in parallel directions, and on opposite sides of the horizontal axis of motion. The pressures upon the pivots will arise from the action of the weight, that of the power, the weight of the wheel, and that of the axle. Call the weight of the axle A, that of the wheel V, the power and the resistance to be overcome being denoted by P and W. Then (fig. 8. pl. V.) the weight A may be considered as acting at the middle of the axle, and the consequent pressure upon each pivot will be $\frac{1}{2}$A. The weight V will act at C, the centre of the wheel: therefore (133. cor. 6.) pressure upon A $= \frac{V \cdot BC}{AB}$, and pressure upon B $= \frac{V \cdot AC}{AB}$. The forces P and W may be conceived to act together at their common centre; that is, if I be the centre of a section of the axle where W acts, and ID : DC : : P : W, D will be the point where P + W will act together: hence, the third pressure upon A $= (P + W)\frac{BD}{AB}$, corresponding pressure upon B $= (P + W)\frac{AD}{AB}$. Consequently, in the case of equilibrium, the aggregate pressure upon A $= \frac{1}{2}A + \frac{V \cdot BC + (P + W) \cdot BD}{AB}$, and the whole pressure upon B $= \frac{1}{2}A + \frac{V \cdot AC + (P + W) \cdot AD}{AB}$.

The pressure upon the pivots when the machine is in motion will be investigated in the second book. (art. 321.)

146. **Prop.** *If a series of wheels and axles be so connected by cords as to act upon one another, the power being applied tangentially to the first wheel, and the weight in like manner to the last axle, there will be an equilibrium when the power is to the weight, as the continual product of the radii of all the axles, to the continual product of the radii of all the wheels.*

Let such a system of wheels and axles be represented in fig. 9, and let the force Q in direction OF balance P acting at G, and the force Q′ in direction NE balance Q : then, by art. 144. we have

P : Q : : OD : DG.
Q : Q′ : : CN : CF.
Q′ : W : : AB : AE.

Whence, by composition of ratios, P : W : : DO · CN · AB : DG · CF · AE. Q. E. D.

Cor. 1. If the wheels and axles, instead of being at a distance from each other, and connected by cords, are placed as in fig. 7, and the axle of one wheel made to act upon the circumference of the next by means of equal teeth, the proportion between the power and the weight will, nevertheless, be as stated in the proposition: for each wheel and axle will act as a lever whose fulcrum is at the centre of the axle, and its arms respectively equal to the radii of the wheel and the axle; and the joint effect will be as above. (141.)

This is generally called *tooth and pinion work*, the axle when its surface is indented being called a *pinion*. And because the number of teeth in the wheels and pinions are to each other as their peripheries, or as their radii, it follows that the power is to the weight, as the continual product of the teeth in the pinions, to the continual product of the teeth in the wheels. Or, if P acts on the wheel A by a cord, and W is suspended from the axle e by another, we shall have P : W :: teeth in a × teeth in b × teeth in c × teeth in d × rad. e : rad. A × teeth in B × teeth in C × teeth in D × teeth in E. If the wheels and pinions, instead of being disposed as in fig. 7. are disposed in any other manner, as in fig. 10. for instance, the ratio of the power and weight will still be determined by a similar process.

COR. 2. If the wheels be equal to each other, and the axles equal to each other, or if each wheel be to the axle on which it is fixed in a constant ratio, $r : s$; then n being the number of wheels, we shall have in the case of equilibrium, P : W :: $s^n : r^n$. Thus, if there be three wheels and axles, the radii of each being as 10 : 1; then will P : W :: 1 : 1000: if there be four such wheels and axles, P : W :: 1 : 10000, and so on.

COR. 3. Since, when a toothed pinion works a wheel the peripheries of both move with the same velocity; and the pinion of the second wheel moves the third wheel with a less velocity than the second moves, being at a less distance from the centre of motion; and the same thing obtains throughout the system; hence, in a combination of wheels in motion the number of revolutions of the wheel where the power acts, is to the number of synchronal revolutions of the wheel where the weight acts, as the product of the radii (or teeth) of the wheels, to the product of those of the axles.

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147. In forming the teeth it is of considerable importance to determine their proper curvature, so that the motion may be communicated equally, and with as little friction as possible. Two methods of accomplishing this end have been recommended: of these the first was originally proposed by M. de la Hire, who affirmed that the pressure would be uniform *if the teeth were formed into epicycloids;* and M. Camus, in his Cours de Mathematiques, has pursued M. de la Hire's principle, and applied it to the various cases which are likely to arise in practice. The construction, however, is subject to a limitation: on which account a second method has been proposed, which secures the perfect uniformity of action without any such limitation. This method consists in making both teeth portions of involutes of circles. Thus, let IHF, KEb, (fig. 1. pl. VI.) be the wheels to which the teeth are to be accommodated: the acting face GCH of the tooth a must have the form of the curve traced by the extremity H of the flexible line FaH, as it is unwrapped from the circumference; and, in like manner, the acting face of the tooth b must be formed by the unwrapping of a thread from

the circumference of the circle E E *b*. The line F C E drawn to touch both circles will cut the surfaces of the two teeth in C, the point where they touch each other; the faces of both teeth will always touch each other at a point in the common tangent to both circles, and the force arising from their mutual pressure will always act in the direction of the circumferences of the wheels at E and F. This form, by allowing the teeth to act on each other through the whole extent of the line F C E, will admit of several teeth to be acting at the same time; and thus, by dividing the pressure among several teeth, will diminish its quantity upon any one of them, and therefore diminish the cause of the indentations they unavoidably make upon each other. Consequently, a considerable number of teeth thus formed acting at once render the communication of motion extremely smooth and regular. But this by the by: the consideration of teeth and pinions, &c. will be farther pursued in the second volume.

III. *Of the Pulley.*

148. Def. The *pulley* is a wheel of wood, brass, or iron, turning on an axis, and enclosed in a kind of frame or case called its *block*, which admits of a rope to pass freely over the circumference of the pulley, in which there is usually a groove to prevent the slipping out of the rope. The pulley is said to be *fixed* or *moveable* according as its block is fixed or rises and falls with the weight. An assemblage of several pulleys is called a system of pulleys, a *muffle*, or *polyspacton*: of which some are in a fixed block, and others in a moveable one.

The nature and effects of the pulley have by some writers been explained by considering a fixed pulley as a lever of the first order, and a moveable pulley as one of the second. But, Professor Hamilton and others have been of opinion that "the "pulley cannot properly be considered as a lever of any kind; "for when any power sustains a weight by means of a system "of pulleys that power will sustain the same weight if the pulleys "be removed, and the ropes be brought over the axles on which "the pulleys turned. If the weight were to be raised up there "would, in this case, be a very great resistance from the friction "of the ropes on the axles; and it is merely to avoid this resistance that pulleys are used, which move round the axles "with but little friction." One of the most simple and natural methods of computing the power and explaining the effects of the pulley, is by considering that every moveable pulley hangs by two parts of the same rope equally stretched, which must sustain equal parts of the weight; and therefore, when one and the same rope goes round several fixed and moveable pulleys, since all its parts are equally stretched, the whole weight must be divided equally among all the ropes by which the moveable pulleys hang. And consequently, if the power which acts on one rope be equal to the weight divided by the number of ropes, that power must sustain the weight. This principle may be applied to many of the cases which occur, with great facility; particularly when the cords run in directions nearly parallel, as in the systems exhibited, fig. 2. pl. VI. But when the ropes are drawn in directions which are not parallel, this method may lead to error, on which account we shall give a general proposition, depending on the pure principles of equilibrium, from which the application to particular cases may be easily deduced.

149. PROP. *Each of two forces in equilibrio about a pulley is to that which retains the axis, as the radius of the pulley, to the chord of that arc with which the rope is in contact.*

Let DEC (fig. 3 pl. VI.) represent a pulley which is kept in equilibrium by three forces P, W, R; of which the two former act by a rope PEGDW, which touches the pulley throughout the arc EGD, the latter acting by a rope or a bar RG whose direction passes through C, the axis of the pulley. Now, since the whole is in equilibrium, the force R is equal and opposite to the resultant of P and W; and since the moment of R with respect to the point C is zero, the moments of P and W must be equal (60. 61.): wherefore, since DC$=$CE, W must be equal to P. Produce, therefore, WD, PE, till they meet CR, and setting off from the point of concourse the equal distances RA, RB, on the directions of the powers to represent them, complete the rhombus RAIB; then we shall have P : W : R :: RB : BI : IR. But the triangles RBI, ECD, having their sides respectively perpendicular to each other, are similar; and consequently P : W : R :: CE : CD : DE. Q. E. D.

COR. 1. When WD and PE are parallel, CD and CE will coincide with the semi-chords FD, FE, and we shall have P$=$W, or the power equal to the weight, as in the fixed pulley with ropes parallel. Or, supposing the figure inverted, the rope fixed at one end as W, and the power acting at the other end P, R would then be the weight and would be equal to 2P.

COR. 2. In this case the pressure is easily estimated from hence, and from (133). Thus in fig. 2. the pressure or stretch upon hook H is $= P = \frac{1}{2}W$; that upon the axis of the pulley C is $= P + \frac{1}{2}W = 2P$; that upon the support X is evidently $= P + W = P + nP = (1+n) P$, where n denotes twice the number of moveable pulleys.

COR. 3. If the angle DRE be denoted by $2a$, then will R$=$

$2P\cos a$. For, since $P : R :: CE : DE$, we have $R = P\frac{DE}{CE}$: and since, moreover, $DE = 2EF = 2CE.\cos CEF = 2CE.\cos a$; consequently $R = P.\frac{2CE.\cos a}{CE} = 2P\cos a$.

Cor. 4. If the angle DRE be equal to 120°, then will $\cos a = \frac{1}{2}$, and $P = W = R$.

150. Prop. *In a system of moveable pulleys, each of which has a separate rope, the power is to the weight, as radius to the continual product of the cosines of the half angles made by the rope sustaining each pulley, into that power of 2 whose exponent is the number of pulleys.*

Let the weight W (fig. 5. pl. VI.) be kept in equilibrio by the power P through the medium of the pulleys A, A′, A″, the angles $n AA'$, $n A'A''$, made at each being denoted by $2a, 2a', 2a''$, &c. and let t, t', t'', &c. be the tensions of the cords by which the pulleys A, A′, A″, are supported. Then by Cor. 2. last prop. we have

$$W = 2t\cos a.$$
$$t = 2t'\cos a'.$$
$$t' = 2t''\cos a''.$$
$$\cdots\cdots\cdots$$
$$t^{n-1} = 2t^{n}\cos a^{n}.$$

Multiplying together the two first, three first, &c. of these we shall obtain the tensions of the several cords in succession; and if we multiply together the whole, we have for the relation between P and W, the following equation

$$W = 2^{n}P(\cos a.\cos a'.\cos a''.\ldots.\cos a^{n}.)$$

Whence the proportion stated in the proposition is manifest.

Cor. 1. If the cords are parallel (as in fig. 4.) then will $\cos a = \cos a' = \cos a''$, &c. $= 1$, and the preceding equation becomes $W = 2^{n}P$, whence $P : W :: 1 : 2^{n}$; that is, *the power is to the weight, as unity, to that power of 2 which is denoted by the number of pulleys.*

Cor. 2. If the angles at A, A′, A″, are equal to each other, then will $P : W :: 1 : 2^{n}\cos^{n} a$, a denoting the angle at each pulley.

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151. A system of pulleys being tolerably portable and not very heavy, can be readily conveyed from place to place, and thus made use of in many instances where other machines cannot be employed. In a judicious combination of them a considerable weight may be moved by a very small power, though in this, as in all other machines, the power must pass over a proportionably greater space. But the chief disadvantages to which pulleys are liable, arise from their great friction,

and the stiffness of the ropes which pass over and under them. Some ingenious contrivances have been devised to remove in a great degree these and other inconveniences; which will be described under the article PULLEY in the second volume of this work.

IV. *Of Inclined Planes.*

152. DEF. An *Inclined Plane*, as its name imports, is one which forms with an horizontal plane any angle whatever. It is reckoned among the mechanical powers because when any body is laid upon it a part of its weight is supported, and it is therefore moved with greater ease. The inclination of the plane is measured by the angle formed by two lines drawn upon the sloping plane and the horizontal plane, perpendicular to their common intersection, and meeting at a point in that intersecting line.

153. Before we demonstrate the general properties of this simple machine we shall premise a few observations on the manner in which *bodies* are supported upon any planes whatever.

I. If a body of any figure touch a plane in a single point, and be solicited by one force only; two conditions are requisite to its continuance at rest on that plane. *First*, that the direction of the single force be perpendicular to that plane. *Secondly*, that its direction pass through the point where the body is in contact with the plane. The necessity of the first of these conditions appears from art. 134. As to the second, it is no less necessary, since if the direction of the power, though perpendicular to the plane, pass not through the point of contact of the plane and the body, the resistance of the plane which is exerted perpendicular to the point of contact, will not be directly opposed to the force, and of consequence they cannot destroy each other although they should be equal.

II. If the body, instead of touching the plane in one point only, touch it in many points, or throughout plane surfaces, then it is not indispensable that the single force which acts on the body should pass through any of the points of contact: but it must be perpendicular to the plane, and so situated that it may be decomposed solely into as many forces perpendicular to the plane as there are points resting on the plane, and the forces resulting from the decomposition must pass through these points.

III. Therefore, if a body which touches a plane in many points be solicited by various forces in different directions, it is necessary, *First*, that these forces may be reduced to one only which is perpendicular to the plane: and, *Secondly*, that the latter in the case in which it does not pass through one of the points of contact, may be decomposed into as many forces parallel to it as there are points of contact, each of these new forces passing through a point of contact.

IV. If the single force which acts upon a body be that of gravity, it follows that the plane must be horizontal; and, if the vertical line drawn through the centre of gravity of the body, does not pass through one of the points of contact, it is farther necessary that all the points of contact should not fall on one and the same side of the said vertical line.

V. If, therefore, the body be solicited by two forces only, it is necessary, *First*, that these two forces are in the same plane: *Secondly*, that this plane be perpendicular to that on which the body lies: *Thirdly*, that the resultant, which ought always to be perpendicular to the latter mentioned plane, leave not all the points of contact on the same side of it. And, if one of the two forces be that of gravity, it is necessary moreover that the plane in which the resultant falls be vertical, and pass through the centre of gravity of the body.

VI. With regard to bodies which rest on many planes at once, whether in virtue of a single force, or of many forces, among which we comprehend that of gravity, the general laws of their equilibrium are, *First*, that the resultant of all these forces may be decomposed into as many forces as there are points on which the body rests: *Secondly*, that these latter are perpendicular to the respective planes at the several points of contact. Whence, we conclude that, in order that a body solicited only by gravity may remain in equilibrio between two inclined planes, it is necessary that there be in the vertical which passes through its centre of gravity, at least one point from which we may let fall a perpendicular on each of these planes; and that each of these perpendiculars have the conditions stated above.

154. Prop. *When two forces acting on a body keep it in equilibrio upon a plane, if we conceive two other planes to which these two forces are respectively perpendicular; the two forces and the pressure upon the plane, are each represented by the sine of the angle comprised between the planes to which the two other forces are perpendicular.*

This is nothing else than a manifest inference from art. 48. It may, however, be demonstrated in a manner immediately applicable to the present case, thus: Let CP, CG, be the directions of the two forces, which sustain the body whose centre of gravity is C at rest upon the plane: and let AD (fig. 6. pl. VI.) be the intersection of the plane of the two forces with that on which the body lies: draw CF perpendicular to AD, and on this line as a diagonal describe the parallelogram CEFG two of whose sides shall fall on the directions of the given powers; then, it is evident from the preceding article, that if the reaction of the plane be represented by FC, the two other forces which sustain the body will be represented by EC and GC respectively: consequently if the pressure on the plane which is equal to its reaction, be called R, and P and G be the powers acting in PC, GC, then will P : G : R : : EC : GC : FC : : FG : EF : FC : : sin GCF : sin FCE : sin GCE. Now from two points A and *i* taken arbitrarily upon AD, draw *i*B, and AB perpendicular to the directions of P and G; so shall the triangle AB*i* be similar to CGF : whence, because P : G : R : : sin GCF : sin FCE : sin GCE or sin GC*p*, we shall have P : G : R : : sin *i*AB : sin A*i*B : sin AB*i*. Q. E. D.

COR. 1. Since the relations just established have place, whatever the forces may be, we may consider one of them, as G for instance, to be the force of gravity: then will AB be parallel to the horizon: *and the power, weight, and pressure upon the plane will be respectively, as the sine of the plane's inclination, the cosine of the angle which the direction of the power makes with the plane, and the cosine of the angle which the direction of the power makes with the horizon:* for sin *i*AB, sin A*i*B, and sin AB*i*, are the same as sin *i*AB, cos CE*i*, and cos *i*BD.

COR. 2. Hence, whether the line of direction of the power be elevated above or depressed below, *if the angles which it makes with the plane are equal, equal powers will sustain the weight.* Thus, if the lines PC*p*, P′C*p*′, be equally inclined to the plane, equal powers acting in those directions will sustain the weight, whether they draw in the directions from C to P or P′, or push from *p* or *p*′ towards C. But the pressures on the planes vary with those directions, being less and less as the direction is more elevated.

COR. 3. *The power* P″ *or p″ is least when its line of direction is parallel to the plane:* for since P : G or W : : sin *i*AB : cos CE*i* we have $\mathrm{P} = \mathrm{W}.\frac{\sin i\mathrm{AB}}{\cos \mathrm{CE}i}$, which is manifestly a *minimum*, when cos CE*i* = radius, that is, when CE*i* vanishes.

The truth of this will also appear by considering the proportion, P : W : : B*i* : BA, in which it is obvious P*i* will be the least possible when it is perpendicular to AD, a perpendicular being the shortest distance from a given point to a line; so that *the most advantageous direction in which the power can act is that which is parallel to the plane.*

COR. 4. In this latter case, because of the similar triangles

ABi‴, ADB, we have P : W : R :: Bi″ : BA : Ai″ :: BD : DA : AB :: sin A : rad. : cos A. That is, *when the direction of the power is parallel to the plane, the power, weight, and pressure on the plane, are respectively as the height, length, and base of the plane; or as the sine of inclination, radius, and cosine of inclination.*

Cor. 5. If two weights W, W′, sustain each other upon two inclined planes AC, CB, which have a common altitude CD, by means of a cord which runs freely over a pulley and is parallel to both planes, then will W : W′ :: AC : CB (fig. 7. p. VI.). For let the power P acting in direction C′W, be that which sustains the weight upon the plane CA, then by reason of the uniform tension of the string, the same power sustains the weight W′ upon the plane CB, acting in a direction parallel to it: hence, Cor. 4.

W : P :: AC : CD,
P : W′ :: CD : CB,
And, compo. W : W′ :: AC : CB.

Cor. 6. Hence also, when one of the planes as CB, becomes vertical, we still have W : W′ :: AC : CB. (fig. 8.)

Cor. 7. If the pulley at C be so elevated that the two parts CW, CW′, of the rope are not parallel to the plane, we shall have an equilibrium when W : W′ :: sin CBD × cos CWC′ : sin CAD × cos CW′C′. For, in this case

W : P :: cos CWC′ : sin CAD,
P : W′ :: sin CBD : cos CW′C′,
Whence, W : W′ :: sin CBD · cos CWC′ : sin CAD · cos CW′C′.

Cor. 8. *If the direction of the power be parallel to the horizon, then will the power, the weight, and the pressure on the plane, be, respectively, as the height; the base, and the length of the plane; or, as the sine of inclination, its cosine, and radius.* For, if CP (fig. 6.) be parallel to AB, then will Bi coincide with BD, and it will be P : W : R :: BD : AB : AD :: sin A : cos A : rad.

Cor. 9. If instead of opposing a single power to the action of the weight we oppose several, then all which we have stated in this proposition and the corollaries as relating to the power P, must be extended to the resultant or equivalent of those powers. For example, if the body W (fig. 10. pl. VI.) is sustained on the inclined plane by the combined action of a power P, and the resistance of a fixed point D to which is attached the rope DKP which embraces the body: then, imagine a line HC to be drawn from the point of concourse of DF and PK, bisecting the angle made by those directions. If this line intersect the vertical drawn through the centre of gravity of the body in a point C, from which we may demit on the plane a perpendicular passing through the point of contact F, the equilibrium will be possible;

and the relation between the weight W, and the effort according to HC, will be determined by what has preceded. As to the ratio of the force in direction HC, to the power P, it will be the same as in the moveable pulley. Thus, if the power P acts parallel to the plane, the weight will be to that power, as the length of the plane to half its height; viz. *the power will be only half that which would have been requisite had the weight been sustained without the aid of the fixed point* D.

155. When two bodies whose weights are represented by w, w', are attached to a thread wCw' (fig. 9. pl. VI.) passing over a pulley at C, and are always in equilibrium upon two curves FA, EB, whatever are their respective positions, it will be easy to determine the necessary conditions of those two curves. In order to this draw the vertical CM through the point C, and suppose that the equations of the two curves with respect to that line as a common axis are $y = fx$, $u = \mathrm{F}t$, f and F being any functions of the axes x, and t, and C their common origin. The weight w is a force acting in the vertical direction wb; and if that force be represented by wb, it may be resolved into the two wd acting in the direction of the thread, and wc acting in that of the normal wM; the parallelogram of forces being cd. Here the similar triangles wdb, CwM, give

$$\mathrm{CM} : \mathrm{C}w :: wb : wd.$$

Likewise, putting Cw = z, we have

$$\mathrm{CM} = x + \text{subnorm.} = \frac{x\dot{x} + y\dot{y}}{\dot{x}} = \frac{1}{2}\frac{\dot{\overline{x^2+y^2}}}{\dot{x}} = \frac{z\dot{z}}{\dot{x}}.$$

Therefore, $wd = \frac{w\dot{x}}{\dot{z}}$. In like manner putting Cw' = z', and operating for the weight w', we have $\frac{w'\dot{t}}{\dot{z}'}$ for the component of that weight acting in the direction Cw'. In the case of an equilibrium these two components must be equal. But the length of the thread wCw' being constant and known, we have $z + z' = \mathrm{C}$, whence $\dot{z} = -\dot{z}'$, and consequently

$$w\dot{x} + w'\dot{t} = 0, \text{ or } wx + w't = \mathrm{A}.$$

From this equation it appears that whatever position we give to the bodies w, w', the centre of gravity of their system will be always on the same horizontal line; because the co-ordinate of that centre with respect to GH is constant: for, from the equation (108. I.) that co-ordinate is

$$\frac{wx + w't}{w + w'} = \frac{\mathrm{A}}{w + w'} = \text{a constant quantity.}$$

The curve FA being given, if we demand what the curve EB ought to be, that the equilibrium should obtain in all its points. 1st, We substitute in the equation for the first $\sqrt{z^2 - x^2}$ for y, and C − z' for z, that is to say, we make $y = \sqrt{(\mathrm{C} - z')^2 - x^2}$.

2dly, we substitute $\frac{\text{A} - w't}{w}$ for x, and thence have the equation of the curve BE in functions of t and z. 3dly, we put $\sqrt{t^2 + u^2}$ for z', and thence determine the constant quantity A from the position of the arbitrary point E where the curve BE must intersect the vertical line CM.

If, for example, the line AF is a right line and the pulley C is placed at its point of intersection F with the vertical, that line has for its equation $y = \frac{s}{c}\,x$, s and c being the sine and cosine of the angle ACD (fig. 7. pl. VI.): and substituting $(\text{C} - z')^2 - x^2$ for y^2, we have $(\text{C} - z')^2 = \frac{s^2}{c^2}\,x^2 + x^2$, or $z' = \text{C} - \frac{x}{c}$, Substituting $\frac{\text{A} - w't}{w}$ for x, and thence will arise

$$z' = \text{C} - \frac{\text{A} + w't}{wc}.$$

This equation is manifestly that of a right line: if we wish it to pass through C likewise, its equation must give at the same time $z' = 0$, and $t = 0$; whence $\text{C} - \frac{\text{A}}{wc} = 0$. Put therefore $u^2 + t^2$ for z'^2, and there will arise

$$u^2 + t^2 = \frac{w'^2 t^2}{w^2 c^2}, \text{ and } u = t\sqrt{\frac{w'^2}{w^2 c^2} - 1}.$$

The last equation leads to the conclusion already announced (154. cor. 5.) For, the angle DCB being supposed to have s' and c' for its sine and cosine, the equation of the right line BC is $u = \frac{s'}{c'}\,t$; therefore, that the equilibrium may obtain we must have $\frac{s'^2}{c'^2} = \frac{w'^2 - w^2 c^2}{w^2 c^2}$, from which results $wc = w'c'$.

156. Prop. *When a heavy body is supported by two planes to determine the relation between the weight of the body and the pressure upon each plane.*

Here we apply the observations laid down in art. 153, VI. in this manner: Let G be the centre of gravity of the body supported by the planes (fig. 11. pl. VI.) through which draw the vertical GF; then if from any point F in that vertical, perpendiculars FC, FE, be demitted upon the two planes, those perpendiculars must pass through points of contact of the body and planes: otherwise a new force will be required to support the body, contrary to the hypothesis. Let the space F*b* on the vertical line represent the weight of the body, on which as a diagonal complete the parallelogram *cfbe*, two of whose sides fall upon FC, and FE; then shall F*c* or *eb*, represent the pressure P upon the plane AB, and F*e* or *cb*, the pressure P′ upon the plane BD. Hence if KL be drawn parallel to HI, the sides

of the triangle BKL will be respectively perpendicular to the sides of the triangle CF*b*, and we shall have
W : P : P′ :: F*b* : FC : *bc* :: KL : BL : BK :: sin ABD : sin ABH : sin DBI. That is, *if the weight of the body be represented by the sine of the angle comprehended between the two planes, the pressures upon them are reciprocally proportional to the sines of the inclinations of those planes with the horizon.*

COR. 1. If the angles of inclination of the planes be each 60 degrees, the sum of the pressures of the body upon both planes will be equal to twice its weight.

COR. 2. If the angle ABD be a right angle, the sum of the pressures upon the two planes shall be as two sides of the right-angled triangle KBL to its hypothenuse.

COR. 3. The more inclined the two planes are, the greater will be the pressure upon them; and *vice versa*.

COR. 4. If one of the planes is horizontal the body cannot remain in equilibrio, independent of friction, except in the case where the vertical drawn through its centre of gravity passes through one of the points with which the body touches the horizontal plane.

COR. 5. In the obtainable case of the last corollary, that is, when BA coincides with BH, and FC becomes vertical, the pressure against BD will vanish, and the horizontal plane will sustain the whole weight. For then, the weight W, and the pressure against BA or BH will be expressed by the equal sines of the supplemental angles DBH, DBI; while the pressure on BD will be expressed by the sine of ABH, an angle which is here nothing, by hypoth.

COR. 6. When a body is sustained by three planes, we must in a similar manner demit perpendiculars from some point in its line of direction (106.) upon those planes, and upon them construct a parallelopipedon whose vertical diagonal shall represent the weight of the body, and its three contiguous edges the pressures upon the planes to which they are perpendicular.

157. PROP. *Given the weight, magnitude, and position of a beam or other heavy body, to find the relative position of two props that may support it in equilibrio, one prop and the lower end of the other having fixed situations.*

Let the body, AD (whether simple or compound, as a beam, or a beam with a load), have its centre of gravity at G; it is required to find the relative position of two props which will support it, their feet standing at C and I (fig. 1. pl. VII.), and the position of one of them as CA being fixed. Produce CA until it meet the vertical passing through G in F; join IF, and it will cut AD in D, the place where the top of the prop ID must be applied. For, drawing AB, BD, perpendicular to CA, DI, they would be sections of inclined planes which would support the body in equilibrio, by the last proposition: and since the action of the body upon those planes would be perpendicular to them, that action would be completely resisted by the props CA, ID, their feet being supposed immoveable.

COR. 1. *The weight of the body, the pressure upon* CA, *and the pressure upon* ID, *are respectively, as sine of* (C + I), *cosine of* I, *and cosine of* C. For, sin (C + I) = sin ABD, cos I = sin DBI, and cos C = sin ABC: consequently this agrees with art. 156. as it ought to do.

COR. 2. The equilibrium will be equally preserved whether the body be sustained by the two props CA, ID; the two inclined planes BA, BD; or by two ropes FA, FD, fixed to a pin or hook at F. For in either case the forces and directions are the same.

The same principles are assumed as the foundation of the more complex investigations relative to the equilibrium of vaults, arches, domes, &c. as will be shewn in their proper place. (Chap. VI.)

V. *Of the Screw.*

158. DEF. The *Screw* is a mechanical power chiefly used in pressing or squeezing bodies close, and sometimes in raising weights. It is a very strong machine, though it cannot be accounted a simple one, as no screw can be made use of without a lever or winch to assist in turning it. The screw is chiefly distinguished by its spiral thread, of which a tolerable conception may be obtained by cutting a piece of paper into the form of an inclined plane, and then wrapping it round a cylinder, as in fig. 2. pl. VII.

159. The screw may be considered as composed of the lever and the inclined plane; as will be evident from a more minute account of the manner in which it may be conceived to be generated. If an isosceles triangle BFG turn about the axe AZ (fig. 5. pl. VII.) there will be generated by that revolution two conic frustums united by their greater ends: conceive now, that besides the motion of rotation, this triangle has also a motion of translation in the direction of the axe AZ, so regulated that while the triangle makes a complete revolution, the point B is moved to G, and the whole triangle is found in the position GF′G′, and so on: the solid thus generated is called the *interior screw;* and the height GB is called *the distance of the threads.* The *exterior screw* is so adapted to the other, as if it were its mould; and is nothing else than the solid generated by the

polygon HGFBC, supposing it to partake of the same motions as the triangle BGF. For the sake of distinction we shall apply the name the *spindle* to the interior screw, calling the exterior one only, the *screw*. The spindle, then, is a cylinder invested with a spiral band of uniform thickness, and of which the inclination with respect to the axe of the cylinder is constant: the screw, on the contrary, is a solid having a corresponding spiral hollow. In some cases the spindle is fixed in a solid block, as AB, fig. 3. while the screw E is moved upon it by means of a lever DC. At other times the screw is fixed and the spindle moveable: but this causes no difference in the theory.

The curve which any one of the points of the generating polygon, as N for instance, describes about AZ, is obviously traced on the surface of a right cylinder whose axis is AZ, and radius of its base EN (fig. 5.) If we develope this, then *dc* (fig. 6.) being the circumference which has EN for its radius, and taking the perpendicular *bc* equal the distance between two contiguous threads, the hypothenusal line *db* will be the developement of an entire revolution of the point N. In effect, the helix being throughout of constant inclination with respect to any position whatever of the generating line of the cylinder, every parallel to AD will make with the developement of that curve the same angle: thus, the developement will be a right line, as *db*; and in like manner the right line *af* will be the developement of a second revolution. This being granted, we may demonstrate in a very satisfactory manner the truth of the following proposition.

160. PROP. *There will be an equilibrium in the screw when the power is to the resistance, as the distance between two contiguous threads in a direction parallel to the axis, to the circumference described by the power.*

Let us suppose the spindle AB to be fixed (fig. 3.) and that the screw is moveable by the aid of a power P applied to the extremity C of a lever CE = R, acting horizontally perpendicular to the lever. Let W be the weight of the screw, or that which the screw supports, or the resistance opposed by the screw to the power P. If the screw pressed only on one of the points of the spindle, suppose it to be at the distance r from the axis, and that its position on the developement *db* of the spiral be at *n* (fig. 6.); then will the pressure on the spindle be exactly the same as on the inclined plane *db*. From the theory (154. cor. 8.) the power M which we suppose applied horizontally in direction M*n*, must, to retain the equilibrium, satisfy this proportion:

$$\mathrm{M} : \mathrm{W} :: cb : cd; \text{ whence } \mathrm{M} = \mathrm{W} \cdot \frac{cb}{cd} = \mathrm{W} \cdot \frac{h}{2\pi r}$$

where $h = bc$, and $\pi = 3{\cdot}141593$, as heretofore. The force M which is supposed applied in n, when the helix is not developed is perpendicular to the edge of the cylinder, or acts in a tangential direction to the cylinder, and of consequence always parallel to the power P.

Now, substituting for this subsidiary power M, the power P, acting at the distance R, we have from the principles of the lever $PR = Mr$, the lengths of the arms being R and r. For M in this equation, substitute its value in the former one, and there arises $2\pi RP = Wh$. This equation, not containing r, is entirely independent of the distance at which the point n is supposed from the axis; it will therefore be the same if we suppose that point any where else on the spindle. Hence we deduce a general result: for this equation will even be true, if the screw, instead of touching the spindle in a single point, as we have hitherto supposed, touch it in any number of points whatever. In this latter case every point on the thread of the spindle bears a portion of the weight W; these portions being denoted by W', W'', W''', &c. give $W' + W'' + W'''$, &c. $= W$. But, on the other hand, the force P which supports the weight of the screw, may be considered as the sum of as many forces P', P'', P''', &c. as there are points of contact, each of which is employed in supporting the weights W', W'', W''', &c. To each of these the last equation applies, we have, therefore, $W'h = 2\pi RP'$, $W''h = 2\pi RP''$, $W'''h = 2\pi RP'''$, &c. their sum manifestly producing the equation $Wh = 2\pi RP$; whence $P : W :: h : 2\pi R$. Q. E. D.

Cor. 1. If the screw had a square or rectangular fillet instead of a triangular one, the conclusion would be the same, for it is independent of the form of the generating polygon.

Cor. 2. In the same screw the effect is always the greater, as the power is applied farther from the axis.

Cor. 3. In two different screws, a force acting with the same distance of lever, produces a greater effect in proportion as the threads of the screw are nearer together.

Cor. 4. In the endless or *perpetual screw* BC (fig. 7.) which drives the teeth of the wheel FD, we shall, in the case of an equilibrium, have $P \times AB \times$ Rad. of $FD = W \times$ dist. of threads $\times$ rad. of axle. For the perpetual screw is a combination of the axis in peritrochio and the screw.

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161. The screw is of very extensive use in Mechanics, its great power rendering it more eligible for compressing bodies together than most other machines, and the great disparity betwixt the velocity of the handle and that of the threads of the screw rendering it proper for dividing space into an almost infinite number of parts. Hence, in the construction of many mathematical instruments, such as telescopes, where it is necessary to

adjust the focus to the eyes of different people, the screw is always made use of in order to move the eye-glass a very little farther from or nearer to the object-glass. In the Philosophical Transactions, vol. 71. a new method of applying the screw, so as to make it act with the greatest accuracy, is described by Mr. Hunter. This method depends upon these general principles, applicable to most machines.

1. That the strength of the several parts of the engine be adjusted in such a manner to the force they are intended to exert, that they shall not break under the weight they ought to counteract, nor yet encumber the motion by a greater quantity of matter than is necessary to give them a suitable degree of strength.

2. That the increase of power by means of the machine be so regulated, that while the force is thereby rendered adequate to the effect, it may not be retarded in procuring it more than is absolutely necessary.

3. That the machine be as simple as is consistent with other conditions.

4. That it be as portable, and as little troublesome as possible in its application.

5. That the moving power be applied in such a manner as to act to the greatest advantage; and that the motion ultimately produced may have that direction and velocity which is most adapted to the execution of the ultimate design of the machine.

6. Of two machines, equal in other respects, that deserves the preference in which the friction least diminishes the effect proposed by the whole.

To attain all these advantages in any one machine is perhaps impossible; but Mr. Hunter's method of applying the screw certainly combines a great portion of them. Let AB (fig. 8. pl. VII.) be a plate of metal in which the screw CD plays, having a certain number of threads in an inch, suppose 10. Within the screw CD there is an exterior screw which receives the smaller screw DE of 11 threads in an inch. This screw is kept from moving about with the former by means of the apparatus at AFGB. If the handle CKL be turned 10 times round, the screw CD will advance an inch upwards; and if we suppose the screw DE to move round along with CD, the point E will advance an inch. If we now turn the screw DE ten times backward, the point E will move downwards $\frac{10}{11}$ths of an inch, and the result of both motions will be to lift the point E an eleventh of an inch upwards. But, if, while the screw CD is turned 10 times round, DE be kept from moving, the effect will be the same as if it had moved 10 times round with CD, and been turned back again 10 times; that is, it will advance $\frac{1}{11}$ of an inch. At one turn, therefore, it will advance $\frac{1}{10}$ of $\frac{1}{11} = \frac{1}{110}$ of an inch. If now the handle be six inches long, the power to produce an equilibrium must be to the weight as 1 to $110 \times 6 \times 2\pi = 4146{\cdot}912$. Thus, the force of Mr. Hunter's screw is greatly superior to the common one; for a common one with a six inch handle must have 110 threads in an inch to produce the same effect, and this great number of threads would render it too weak to resist any considerable violence.

With regard to the second general maxim, both kinds of screws are equally applicable; only that the more complicated structure, and consequently greater expence of Mr. Hunter's screw, renders it convenient to use the common screw where only a small increase of power is requisite, and the improved one where a great power is wanted. The handle being short makes this machine accord with the fourth maxim.

To answer the fifth both seem equally proper; but for the sixth the preference must be given to such as best answer the specific purpose proposed. Thus, if the screw DE be designed to carry an index which must turn round at the same time that it rises upward, the common screw seems preferable; though Mr. Hunter proposes a method by which his may answer the same purpose: with this view a still smaller screw ought to play within DE, and be connected with CD, so as to move round along with it. It must have, according to the foregoing proportions, 111 threads in an inch: and they must lie in a contrary direction to those of CD; so that when they are both turned together, and CD moves upwards, this other may move downwards. At one turn this will move upwards $\frac{1}{12210}$th part of an inch, and at the same time will move in a circular direction. Similar methods may be applied in many other cases: indeed they have lately been applied very frequently; though few of those who have adopted them have acknowledged by whom they were first proposed; on which account we have given this brief description of Mr. Hunter's contrivance, and of his judicious practical maxims.

VI. *Of the Wedge.*

162. Def. A *Wedge* is a triangular prism, or a solid conceived to be generated by the motion of a plane triangle parallel to itself upon a straight line which passes through one of its angular points. The wedge is called *isosceles, rectangular*, or *scalene*, according as the generating triangle is isosceles, right-angled, or scalene.

It is very frequently used in cleaving wood, as represented fig. 12. pl. VII. and often in raising great weights. The theory of knives, swords, coulters, nails, &c. is generally reduced to that of the wedge. The doctrine of the wedge is very imperfect, and can only be exhibited at all by making gratuitous assumptions: such of those as are most likely to obtain in practice are made the basis of the three succeeding propositions.

163. PROP. *When a resisting body is sustained against the face of a wedge, by a force acting at right angles to its direction, in the case of equilibrium the power is to the resistance, as the sine of the semi-angle of the wedge, to the sine of the angle which the direction of the resistance makes with the face of the wedge; and the sustaining force will be as the cosine of the latter angle.*

Let ABC (fig. 9. pl. VII.) be a rectangular wedge, whose edge is C, face BC, and back AB. Let this wedge slide freely along the plane LN; let a body E be drawn or urged in the direction KE against the face of the wedge, and let it be kept in that direction by a force acting in the direction DE, at right angles to KE. There are now three forces acting on the body E, viz. the resisting force KE,[1] the sustaining force DE, and the re-action of the wedge in the direction AE, perpendicular to the surface BC. On ED demit the perpendicular AG; and since the three forces are in equilibrio, they will be to each other as the sides of the triangle AEG drawn parallel to their directions. Draw EF perpendicular to AC, and the force AE will be resolved into two, one of which EF presses the wedge perpendicularly against the plane LN, and is balanced by the reaction of the plane; the other FA endeavours to move the wedge upwards along the plane LN, and is balanced by the power on the back of the wedge. If, therefore, AG represent the force KE, EG will be the sustaining force, and AF the power applied on the back of the wedge, when these forces balance each other. Hence, making AE radius, AF is the sine of the angle AEF or ACB; and AG is the sine of the angle AEG or KEC, these two angles being the complements of AEK.

If the wedge be isosceles, or composed of two rectangular wedges, the force EF, which in the former case was counteracted by the plane, will now be counteracted by the other half of the wedge: and the power, resistance, and sustaining force, will remain in the same ratio as before. Q. E. D.

COR. 1. When EK is parallel to BA, AG becomes equal and parallel to EF; and EG equal and parallel to AF; and the power is to the resistance as AF to EF, or AB to AC, and equal to the sustaining force.

COR. 2. If EK be perpendicular to BA, the direction of the resisting force will be parallel to AB; therefore the resisting and sustaining forces changing denominations, this will be a case corresponding with the former.

Cor. 3. When KE is perpendicular to BC, the sine of the angle KEC is radius: and its cosine, which represents the sustaining force, vanishes: therefore, the power is to the resistance, as the sine of the semiangle of the wedge, to radius.—See Ludlam's 2d Essay.

164. Prop. *When the resistance is made against the face of a wedge by a body which is not sustained, but will adhere to the place to which it is applied without sliding, the power is to the resistance, in the case of equilibrium, as the cosine of the difference between the semiangle of the wedge and the angle which the direction of the resistance makes with the face of the wedge, to radius.*

From any point K (fig. 10. pl. VII.) draw the line KE through the middle point of the back, meeting the face of the wedge in E; let E be the unsliding body, which acts in the direction EK, and let the magnitude of the force with which it is urged be represented by AE: from E let fall the perpendicular EF upon AC; and the force AE will be resolved into two, one of which EF will be balanced by the opposite half of the wedge, and the other AF will be counteracted by the power; therefore the power is to the resistance as AF to AE, that is, making AE radius, as the cosine of the angle EAF to radius.

Cor. 1. When KE is perpendicular to BC, the power is to the resistance as AF to AE, that is, as the sine of the semi-angle of the wedge to radius.

Cor. 2. When KE is parallel to AB, AF vanishes, that is, the power is indefinitely less than the weight.

Cor. 3. When KE is perpendicular to AB, EF vanishes, and AF and AE, which represent the power and resistance, become equal.

165. Prop. *When the resisting body is neither sustained nor adheres to the point to which it is applied, but slides freely along the face of the wedge, the power is to the resistance, as the product of the sines of the semiangle of the wedge and the angle in which the resistance is inclined to its face, to the square of radius.*

Let AE (fig. 11. pl. VII.) be perpendicular to BC, and let the body E be urged against the face of the wedge in the direction KE; and let KE represent the magnitude as well as the direction of that force. On AE produced let fall the perpendicular KO, which will be parallel to BC; thus will the force KE be resolved into two, one of which KO will carry the body down along the face of the wedge, and the other OE will propel it perpendicularly against it. Now in the case of equilibrium, the power is to OE, that part of the resistance which acts perpendicularly against the face of the wedge, as the sine of the angle ACB to radius; and OE is to the whole resistance, as OE to KE: that is, making KE radius, as the sine of the angle OKE, or its alternate KEB, to radius. Therefore, *ex æquo & componendo*, the power is to the resistance, as sin ACB × sin KEB, to the square of radius.

COR. 1. When KE is perpendicular to BC, the sine of the angle in which the resistance is applied, is radius; therefore the power is to the resistance, as the sine of the semiangle of the wedge to radius.

COR. 2. When KE is parallel to AB, the angle of inclination is the complement of the semiangle of the wedge; and therefore, the power is to the resistance, as the product of the sine and cosine of the semiangle of the wedge to the square of radius.

COR. 3. When KE is perpendicular to AB, the angle of inclination is equal to the semiangle of the wedge, and the power is to the resistance in a duplicate ratio of the sine of the semiangle of the wedge to radius. See Dr. M. Young's Lectures.

SCHOLIUM.

166. The theory of the equilibrium of the wedge has greatly engaged the attention of many philosophers, as Mr. *Ludlam*, &c. but it is not of very great use in practical mechanics, because the wedge is scarcely ever otherwise urged than by percussion. In cleaving of wood, the resistance opposing the force of the mallet (supposing the sides of the wedge perfectly polished, and its edge a line without breadth), is the cohesion of the particles of the wood which are about to be separated; and this being a kind of pressive force acting against the sides of the wedge, it is by many philosophers thought absurd to attempt to compare it with the percussive force of the mallet. For the greatest finite pressive force must, in their opinion, give way to the least percussive one, and there cannot be an equilibrium between two such different forces. " Any percussive force (say they) acting on a moveable body, generates a finite quantity of motion in an indefinitely small portion of time; but the time will be finite in which any given pressive force whatever, acting on the same body, can generate or destroy the same quantity of motion. Therefore, a body being urged in a certain direction by any pressive force whatever, and in the contrary direction by any percussive one, the pressive force will be some finite time in destroying the quantity of motion which the percussive one generated in an instant. Consequently, how great soever the pressive force may be, and how small soever the percussive one, the body will be moved (at least for some short time) by this

last force." Indeed, after the stroke is given, the pressive force may quickly prevail and force back the body, which the impulse of the other force had driven forward. And so it would frequently be in the operation of cleaving wood, if the sides of the wedge were perfectly smooth. For, after the stroke of the mallet, the wedge, unless its weight were equivalent to the attraction of the parts of the wood about to be separated, would presently be forced back from the place to which it had been driven by the mallet. And it is chiefly the roughness of the sides of the wedge, and of the parts of the wood in contact with it, which, in that operation, keeps the wedge from receding. It is that roughness too, and the bluntness of the edge, which sometimes prevent the wedge from being moved by the stroke of the mallet. For were it not obstructed by such roughness and bluntness, it would, according to what we just now observed, be always driven forward, even by the least percussive force.

Several of these remarks, it will be seen, rest upon the commonly received doctrine of percussion: a doctrine, however, which, in our humble opinion, has no very durable basis. It is manifest that a blow may perform many things (particularly in effecting fractures or breaches) which a considerable pressure cannot accomplish: but this is probably owing to the circumstance that the rapidity with which the excited pressure increases to its maximum does not leave sufficient time for the forces which connect the particles of the body struck to be excited throughout to its more remote parts.—We hope to place this matter in a rather better light when the subjects of Collision and Percussion come under consideration. See Book II. Chap. 5.

CHAPTER V.

On the Strength and Stress of Materials.

167. THE *Resistance* of solids, or that force with which the quiescent parts of solid bodies oppose the motion of others contiguous to them, is generally considered as of two kinds, in one of which the resisting and resisted parts, though contiguous, constitute separate masses; this will be considered in another place, under the title of *Friction:* in the other kind the resisting and resisted parts are not only contiguous but cohere, being parts of the same body or mass; and it is this which we now propose to consider.

This kind of Resistance has exercised the sagacity of some of the most eminent philosophers from the time of Galileo down to the present period; and different theories have been proposed by Mariotte, Leibnitz, Varignon, Buffon, Euler, Lagrange, and Girard (perhaps others which have not come to our knowledge), but none of them are so free from objection and from error as might be wished. Indeed, the figure and constitution of bodies are so variable and irregular, that we cannot with the desirable precision determine those elements which should precede and regulate this discussion. Of the theories above adverted to some are certainly very ingenious; but at the same time they are very complex and intricate, and cannot by any means be relied upon independent of experiment: we therefore prefer the comparatively simple theory of Galileo, originally laid down in his dialogue "*On the Cause of the Coherence of Solids,*" with which the other hypotheses agree in the most essential particulars, and which, when aided by proper experiments, may serve as a safe approximation to the strength and stress of the different parts of machines.

"That the resistance of solids might be subjected to calculation, Galileo supposed first that bodies were composed of solid fibres, parallel to one another; he then enquired what was the force with which they resist the action of a power stretching them in a direction parallel to their length, and found that it was proportional to the number of integral fibres: next, considering the fibres as subjected to an effort perpendicular to their length, he found that the resistance of the integral fibres was proportional to their sum multiplied by an arm of a lever, which is always at a certain part of the vertical dimensions of a solid in the plane of its rupture." The length of this arm of lever was regulated, according to Galileo, by the position of the centre of gravity of the plane of rupture, according to others, by the centre of percussion, &c. But "the distinctive character of Galileo's hypothesis consists in this, that the resistance of each of the fibres is independent of their quantity of extension at the instant of their rupture." Galileo's reasoning on this interesting topic is comprised in 17 propositions: his *mode* of discussion will not be entirely adopted; but, that the reader may know how far this philosopher advanced the subject, we shall distinguish by a [G] such of the following propositions and corollaries as are found in his Treatise.

168. Def. *Strength*, and *Stress*, or *Strain*, are terms, the former of which is used to denote the force or power with which any mass or body resists a breach or change in its state, which a pressure or stroke upon it has a tendency to produce; and the latter are used indiscriminately to express the force which is excited in any such mass and tending to break it. Thus, every part of a pillar is equally *strained* by the load which it supports. Hence, it is evident that we cannot make any structure fit for its purpose, unless the *strength* in every part be at least equal to the *stress* laid on, or the *strain* excited in that part: and hence the necessity of an acquaintance with the nature of the resistance of bodies, so that there shall be neither a surplus nor a deficiency of materials in any machine.

169. Prop. *The strength of a beam or bar to resist a fracture by a force acting laterally, is as the solid, made by a section of the beam in the place where the force is applied, into the distance of its centre of gravity from the point or line where the breach will end.* [G.]

Suppose AB (fig. 13. pl. VII.) to be the beam (of uniform matter throughout) fixed firmly at its two ends A, B, at the middle of which is laid the weight W. In the case of a rupture we conceive the beam will be separated first in the line *cd* opposite to W and farthest from it, and the separation be gradually continued till it arrives at *ab*, which may, therefore, be considered as a fixed line till the termination of the fracture. Now the area *abcd* represents the sum of all the fibres to be broken or torn; and as they are equal to each other both in magnitude and strength (by hypothesis), this area will likewise express the aggregate of the strength of the fibres in the longitudinal direction. But with respect to lateral strength, we must consider each fibre as acting at the extremity of a lever whose centre of motion is on the line *ab*: thus, each fibre in the line *cd*, will resist the breach by a force proportional to the product of its individual strength into its distance *db* from the centre of motion; and consequently the resistance of all the fibres in *cd* will be represented by $cd \times bd$. In like manner, the aggregate resistance of another course of fibres parallel to *ab*, as *oo*, will be represented by $oo \times bo$: and of a third, as *ii*, by the rectangle $ii \times bi$, and so throughout. Therefore the sum of all these products will express the total strength or resistance of the beam in that part. But (108.) the sum of all these products is equal to the product of the area *abcd*, into the distance of its centre of gravity from *ab*: whence the proposition is manifest.

COR. 1. *In square beams the lateral strengths are as the cubes of the breadths or depths.* [G.]

COR. 2. *In cylindric beams the lateral strengths are as the cubes of the diameters.* [G.]

COR. 3. *The lateral strengths of any beams whose sections are similar figures, are as the cubes of corresponding dimensions of the sections.*

COR. 4. *In rectangular beams the lateral strengths are conjointly as the breadths and squares of the depths.* [G.]

For the areas are $\propto$ breadth $\times$ depth, and the distances of the centre of gravity are $\propto$ depth: consequently, strength $\propto$ breadth $\times$ depth^2.

COR. 5. *The lateral strength of a beam with its narrower face upwards, is to its strength with the broader face upwards, as the breadth of the broader face, to the breadth of the narrower.* [G.] For $bd^2 : db^2 :: d : b$.

COR. 6. If a beam were fixed firmly at one end into a wall, and the fracture were caused by a weight suspended at the other end, the process of nature would be similar, only that the breach would terminate at the lower part of the beam; and the proposition and five first corollaries would still obtain.

170. PROP. *The lateral strengths of prismatic beams of the same materials are as the areas of the sections and the distances of their centres of gravity, directly, and as their lengths and weights, inversely.*

Let BC, GH (fig. 14. pl. VII.), be two beams of like materials fixed in horizontal positions to the upright wall AB, by their ends B, G. Let A be the area of the end G of the beam GH, G the distance of the centre of gravity of that end from its lowest point, L its length, W its weight, and S its strength: and let *a*, *g*, *l*, *w*, and *s*, be corresponding particulars in the beam CB. Then $S : s :: \frac{AG}{LW} : \frac{ag}{lw} : AGlw : agLW$. For, the direct strength, or effort tending to preserve the adhesion of the fibres, varies as the product of AG, *ag*, by the last prop. while the efforts tending to destroy their adhesion, and which are therefore in the inverse ratio of the strengths, vary both in proportion to the weights of the beams, and the distances at which those weights act: but the weights of the beams may evidently be considered as acting at their centres of gravity, the distances of which from the end supported vary as the length of the beams; and consequently the efforts tending to destroy the adhesion of the beams are as $\mathrm{L}w$, lw. Whence, by incorporating the direct and inverse ratios, we obtain that stated in the proposition.

Cor. 1. Had the beams been considered as fixed at both ends, the same thing would follow, *with this difference only, that a beam when fixed at both ends is as strong as one of equal breadth and depth, and but half the length, which is fixed only at one end.* For, if the longer beam were bisected, each of its halves would be situated with respect to its fixed end, in the same manner as the shorter beam with respect to its fixed end.

Cor. 2. When the strength of a beam is very considerable in relation to its weight, we may, instead of the proposition, take $\mathrm{S} : s :: \frac{\mathrm{AG}}{\mathrm{L}} : \frac{ag}{l}$.

Cor. 3. *Cylinders and square prisms have their lateral strengths proportional to the cubes of the diameters, or depths, directly, and their lengths and weights, inversely.* [G.]

Cor 4. *Similar prisms and cylinders have their strengths inversely proportional to their like linear dimensions.* [G.] For the cubes of the diameters or depths vary as the cubes of the lengths, and the weights and lengths are as the cubes of the lengths and the lengths conjointly, or in the quadruplicate ratio of the lengths: therefore, the strengths are as L^3 to l^3 directly, and L^4 to l^4 inversely, or inversely as the lengths, L, and l.

SCHOLIUM.

171. From the preceding deductions it follows that greater beams and bars must be in greater danger of breaking than the less similar ones; and that, though a less beam may be firm and secure, yet a greater similar one may be made so long, as necessarily to break by its own weight. Hence Galileo justly concludes that what appears very firm, and succeeds well in models, may be very weak and unstable, or even fall to pieces by its weight, when it comes to be executed in large dimensions, according to the model. From the same principles he argues that there are necessarily limits in the works of nature and art, which they cannot surpass in magnitude: that immensely great ships, palaces, temples, &c. cannot be erected, their yards, beams, bolts, &c. falling asunder by reason of their weight. Were trees of a very enormous magnitude, their branches would, in like manner, fall off. Large animals have not strength in

proportion to their size; and if there were any land animals much larger than those we know, they could hardly move, and would be perpetually subjected to most dangerous accidents. As to marine animals, indeed, the case is different, as the gravity of the water sustains those animals in great measure; and in fact these are known to be sometimes vastly larger than the greatest land animals. It is, says Galileo, impossible for Nature to give bones for men, horses, or other animals, so formed as to subsist, and proportionally to perform their offices, when such animals should be enlarged to immense heights, unless she uses matter much firmer and more resisting than she commonly does; or should make bones of a thickness out of all proportion; whence the figure and appearance of the animal must be monstrous. This he supposes the Italian poet hinted at, when he said,

Non si può compartir quanto sià lungo,
Si smisuratamente è tutto grosso.

Whatever height we to the giant give,
He cannot without equal thickness live.

And this sentiment being suggested to us by perpetual experience, we naturally join the idea of greater strength and force with the grosser proportions, and that of agility with the more delicate ones. The same admirable philosopher likewise remarks, in connection with this subject, that a greater column is in much more danger of being broke by a fall than a similar small one: that a man is in greater danger from accidents than a child: that an insect can sustain a weight many times greater than itself; whereas a much larger animal, as a horse, could scarcely carry another horse of his own size. The ingenious student may easily extend these practical remarks, to any cases which may come before him.

172. PROP. *The lateral strengths of two cylinders (of the same matter) of equal weight and length, one of which is hollow and the other solid, are to each other as the diameters of their ends.* [G.]

Let ABE, HIK (figs. 1 and 2. pl. VIII.), be the ends of two cylinders of equal length, and containing equal quantities of matter; the former being the section of a tube constituted of cylinders having a common axis; then is the strength of the tube to that of the solid cylinder as AB to HI. For the lateral strengths (169.) are conjointly as the areas and the distances of the centres of gravity of the sections from A or from B, according as the fractures terminate at the one or the other point:

but the areas of the annulus in the first fig. and of the circle in the second, are here equal, and the centres of gravity of both are at their centres of magnitude; wherefore, since the radii vary as the diameters, the strengths in this case vary in the same ratio.

Cor. 1. Since, when the area of a circular section is given, its diameter is greater when the section is an annulus, than when it is a circle without any cavity; and since the power with which the parts of the cylinder resist extraneous force is greater in the same proportion, it follows according to the theory thus stated, that the strength may be increased indefinitely without augmenting the quantity of matter.

This conclusion is, however, manifestly erroneous: because after the diameter of the tube exceeds a certain magnitude (which can only be ascertained experimentally) it will become flaccid, and bend under the smallest additional weight. The fact is, the reasoning in the proposition is founded upon the presumption that the figure of the section will be *constantly circular;* and will, therefore only hold true under those limits in which the pressure or stroke upon the tube will not cause its section to degenerate from the circle to an ellipsis or any other form.

Cor. 2. When the two diameters of the end of a tube are given, the diameter of a solid cylinder of equal weight may be easily found (the lengths being supposed the same), by taking the square root of the difference of the squares of the diameters of the tube; or the square root of the product of their sum and difference.

Or, the same may be effected geometrically by this simple process: from one end of the exterior diameter AB set off AE equal to the interior diameter CD, and join EB, which will be the diameter of the section sought. For EB^2 must be equal to $AB^2 - CD^2$, which it is by this construction and Euc. I. 47. and III. 31.

Or, if BE be drawn a tangent to the inner circle till it cuts the exterior one in two points E and B, it will be the diameter sought. For the triangles BEA, BFG, are then similar, the angles at E and F being right angles; consequently $EA = 2FG = CD$; as in the preceding construction.

Cor. 3. The lateral strengths of tubes and solid cylinders of equal length and similar materials, are as the areas of their ends and their diameters conjointly.

SCHOLIUM.

173. From this proposition Galileo justly concludes, that Nature in a thousand operations greatly augments the strength of substances without increasing their weight: as is manifested

in the bones of animals and the feathers of birds, as well as in most tubes, or hollow trunks, which though light, greatly resist any effort to bend or break them. "Thus (says he) if a wheat "straw which supports an ear that is heavier than the whole "stalk were made of the same quantity of matter, but solid, it "would bend or break with far greater ease than it now does. "And with the same reason Art has observed and Experience "confirmed that a hollow cane, or tube of wood or metal, is "much stronger and more firm than if, while it continued of "the same weight and length, it were solid, as it would then, of "consequence, be not so thick: and therefore Art has contrived "a method to make lances hollow within, when they are re"quired to be both light and strong," in this instance, as in many others, imitating the wisdom of Nature.

In all such instances, however, there is an obvious distinction between the works of Nature and those of Art; "in the former (as M. *Girard* remarks, when treating the same subject), the cause and the effect essentially agrees; the one cannot undergo any modification without the other experiencing a correspondent change: or, to speak more precisely, a new effect always results from a new cause:—in the productions of human industry, on the contrary, there is no necessary proportion between the effect and cause: if, for example, a determinate weight is to be raised, it is indifferent whether we use the thread which has precisely the adequate force, or the cable which has a superabundant one; while, if the same weight had rested naturally suspended, it would have done so by means of fibres peculiarly appropriated in their organization to the object, and whose disposition would have presented the most advantageous form. Perfection resides in a single point, at which Nature arrives without effort; while man is obliged, by repeated trials, to pass over an immense space which separates him from it."

174. PROP. *Of all hollow cylinders whose lengths, and the diameters of the exterior and interior circles, continue the same, those have the greatest lateral strength, in which the interior touches the exterior circle, in the highest part, provided the cylinders are fixed at both ends and in a horizontal position; or when they touch in the lowest part, if the cylinders are fixed only at one end: the cylinders in both cases being conceived to exert their strength against weights acting vertically.*

Here, since the diameters of the exterior and interior circles are supposed invariable, the area of the space they include will be likewise invariable, so that the strengths of the cylinders will be proportional to the distances of the centres of gravity of their sections from the point where the fracture would end on the supposition the cylinders were broken. Now, when a beam is fixed at both ends, and broken by a weight laid between those ends, the breach will manifestly commence at the lower part and terminate at the upper; and, when it is fixed at one end only, and the weight acts at the other, the breach will commence on the upper side, and terminate at the lower: consequently, in the first instance, the centre of gravity of the section must be at the greatest distance from its highest point, and in the second instance at the greatest distance from the lowest point, to ensure a maximum of strength. Let figs. 3 and 4. represent the section of the tubes, BC and AD being the diameters of the hollow part of both: let the area of the circle whose diameter is AB be M, the distance of its centre of gravity from A being called G, the area of the smaller circle m, the distance of its centre of gravity from A, g, the area of the space between the two circles $(=\mathrm{M}-m)$ μ, the distance of its centre of gravity from A, γ; then, by the nature of the centre of gravity, will $\mathrm{MG}=mg+\mu\gamma$. In this equation M, G, m, and μ, are invariable; therefore γ must vary inversely as g: and consequently, when γ is a maximum g is a minimum, and *vicè versa*. But g is a minimum when the inner circle touches the outer one in A, as in fig. 4.; and it is a maximum when the interior and exterior circles touch in B, as in fig. 3. Therefore the maximum and the minimum values of γ obtain when the circles touch in A and B respectively; and the comparative strengths of the tubes are as expressed in the proposition.

175. In cases actually arising in practice, this proposition, as well as that laid down in art. 172. will require some modification, and for the same reason as is there stated. The strengths of the tubes, however, will increase as the circles approach nearer to each other, until they reach a certain limit, which can only be determined by experiment for each different kind of resisting body, and for various proportions of the exterior and interior circles.

176. **Prop.** *The strongest rectangular beam which can be cut out of a given cylinder, is that of which the squares of the breadth, and depth, and the square of the cylinder's diameter, are respectively as the numbers* 1, 2 *and* 3.

In fig. 6. pl. VIII. let BC the breadth of the beam be denoted by b, AC the depth by d, the diameter AB being denoted by D; then when BC is horizontal, the lateral strength will be represented by bd^2 (art. 169. cor. 4.), which is to be a maximum. But $\mathrm{AC}^2=\mathrm{AB}^2-\mathrm{BC}^2$, or $d^2=\mathrm{D}^2-b^2$; therefore $(\mathrm{D}^2-b^2)\,b=b\mathrm{D}^2-b^3=$ a max. In fluxions $\mathrm{D}^2\,\dot{b}=3b^2\,\dot{b}$: whence $\mathrm{D}^2=3b^2$;

and $d^2 = D^2 - b^2 = 3b^2 - b^2 = 2b^2$. Consequently, $b^2 : d^2 : D^2 :: 1 : 2 : 3$. Q. E. D.

COR. 1. Hence arises this easy practical construction: divide the diameter AB into three equal parts in E, F; erect the perpendiculars ED, FC; and join the points C, D, to the extremities of the diameters: so shall ACBD, be the rectangular end of the beam required. For, because AE, AD, AB, are in continued porportion, we have AE : : AB : : $AD^2 : AB^2$: and, in like manner AF : AB :: $AC^2 : AB^2$. Hence AE : AF : AB :: $AD^2 : AC^2 : AB^2$ :: 1 : 2 : 3.

COR. 2. The ratio of b to d is nearly that of 5 to 7, or more nearly that of 12 to 17.. For $5^2 : 7^2 :: 25 : 49 :: 1 : 2$ nearly; and $12^2 : 17^2 :: 144 : 289 :: 1 : 2$ more nearly.

COR. 3. A square beam from the same cylinder would have its side $= D\sqrt{\frac{1}{2}} = \frac{1}{2}D\sqrt{2}$. Its solidity would be to that of the strongest beam, as $\frac{1}{2}D^2$ to $\frac{1}{3}D^2\sqrt{2}$, or as $\frac{1}{2}$ to $\frac{1}{3}\sqrt{2}$, or as 5 to 4·714; while its strength would be to that of the strongest beam, as $(D\sqrt{\frac{1}{2}})^3$ to $D\sqrt{\frac{1}{3}} \times \frac{2}{3}D^2$, or as $\frac{1}{4}\sqrt{2}$ to $\frac{2}{9}\sqrt{3}$, or as 3560 to 3849.

COR. 4. Either of these beams will exert the greatest lateral strength, when the diagonal of its end is placed vertically. For the area being the same in both positions, the strength will vary as the distance of the centre of gravity from the base of fracture: but when one of the sides is vertical, as in fig. 7. the distance of the centre of gravity of the end will be $=$ AI, or ID, equal to half the side; whereas, when the diagonal is vertical as in fig. 5. that distance will be CE or ED, half the diagonal.

COR. 5. The strength of the whole cylinder will be to that of the square beam when placed with its diagonal vertically, as the area of the circle to that of its inscribed square. For CE or ED (fig. 5.) is the distance of the centre of gravity in both cases; therefore the strengths vary as the areas.

177. PROP. *When a triangular beam is supported at both ends, its strength when the edge of the beam is uppermost is, to the strength when the opposite side is uppermost, as 2 to 1.*

For in this case the fracture will terminate at the top of the beam; and as the area of the end continues the same whether side or edge be uppermost, the strength will vary as the distance of the centre of gravity from the uppermost point. Now, in the triangle ABC, the distance CG of the centre of gravity from the vertex, is double the distance GD from the base (fig. 8. pl. VIII.): therefore when the triangle is inverted as *abc*, the distance of the centre of gravity from *d* the highest point, is only half its distance from the highest point C of the triangle before it is inverted. Whence the proposition is manifest.

Cor. If the beam be supported at one end only, it would if broken have the breach commence at the upper part and terminate at the lower: and in that case the beam will be strongest when the edge is downward, and only half that strength when its opposite face is downward and horizontal.

178. Prop. *If a weight be placed upon any part of a horizontal beam fixed at both ends, the stress of the beam at that part will be as the rectangle of its distances from the supported ends.* [G.]

Let the weight W press upon the beam at C (fig. 5. pl. IV.), then is the weight equal to the pressures upon A and B, and pressure upon A $= \frac{W\cdot BC}{AB}$, while pressure upon B $= \frac{W\cdot AC}{AB}$. (See Cor. 6. art. 133.) But the reaction of either point of support is equal to the pressure upon it (ax. 3.); and this may be considered as a force acting at the point C as upon the arm of a lever: so that the stress at C is as the pressure at either point of support into its distance from C. That is, the pressure is as $\frac{W\cdot BC}{AB}\cdot AC$, or as $\frac{W\cdot AC}{AB}\cdot BC$, which are manifestly equal, the one to the other. But W and AB are given; therefore the stress varies as the rectangle $AC\cdot CB$.

Cor. 1. The same thing will obtain if the weight be equally diffused through the whole of the beam. For, in this case as in the former the sum of the pressures upon A and B will be equal to the whole weight: and if w be the weight of the part BC, its pressure upon A, will be $\frac{w\times\frac{1}{2}BC}{AB}$; and this referred back to the point C will give $\frac{w\times\frac{1}{2}BC}{AB}\cdot AC$, for the stress; which varies as the rectangle $AC\cdot BC$, as before.

Cor. 2. The greatest stress of a beam is in the middle between its supports, whether the weight be applied there or equally diffused over its whole length. [G.]

For the rectangle of the two parts of a line is a maximum, when those parts are equal to each other.

Cor. 3. Hence in all structures we should as far as possible avoid placing weights in the middle of beams: thus, in roofing, it is better to use prick-posts than king-posts, unless where there is a pillar or a partition wall to support the beam in the middle.

Cor. 4. If w be the greatest weight a beam will sustain at its middle point, and it be required to support by that beam a greater weight W; the point C may be found by making $W : w :: \frac{1}{2}AB\times\frac{1}{2}AB : AC\cdot CB$.

To perform this geometrically Galileo proceeds thus: let W and *w* be represented by lines, their mean proportional being a line M : find a fourth proportional S, to W, M, and AD (fig. 9. pl. VIII). Let AD be the diameter of the semicircle AHD, in which draw the line AH = S; join H, D, and make DC = DH, then is C the point sought. For, on BA describe the semicircle ANB, raise the perpendicular NC and join N, D: then, because $NC^2 + CD^2 = ND^2 = AD^2 = AH^2 + HD^2$, and $HD^2 = DC^2$, we have $NC^2 = AC \cdot CB = AH^2$, that is $= S^2$. But $S^2 : AD^2 :: w : W$; whence the truth of the construction is obvious.

COR. 5. Since the rectangle $AC \cdot CB$ diminishes as C approaches to A or B, and the stress varies as that rectangle, it follows that much may be taken from the thickness of beams towards each end without rendering them too weak for the load. [G.]

COR. 6. The strains at any two points C, and D, in a beam, are as the rectangles $AC \cdot CB$, $AD \cdot DB$; and therefore when weights are laid on at C, and D, in the proportion of those rectangles the beam will be no more liable to break at one point than at the other. [G.]

COR. 7. The strain at any point D caused by a weight at C, is equal to the strain at C occasioned by the same weight at D.

For, when the weight W is at C, the stress there is as $AC \cdot CB$, and the stress at $D = \frac{BD}{BC} \times$ strain at $C = \frac{BD}{BC} \times AC \cdot CB = BD \cdot AC$. And, when the weight is at D, the strain there is as $AD \cdot DB$; while the strain at C is as $\frac{AC}{AD} \times$ stress at $D = \frac{AC}{AD} \times AD \cdot DB = AC \cdot BD$, the same as before.

179. PROP. *If a beam in form of an isosceles wedge be fixed by its base to a vertical wall, its faces being in a vertical position, and a weight be attached to its vertex, such a beam will be equally strong throughout.*

Let the wedge BA (fig. 10. pl. VIII.) be fixed to the vertical wall BE, and a weight W be placed at A. The effort of the weight W upon any point D of the beam will, by the nature of the lever, be as the rectangle $W \cdot AD$, or as AD, because W is constant. And the strength at any point D, will be as the breadth into the square of the depth at that place (all the vertical sections being rectangles), or as the breadth CD, the depth being constant. Therefore, since when the beam is equally strong throughout, the strength and stress are in an invariable ratio, we shall have CD constantly as AC; and, of consequence, ACD must be a rectilinear triangle, and the beam a wedge.

180. PROP. *If a beam is placed horizontally with one end fixed to a wall, and a weight hung at the other, then if its breadth be the same from one end to the other, it will be equally strong*

throughout when the vertical sides are in form of a parabola. [G.]

For the stress is as AD (fig. 11.) as in the last prop. and the strength is as the breadth into the square of the depth, or, because the breadth is constant, the strength is as CD^2. But the stress and strength must remain in a constant ratio; therefore AD must vary as CD^2 throughout the figure; which is the well known property of a parabola whose vertex is A.

COR. 1. Since the parabola is $\frac{2}{3}$ of its circumscribing parallelogram, parabolic beams require $\frac{1}{3}$ less matter than prismatic ones; a circumstance which may be beneficially attended to, particularly when iron is used.

COR. 2. In the beams of balances for very great weights, each arm may be constructed of a parabolic shape as in fig. 13. P being the centre of motion: which will cause a saving of materials without any diminution of *useful* strength.

181. PROP. *If a beam has one end fixed to a wall, and gradually diminishes towards the other end where a weight is placed, so that all its vertical sections are squares, then, in order that it may be equally strong throughout, the bounding curve must be in the form of a cubic parabola.*

In fig. 12. Pl. VIII. the stress or effort of the weight upon any point F will be as AF: and, because the sections are all similar, the strengths will vary as the cubes of the depths. Hence, in this case $AF \propto DC^3$, which is a well-known property of a cubic parabola.

182. PROP. *When a beam whose vertical sides are parallel planes is fixed at both ends, it will be equally strong throughout, if either the tops of those sides or the bottoms, or both, be terminated by ellipses.*

For, the sides being parallel planes, the beam will be of equal thickness throughout, and consequently the strength at any point DC, will be as CD^2, or as Cc^2, according as ADB, or AcB, is the bottom of the beam (fig. 13. Pl. VIII.) Now, the stress at the point D, is as the rectangle $AD \cdot DB$ (art. 178.); therefore CD^2, or Cc^2 must vary as $AD \cdot DB$, to ensure equal strength throughout. And this is the fundamental property of the ellipse whose vertices are A and B.

183. PROP. *Given the length and weight of a cylinder or prism, which is placed horizontally with one end fixed firmly, and will just support a given weight at the other end without breaking, to find the length of a similar prism or cylinder, which when supported in like manner at one end, shall just bear without breaking a second given weight at the unsupported end.*

Let l denote the length of the given cylinder or prism, d the diameter or depth of its end, w its weight, and u the weight hanging at the unsupported end: the capitals L, D, W, and U, representing corresponding particulars with respect to the other prism. Then the weights of similar solids of the same matter being as the cubes of their lengths, we shall have $l^3 : \mathrm{L}^3 :: w : \frac{\mathrm{L}^3}{l^3}w$, the weight of the prism whose length is L. We may either suppose the whole weight of each beam to act at its centre of gravity, or half its weight at the unsupported end: for, on both these suppositions the stress upon the supported or fixed end arising from the weight of the beam will be the same: hence then, the stress upon the fixed end of the beam whose length is l, arising both from its own weight and the weight u attached to it, will be $(\frac{1}{2}w+u)l$; and the stress upon the other beam will be $\left(\frac{1}{2}w\frac{\mathrm{L}^3}{l^3}+\mathrm{U}\right)\mathrm{L}$, or $\left(\frac{\mathrm{L}^3}{2l^3}w+\mathrm{U}\right)\mathrm{L}$. But the lateral strength of the first beam is to that of the second, as d^3 to D^3 (art. 169. cor. 2.) or as l^3 to L^3. Wherefore, since the strengths and stresses of the two beams in their respective circumstances must be in the same ratio, to answer the conditions of the proposition, we have $(\frac{1}{2}w+u)\,l : \left(\frac{\mathrm{L}^3}{2l^3}w+\mathrm{U}\right)\mathrm{L} :: l^3 : \mathrm{L}^3$. This analogy converted into an equation, gives, after a little reduction, $\mathrm{L}^3-\frac{w+2u}{w}l\mathrm{L}^2+\frac{2}{w}\mathrm{U}l^3=0$: a cubic equation from which the numeral value of L may be determined, when those of the other quantities are specified.

COR. 1. When U vanishes the equation becomes $\mathrm{L}^3=\frac{w+2u}{w}l\mathrm{L}^2$; whence $\mathrm{L}=\frac{w+2u}{w}l$, or, $w : w+2u :: l : \mathrm{L}$. From which the length of the beam which will just break by its own weight, may be readily found.

COR. 2. Hence of all beams of similar shape and materials, there is one, and only one, that will merely sustain itself when fixed at one end, being just on the point of breaking. [G.]

COR. 3. If a beam break by its own weight, when fixed at one end, a beam of twice its length fixed in a similar manner at both ends will also break by its own weight: or if one sustain itself the other will.

For the strain or stress is the same in both of them; each being equal to the stress of a beam of the double length, and supported at its middle point.

184. PROP. *Given the length and weight of a cylinder or prism, which is fixed horizontally as in the foregoing proposition, and a weight which when hung at a given point breaks the*

prism, to find how much longer a prism of equal diameter or of equal breadth and depth, may be extended before it break either by its own weight, or by the addition of any other adventitious weight.

Here let l denote the length of the given prism, w its weight, u a weight attached to it at the distance d from the fixed end; L the length of the required prism, and U the weight attached to it at the distance D. Then $l : L :: w : \frac{Lw}{l}$, the weight of the prism whose length is L: therefore the strain it will occasion upon the fixed end will be $\frac{Lw}{l} \cdot \frac{1}{2} L = \frac{wL^2}{2l}$; and the strain arising from the weight U acting at the distance D will be DU; so that the whole strain occasioned by the longer beam and its weight will be $\frac{wL^2}{2l} + DU$. The strain occasioned, in like manner, by the weight of the original beam will be $\frac{1}{2}wl$, and by its weight u at the distance d, will be du, their sum being $\frac{1}{2}wl + du$. Now the strength of the beam which is just sufficient to resist these strains is the same in both instances; consequently $\frac{wL^2}{2l} + DU = \frac{1}{2}wl + du$; and by reduction $L = \sqrt{\frac{2l}{w}\left(\frac{1}{2}wl + du - DU\right)}$.

Cor. 1. If the lengthened beam just breaks with its own weight, then U vanishes, and we have $L = \sqrt{\left(l^2 + \frac{2dul}{w}\right)} = \sqrt{l \times \left(l + \frac{2du}{w}\right)}$.

Cor. 2. If when U vanishes d becomes equal to l, we have $L = \sqrt{l \times \frac{wl + 2lu}{w}} = l\sqrt{} + \frac{w + 2u}{w}$.

185. Prop. *The strength of a rectangular beam in an inclined position is to the strength of the same beam in an horizontal position, to resist a vertical pressure, as the square of the radius to the square of the cosine of elevation.*

For a transverse vertical section of the beam, will be rectangular, whether the beam be in a horizontal or oblique position; and consequently, on the principles we have assumed, the strengths in both cases will be as the squares of the depths. Now AB (fig. 14. Pl. VIII.) being a longitudinal vertical section of the beam, the depth of the beam when inclined will be CD, and Cd when it is horizontal. Hence the strength in the one case to that in the other is as CD^2 to Cd^2, or because of the similar triangles CDd, ADO, as AD^2 to AO^2, that is, making AD radius, as rad^2 to $\cos^2$ DAO. Q. E. D.

COR. 1. The strength of any beam resisting a vertical pressure is greatest when it is in a vertical position; for then the cosine of inclination vanishes.

COR. 2. The same prop. and corol. will hold, with respect to a pressure in any other direction, provided AH, instead of being horizontal, be perpendicular to the pressure.

186. PROP. *When two beams standing obliquely bear two weights upon them, either at the middle points or in any similar situations, or equally diffused over their whole lengths; the strains upon them will be directly as the weights, the lengths, and the cosines of elevation conjointly.*

For (art. 154. cor. 4.) the weight is to the pressure upon the plane, as radius to the cosine of elevation: therefore the pressure is as the rectangle of the weight and cosine of elevation; and this is the force acting against the beam perpendicularly. Whence the stress will universally be as the length of the beam and this force, that is, as the length × weight × cosine of elevation.

COR. 1. If the lengths of the beams, and their weights, be the same, the stress will be as the cosine of elevation; and consequently it will be greatest when the beam is horizontal.

COR. 2. If the beams are horizontal, or at any equal inclinations, and the weights vary as the lengths, or the beams are uniform, then will the stress vary as the squares of the lengths.

COR. 3. If the weights are equal on the horizontal beam AC, and the inclined one AB (fig. 15.) and BC be vertical, the stress upon both beams will be equal.

For the length into the cosine of elevation is the same in both.

COR. 4. But if the weights on the beams vary as their lengths, the strength will also vary in the same ratio.

COR. 5. And universally the stress upon any point of an oblique beam, is as the rectangle of the segments of the beam, the weight and cos inclination directly, and the length of the beam reciprocally.

187. PROP. *Let* AC (*fig.* 1. *Pl.* IX.) *represent a beam moveable about a centre* C, *so as to make any angle* ACB *with the plane of the horizon* CB; *to determine the position of a prop or supporter* OS *of a given length, which shall sustain it with the greatest ease in any given position; also to ascertain the inclination of* AC *to the horizon when the least force that can sustain it is greater than the least force in any other position.*

Let G be the centre of gravity of the beam AC, and draw Gn, Gm, and CD, perpendicular to AC, CB, and OS respectively.

Put SO $= p$, CG $= r$, C$m = x$, and the weight of the beam $= w$. Then, by the nature of the parallelogram of forces, we shall have Gm : Gn, or sim. triangles, as CG $(= r)$: Cm $(= x)$: : w : $\frac{wx}{r}$, the force which acting at G in the direction Gn, is sufficient to sustain the beam: and, by the nature of the lever, CO : CG $(= r)$: : $\frac{wx}{r}$ (the requisite force at G) : $\frac{wx}{\mathrm{CO}}$, the force capable of supporting it at O in a direction perp. to AC or parallel to Gn : and, again, as CD : CO : : $\frac{wx}{\mathrm{CO}}$ (the force acting at O in direction $\perp$ to CO) : $\frac{wx}{\mathrm{CD}}$, the force or weight actually sustained by the given prop SO in a direction $\perp$ to CD. This latter force will manifestly be the least possible when the perpendicular CD upon OS is the greatest possible, let the angle ACB be what it may. But of all triangles, having the same base OS and vertical angle SCO that which is isosceles is known to have the greatest perpendicular [being an obvious corol. from the latter part of prob. VI. p. 171. Vol. iii. Hutton's Course]: therefore the triangle OSC will be isosceles, and the angle S = angle O, when the weight sustained by the prop OS is a *minimum*.

Secondly, in order to give a solution to the latter part of the problem, we have to find when $\frac{wx}{\mathrm{CD}}$ is a maximum, the angles S and O being always = each other, while they vary in magnitude in consequence of the change of the inclination ACB. Let CD produced meet Gm in I; then, because of the similar triangles CDS, CmI, we shall have CD : Cm $(= x)$: : SD $(= \frac{1}{2}p)$: mI, whence $\frac{x}{\mathrm{CD}} = \frac{m\mathrm{I}}{\frac{1}{2}p}$; and consequently $\frac{x}{\mathrm{CD}} \times w = \frac{m\mathrm{I}}{\frac{1}{2}p} \times w$. But since CI bisects the angle mCG, we also have CG + Cm $(= r + x)$: Cm $(= x)$: : Gm $(= \sqrt{r^2 - x^2})$: I$m = \frac{x}{r+x} \sqrt{r^2 - x^2} = x \sqrt{\frac{r-x}{r+x}}$: consequently the force $\frac{m\mathrm{I}}{\frac{1}{2}p} \times w$, acting upon the prop, is likewise truly expressed by $\frac{wx}{\frac{1}{2}p} \sqrt{\frac{r-x}{r+x}}$. The fluxion of this expression being taken and put equal to *zero*, we obtain $x = \frac{\sqrt{5} - 1}{2}$ r: therefore CG : Cm : : $1 : \frac{1}{2}\sqrt{5} - \frac{1}{2}$: : radius : cosine of GCB $= 51° 50'$, the inclination required.

188. PROP. *Suppose the beam* AC *instead of being moveable about the centre* C, *to be supported in a given position by means of the given prop* OS ; *it is required to determine the position of*

that prop so that the prismatic beam CB *on which it stands may be the least liable to breaking, this latter beam being only supported at its ends* C *and* B.

Let CB $= b$, OS $= p$, CG $= r$, weight of CA $= w$, CO $= z$, sine and cosine of angle C $= s$ and c respectively, sine $\angle$ O $= x$, sine $\angle$ S $= y$. Then, by trig. $z : y :: p : s$, or $\frac{y}{z} = \frac{s}{p}$, and CS $= \frac{px}{s}$: also the force of the beam at G in direction G$n = cw$. Let F denote the force sustaining the beam at O in the direction SO; then, because action and reaction are equal and opposite, the same force will be exerted at S in the direction OS: therefore CG $\times cw = \text{F}x$, and $\text{F} = \frac{rcw}{zx}$. Again, the vertical stress at S, will $\propto \text{F} \times \sin \text{S} \times \text{CS} \cdot \text{SB} = \text{F}y \cdot \text{CS} \cdot \text{SB} = \frac{rcwy}{zx} \times (b - \frac{px}{s}) \times \frac{px}{s} =$ (substituting $\frac{s}{p}$ for its equal $\frac{y}{z}$) $\frac{rcws}{px} \times \frac{px}{s} \times \frac{bs - px}{s} = rcw \times \frac{bs - px}{s} = \frac{rcwp}{s} \times \left(\frac{bs}{p} - x\right) =$ a min. by the prop. Consequently $\frac{bs}{p} - x =$ a min. or x a max.: that is, $x = 1$, and the angle COS a right angle. Hence the point O is readily found by this proportion, sin C : cos C :: OS : OC*.

GENERAL SCHOLIA.

189. We have already adverted to a general maxim, which, on account of its great importance, we beg to state again: it is this:—*When several pieces of timber, iron, or any other materials are introduced into a machine or structure of any kind, the parts not only of the same piece, but of the different pieces in the fabric, ought to be so adjusted with respect to magnitude that the strength may be in every part as near as possible in a constant proportion to the strain to which they will be subjected.* Thus, in the construction of any engine, the weight and pressure upon every part should be investigated, and the strength should be apportioned accordingly. All levers, for instance, should be made strongest where they are most strained: as levers of the first kind, at the fulcrum; levers of the second kind, where the weight acts; and those of the third kind, where the power is applied. The axles of wheels and pulleys, the teeth of wheels, ropes, &c. must be made stronger or weaker, as they will be more or less acted upon. Let the strength allowed be more than fully competent to the stress to which the parts can ever be liable; but let not the surplus be extravagant: for such an excess of strength in any part, instead of being serviceable, is

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* A geometrical solution of this problem was given in the Ladies' Diary for 1757. See Hutton's Diarian Miscellany, vol. iii. p. 38.

injurious by increasing the resistance the machine has to overcome, and thus encumbering, impeding, and often destroying the requisite motion: while, on the other hand, a defect of strength in any one part will cause a failure there, and either render the whole useless, or call for frequent repairs.

190. The propositions we have given on the strength and stress of materials, however true, according to the principles assumed, are of no use in practice till the comparative strength of different substances is ascertained. And even then they will apply more accurately to some substances than others. Hitherto they have been almost exclusively applied to the resisting force of beams of timber; though it is probable no materials whatever accord less with the theory than timber of all kinds*. The resisting body is supposed in the theory to be perfectly homogeneous, or composed of parallel fibres, equally distributed around the axis, and presenting uniform resistance to rupture. But this is not the case in a beam of timber: for, by tracing the process of vegetation, it has been found that the ligneous coats of a tree, formed by its annual growth, are almost concentric; and that they are like so many hollow cylinders thrust into each other, and united by a kind of medullary substance which offers but little resistance: these hollow cylinders, therefore, furnish the chief resistance to the force which tends to break them. Now, when the trunk of a tree is squared in order that it may be converted into a beam, it is evident that all the ligneous cylinders greater than the circle inscribed in the square or rectangle, which is the section of the beam, are cut off at the sides; and therefore, as Montucla remarks, almost the whole resistance arises from the cylindric trunk inscribed in the solid part of the beam. The portions of the cylindric coats which are towards the angles add a little, it is true, to the strength of that cylinder, as they cannot fail to oppose *some* resistance to the straining force; but it is far less than though the ligneous cylinder were entire. Hence we cannot by legitimate comparison accurately deduce the strength of a joist cut from a small tree, by experiments on another which has been sawn from a much larger tree or block: the latter is generally weak, and very liable to break. As to the concentric cylinders we have been speaking of, they are evidently not all of equal strength. Those nearest the centre being the oldest, are likewise the hardest: which again, is contrary to the theory, in which they are supposed uniform throughout. After all, however, it is still found that in some of the most

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* The propositions which relate to the equilibrium and strength of substances perfectly and imperfectly elastic, depend upon principles which cannot naturally be introduced in this place. The reader may be referred to Dr. Young's Natural Philosophy, vol. ii. p. 46—50.

important problems the results of the theory and well-conducted experiments coincide, even with regard to timber: thus, for example, the experiments of Duhamel on rectangular beams afford results deviating but in a slight degree from the theorem of Galileo, that the strength is proportional to the product of the breadth into the square of the depth.

190. A. Experiments on the strength of different kinds of wood, are by no means so numerous as might be wished. The most useful seem to be those made by Emerson, Parent, Banks, and Girard: but it will be at all times highly advantageous to make new experiments on the same subject; a labour especially reserved for engineers who possess skill and zeal for the advancement of their profession. It has been found by experiments that the same kind of wood, and of the same shape and dimensions, will break with very different weights: that one piece is much stronger than another not only cut out of the same tree, but out of the same rod: and, that if a piece of any length planed equally thick throughout, be separated into three or four pieces of an equal length, it will be found that these pieces require different weights to break them. Emerson observes that wood from the boughs and branches of trees is far weaker than that of the body: the wood of the great limbs stronger than that of the small ones: and the wood in the heart of a sound tree strongest of all. He also observes that a piece of timber which has borne a great weight for a small time, has broke with a far less weight, when left upon it for a much longer time. Wood is likewise weaker when it is green, and strongest when thoroughly dried; and should be two or three years old, at least. Knots in wood often weaken it very much. And when wood is cross-grained, as often happens in sawing, this will weaken it in a greater or less degree, according as the cut runs more or less across the grain. From all which it follows that a considerable allowance ought to be made for the strength of wood, when applied to any use where strength and durability are required.

Iron is generally much more uniform in its strength than wood: yet experiments shew that there is some difference occasioned by different kinds of ore; the difference is not only found in iron from different furnaces; but from the same furnace, and the same melting; this may arise partly from the different degrees of heat which it has when it is poured into the mould, and partly from the different intensities of heat acquired by the fused metal in different parts of the same furnace.

Every beam or bar, whether of wood, stone, or iron, is more easily broken by any transverse strain, when it is sustaining any very great compression endways. Several experiments have been made on this kind of strain: a piece of white marble $\frac{1}{4}$ inch square and three inches between the props, bore 38lbs: when compressed endways with 300lbs. it broke with $14\frac{1}{2}$lbs. The effect is much more remarkable in timber, and more elastic bodies; but is considerable in all. This, therefore, is a point which must be attended to in all experiments: as must likewise the following, namely, that a beam supported at both ends, will carry *twice* as much when the ends beyond the props are kept from rising, as when the beam rests loosely on the props. The demonstration of this is given by *Girard;* and many experiments furnish nearly the same result.

Some writers, however, seem to doubt its truth: and as every one has not opportunities of trying the matter experimentally, *Professor Robison,* in order to remove any doubts, gives this familiar representation of the case. Let LM (fig. 2. pl. IX.) be a long beam divided into six equal parts, in the points D, B, A, C, E; and firmly supported at L, B, C, M. Let it be cut through at A, and have compass-joints at B and C: FB and GC are two equal uprights, resting on B and C, but without any connection; AH is a similar and equal piece occasionally applied at the seam A. Now let a thread or wire AGE be extended over the piece GC, and made fast at A, G, and E: and let the same thing be done on the other side of A, as at A, F, D. If a weight be now laid on at A, the wires AFD, AGE, may be strained till they are broken. In the instant of fracture we may suppose their strains to be represented by A*f* and A*g*. Complete the parallelogram, and A*a* represents the magnitude of the weight. Nothing, it is plain, is concerned here, but the cohesion of the wires; for the beam is sawed through at A, and its parts move with perfect freedom round B and C. Instead of this process apply the piece AH below A, and keep it there by straining the same wire over it in the position BHC. If a weight be now laid on, it must press down the ends of BA and CA, and cause the piece AH to strain the wire BHC. In the instant of fracture of the same wire, similarly posited, its resistance H*b*, H*c*, must be equal to A*f*, A*g*, and the weight *h*H which breaks them must be equal to A*a*. Lastly, employ all the three pieces FB, AH, GC, with the same wire attached as in the two cases combined: there can be no doubt that the weight which breaks all the wires must be $=$ A*a* $+$ *h*H, or twice A*a*. And it is exceedingly manifest that the wires perform the very same office with the fibres of an entire beam LM held fast in the four holes D, B, C, E, of some upright posts.

The same thing is shewn rather differently at pa. 464, Emerson's Algebra.

The relative strengths of several sorts of wood, and of other bodies, as determined by Mr. Emerson, are as follow:

Box, yew, plumtree, oak	11
Elm, ash	8½
Walnut, thorn	7½
Red fir, hollin, elder, plane crabtree, apple-tree	7
Beech, cherry-tree, hazle	6⅔
Alder, asp, birch, white fir, willow or saugh .	6
Iron	107
Brass	50
Bone	22
Lead	6½
Fine free-stone	1

A cylindric rod of good clean fir, of an inch circumference, drawn in length, will bear at extremity 400 lbs., and a spear of fir 2 inches diameter, will bear about 7 tons.

A rod of good iron of an inch circumference will bear nearly 3 tons weight.

A good hempen rope of an inch circumference will bear 1000 lbs. at its extremity.

Hence this author concludes that if a rod of fir, or a rope, or a rod of iron, of *d* inches diameter were to lift ¼ the extreme weight, then

The fir would bear 8⅘ *dd* hundred weights.
The rope . . 22 *dd* ditto.
The iron . . 6¾ *dd* tons.

191. Mr. Banks, an ingenious lecturer on Natural Philosophy, has at various times made many experiments on the real and comparative strengths of oak, deal, and iron. He found that the worst or weakest piece of dry heart of oak, 1 inch square, and 1 foot long, bore 660 lbs. though it was much bent; and 2 pounds more broke it. The strongest piece he tried of the same dimensions, broke with 974 lbs. The worst piece of deal bore 460 lbs. but broke with 4 more. The best piece bore 690 lbs; but broke with a little more. And with respect to *cast* iron, he concludes that a bar of the weakest kind, an inch square and a foot long, would break with about 2190 lbs. The following are some of the experiments he mentions. [Banks on Power of Machines, pa. 89.]

" *Experiments on the strength of cast iron, tried at Ketley,*
" *March* 1795.

☞ " The different bars were all cast at one time out of the same air furnace, and the iron was very soft, so as to cut or file easily.

" *Exp.* I. Two bars of iron, one inch square, and exactly three feet long, were placed upon an horizontal bar so as to meet in a cap at the top, from which was suspended a scale; these bars made each an angle of 45° with the base plate, and of consequence formed an angle of 90° at the top: from this cap was suspended a weight of seven tons, which was left for 16 hours, when the bars were a little bent, and but very little.

" *Exp*. II. Two more bars of the same length and thickness, were placed in a similar manner making an angle of $22\frac{1}{2}^\circ$ with the base plate; these bore four tons upon the scale: a little more weight broke one of them, which was observed to be a little crooked when first put up. In this case the pressure would be as the sines of the angles of elevation, viz. as 3826 to 7071; and as 3826 : 4 tons : : 7071 : 7·6 tons; that is, if the second bars broke with four tons, the first ought to have taken 7·6 tons to break them; and it is likely that would if tried have been the case.

" *Exp*. III. Another bar was placed horizontally upon two supporters, exactly three feet distant; it bore 6 cwt. 3 qrs. but broke when a little more was added.

" *Exp*. IV. The same experiment repeated with the same result.

" *Exp*. V. The bearings were 2 feet 6 inches apart, the bar bore 9 cwt. and broke." " Three more experiments were tried the next day with the prisms 3 feet distant; the average result was 6 cwt. 2 qrs. $7\frac{1}{3}$ lbs."

" *Experiments tried at Colebrook-dale, on curved bars or*
" *ribs of cast iron, April* 1795.

" Rib 29 feet 6 inches span, and 11 inches high in the centre; it supported 99 cwt. 1 qr. 14 lbs.: it sunk in the middle $3\frac{7}{8}$, and rose again $\frac{3}{4}$ when the weight was removed. The same rib was afterwards tried without abutments, and broke with 55 cwt. 0 qrs. 14 lbs.

" Rib 29 feet 3 inches in span, a segment of a circle, 3 feet high in the centre; it supported 100 cwt. 1 qr. 14 lbs. and sunk $1\frac{3}{16}$ in the middle. The same rib was afterwards tried without abutments, and broke with 64 cwt. 1 qr. 14 lbs."

The thickness of these ribs, unluckily, is not specified; but the experiments prove that each rib exerted little more than half the strength when the abutments were removed. This corresponds with what we have stated in the last article relatively to beams being fixed at the ends, as nearly as can be expected, considering the necessary difference between prisms and these arch ribs.

Mr. Banks made some experiments on the strength of cast iron, at Messrs. Aydon and Elwell's foundry, Wakefield. " The iron came from their furnace at Shelf, near Bradford, and was cast from the air furnace; the bars one inch square, and the props exactly a yard distant. One yard in length weighs exactly 9lbs. or one was about half an ounce less, and another a very little more: they all bent about an inch before they broke.

lbs.
"1. The first bar broke with 963
"2. Bar broke with 958
"3. Bar broke with 994
"4. Bar made from the cupola, broke with 864
"Bar equally thick in the middle, but the ends formed into a parabola, and weighed 6 lbs. 3 oz. broke with . . . . . 874."

This gentleman made many other experiments. He concludes from the whole that cast iron is from 3⅓ to 4½ times stronger than oak of the same dimensions, and from 5 to 6½ times stronger than deal.

With respect to the twist to which bars or shafts placed in an upright position are liable, between the wheel which drives them and the resistance they have to overcome, he observes that a cast iron bar an inch square and fixed at one end, and 631 pounds suspended by a wheel of 2 feet diameter fixed on the other end, will break by the twist: though some have required more than 1000 lbs. in similar situations to break by the twist. The strength to resist the twisting strain is as the cube of like lateral dimensions.

191. A. Professor Robison found by many trials that a prism of white marble an inch square and a foot long, bears about 500 lbs.: and that a cylinder of an inch in diameter loaded to one-fourth will carry, if of iron 135 hundred weight, of rope 22, oak 14, fir 9.

Professor Robison also collected from the experiments of Muschenbroek and others, a table of the absolute strengths of several substances: from which the following is selected. The specimens are supposed to be prisms or cylinders of one square inch transverse area; and to be stretched or drawn lengthwise by suspended weights, gradually increased till the bars were torn asunder. The avoirdupoise pounds, which on a medium of many trials effected this, are set against each name.

1st. METALS.

	lbs.		lbs.
Gold, cast	22,000	Tin, cast	5,000
Silver, cast	42,000	Lead, cast	860
Copper, cast . . .	34,000	Regulus of Antimony	1,000
Iron, cast	50,000	Zinc	2,600
Iron, bar	70,000	Bismuth	2,960
Steel, bar	135,000		

It is a circumstance deserving notice that almost all the metallic mixtures have greater tenacity than the metals themselves. The change of tenacity depends much on the relative proportions of the ingredients; yet the law is extremely variable, differing much in different metals. The proportions of constituent parts selected are those which produce the greatest strength.

	lbs.
2 parts Gold with 1 of silver	28,000
5 Gold, 1 copper	50,000
5 Silver, 1 copper	48,500
4 Silver, 1 tin	41,000
6 Copper, 1 tin	60,000
Brass, of copper and tin	51,000
3 Tin, 1 lead	10,200
8 Tin, 1 zinc	10,000
4 Tin, 1 regulus antimony	12,000
8 Lead, 1 zinc	4,500
4 Tin, 1 lead, 1 zinc	13,000

These mixtures are of considerable use in the arts. The mixtures of copper and tin are especially important in the fabric of great guns. By mixing copper whose strength, at greatest, does not exceed 37,000, with tin, which in like manner is less than 6000; a metal is produced whose tenacity is almost double that of copper, at the same time that it is harder and more easily wrought: it is, however, more fusible. It appears, also, that a very small addition of zinc almost doubles the tenacity of tin, and increases the tenacity of lead five times. An addition of from one fourth to one third of lead doubles the tenacity of tin. These are economical mixtures, and afford valuable information to plumbers, as to the manner of augmenting the strength of water pipes. By having recourse to these tables, also, the engineer can proportion the thickness of his pipes, of whatever metal, to the pressures they are intended to sustain.

2dly. WOODS, &c.

	lbs.		lbs.
Locust tree	20,100	Elder	10,000
Jujeb	18,500	Pomegranate	9,750
Beech and Oak	17,300	Lemon	9,250
Orange	15,500	Tamarind	8,750
Alder	13,900	Fir	8,330
Elm	13,200	Walnut	8,130
Mulberry	12,500	Pitch pine	7,650
Willow	12,500	Quince	6,750
Ash	12,000	Cypress	6,000
Plum	11,800	Poplar	5,500

	lbs.		lbs.
Cedar	4,880	Horn	8,750
Ivory	16,270	Whalebone . . .	7,500
Bone	5,250	Tooth of sea-calf .	4,075

These numbers express something more than the utmost cohesion; the weights being such as will very soon tear the rods asunder. It will be found generally that $\frac{2}{3}$ of these weights will sensibly impair the strength after acting a considerable while; and that $\frac{1}{2}$ is the utmost that can remain permanently suspended from the rods with safety. Wood, however, of straight fibre, such as fir, will bear a greater proportional load than other timber.

The strength of some metals is doubled or tripled by the operation of forging and wire-drawing; and the cohesive as well as the repulsive force of wood is often increased by moderate compression. Oak will *suspend* much more than fir, but fir will *support* twice as much as oak; the curvature of the fibres of oak appears to be the reason of the difference; yet oak has been known to support, with safety, more than 2 tons for every square inch. Stone will support from 250 to 850 thousand pounds on a foot square, brick 300; and sometimes they are practically made to support one-sixth as much. Stone is said to be capable of bearing a much greater weight in that position with respect to the horizon in which it is found in the quarry, than in any other position.

192. Having now given, in addition to the theoretic propositions, the results of experiments made at different times by various persons, it remains to exhibit a few examples of the practical application of the whole.

I. Let it be required to find what weight suspended from the middle of an oak beam (supported at each end) will break it, the length of the beam being 8 feet, its end a square, and each side six inches.

According to Mr. Banks's experiments a bar of oak an inch square and a foot long lying on a prop at each end, will break with 660lbs. on its middle, taking the lowest number. And in all cases where we make such comparison of strength, *the breadth multiplied into the square of the depth and divided by the product of the length and weight must be a constant quantity:* that is, taking b, d, l, and w, for the breadth, depth, length, and weight, used in the experiment, and B, D, L, W, those proposed in the example, we have $\frac{bd^2}{lw} = \frac{\mathrm{BD}^2}{\mathrm{LW}}$. Hence $\mathrm{W} = \frac{\mathrm{BD}^2 lw}{bd^2 \mathrm{L}}$, which in the present instance gives $\mathrm{W} = \frac{6 \times 6^2 \times 1 \times 660}{1 \times 1^2 \times 8} =$ 17820lbs.

II. Let the length of the beam be as above, the breadth 3 inches, and the weight 17820, what must be the depth of the beam?

Here b, d, l, and w, also L, and W, are as in the preceding example. From the general equation we obtain $D = \sqrt{\frac{bd^2LW}{Blw}}$ $= \sqrt{\frac{1 \times 1 \times 8 \times 17820}{3 \times 1 \times 660}} = 8{\cdot}45$ inches, nearly.

Cor. The latter beam, though as strong as the former, is but little more than $\frac{2}{3}$ of it in size.

III. Required the breadth of a bar of iron 8 feet long, and 4 inches in depth, to sustain the same weight at its middle point?

Here b, d, l, L, W, as before, $w = 2190$, $D = 4$; and from the theorem $B = \frac{bd^2LW}{D^2lw} = \frac{1 \times 1 \times 8 \times 17820}{4^2 \times 1 \times 2190} = 4{\cdot}07$ inches, nearly.

IV. Required the length of a piece of oak an inch square, so that when propped at both ends it may just break with its own weight?

Here we adopt the notation in art. 184. cor. 2. and have $l = 1$, $w = \frac{2}{5}$ of a lb. $u = 660$ lbs. Then $L = l\sqrt{\frac{w + 2u}{w}} = 57{\cdot}45$ feet, nearly.

V. Find the length of an iron bar an inch square, that it may break with its own weight, when it is supported at both ends.

Here l as before, $u = 2190$, $w = 3$. Hence $L = l\sqrt{\frac{w + 2u}{w}}$ $= 38 \cdot 223$ feet nearly.

Cor. It might have been supposed this result should exceed the preceding one: but it must be considered that while iron is only $4\frac{1}{2}$ times stronger than oak, it is about $7\frac{2}{5}$ times heavier.

VI. When a weight W is suspended from E on the arm of a crane ABCDE (fig. 3. Pl. IX.), it is required to find the pressure at the end D of the spur, and that at B against the upright post AC.

Here, by the nature of the lever $\frac{CE}{CD}W =$ the pressure at D in the vertical direction DG: but the pressure at D in the direction DB will be to the vertical pressure as DB to DG: hence $DG : DB :: \frac{CE}{CD}W : \frac{DB \cdot EC}{DG \cdot CD}W$. And the horizontal pressure against the upright post CA at B, will be to that in direction DB, as GB to DB; therefore $DB : GB$, or $DB : CD :: \frac{DB \cdot EC}{DG \cdot CD}W : \frac{EC}{DG}W = \frac{EC}{BC}W$.

Mr. Banks gives the following among other instances of the use of this: "Given EC = 12 feet, BC = 6, DC = 6·7, DB = 9, W = 4. Required the pressure on the spur, and the horizontal pressure against the upright. 1. $\frac{\text{DB} \cdot \text{EC}}{\text{DG} \cdot \text{DC}}$ W = 10·74, the pressure against the end of the spur in direction DB. The pressure against the post is $\frac{\text{EC} \cdot \text{W}}{\text{BC}} = \frac{12 \times 4}{6} = 8$. In this example, let AC and CE be oak beams, each 10 inches square, and the spur DB 6 inches square. The strength of EC is $\frac{1000}{10 \cdot 6}$, or $94\frac{1}{3}$; which multiplied by 660 gives 31132 pounds, which suspended at E would break the beam CE at D. The length of the upright AC is 12 feet, and has its strength expressed by $\frac{1000}{12}$; which multiplied by 660, produces 55000 pounds, the weight which would break it at B. But $\frac{31132 \times 12}{6} = 62264$, the pressure at B, being 7264 pounds more than the beam AC can support. The strength of the spur BD is $\frac{6 \times 6 \times 6}{9} = 24$, which multiplied by 2 gives 48 tons for the strength, or 107520 pounds. But $\frac{\text{DG} \cdot \text{EC}}{\text{DG} \cdot \text{DC}}$ W $= \frac{9 \times 12 \times 31132}{6 \times 6 \cdot 7} = 83638$ pounds, which is 23882 pounds less than the force requisite to break the spur. From the above it appears that the upright AC is the weakest part; but from the principles already explained, the ingenious mechanic will easily proportion the parts so as to be equally strong."

VII. "Let it be required to make a crane of cast iron to bear 4 cwt. but that it may be perfectly safe let it be calculated for 10 cwt. and let AC = CE = 3 feet, also BC = CD = $1\frac{1}{2}$ foot.

"Let the thickness of the iron be half an inch, and put z = depth of CE. Then as $1 : 2190 :: \frac{z^2 \times \frac{1}{2}}{3} : 1120$, from which we find the thickness $z = 1{\cdot}75$ inches. The pressure upon the spur at D, in the direction DG = 1120 pounds; the length of the spur is 2·12 feet, and as DG (15) : DB (2·12) : : 1120 : 1583 for the pressure in the direction DB. As a bar 1 inch square, and 1 foot long, will bear 15 tons, or 33600 pounds (at the end), we say as $1 : 33600 :: \frac{z^3}{2{\cdot}12} : 1583$, from which we find z the side of the prop or spur = ·46385 of an inch. Next, for the upright, we have $\frac{\text{CE} \times \text{W}}{\text{BC}} = \frac{560 \times 3}{1 \cdot 5} = 1120$ pounds, the pressure against B, then as $1 : 2190 :: \frac{x^2}{3 \times 2}$ (the square of the breadth) : 1120 pounds, the same as CE, as they are of the same length, and the breadth will be the same, that is 1·75 inches."

Many other examples, particularly of the strength of beams to work cylinders, and of cast iron shafts or axles for mills, may be seen in Mr. Banks's little Treatise on the Power of Machines, &c.

CHAPTER VI.

On Cords, Arches, and Domes.

193. We have already spoken of the use of cords in their connection with the lever, the pulley, and the axis in peritrochio: but they are sometimes used in a manner more complicated than what we have yet considered, and are by some authors regarded as a distinct machine under the name of *Funicular machine.* We shall, therefore, treat of them a little more particularly, and consider them as perfectly flexible, without gravity and reduced to their axes, unless the contrary is stated. The *tension* of a cord is the force which acts at one end of it when the other is fixed, or it is equivalent to that force: thus in the case of the equilibrium of powers applied to a physical point, if we regard that point as fixed, the tension of each cord is precisely the force applied at each cord to move the point: but if the equilibrium does not obtain, as when, for example, a cord has two unequal powers acting at its extremities, the tension is the least of the two forces, for the tension will obviously be the same as if the one of the extremities were fixed, and the least of the two forces acted solely at the other end.

194. When three, four, or more powers, act at the extremities of different cords, all united at one node, the conditions of equilibrium will be the same as we have exhibited in Chap. II. between arts. 46. and 70. whether the cords are all situated in one plane, or several. But if two forces C, C′ (fig. 3. pl. I.) act at the extremities of a cord CPC′ passing through a ring at P fastened to a cord OP, and retained by a power O, the conditions of equilibrium are these:

1. The line OP when produced must bisect the angle CPC′.
2. The forces C, C′ must be equal to each other. For, if the angles CPD, C′PD, are unequal, the cord will slide along the ring; and this condition combined with that of the parallelogram of forces requires the equality of C, and C′. This being premised, we proceed to a few useful problems.

195. Prop. *Given the length* l *of a cord* EAH, *and the position of the points* E, H, *to which its ends are fixed, and suppose a given weight* P *hangs by a cord* AP, *the latter cord running freely along the former by a ring at the end* A (fig. 4. pl. IX.),

to determine the position of the whole when in equilibrio, and the tension of the cord EAH *at its extremities.*

Draw through E the horizontal line EG, and through H the vertical FK. Since AL divides the angle EAH into two equal parts, if EA produced meet FK in K, we shall, because of the parallels LA, FK, have K = EAL = LAH = AHN; therefore the triangle HAK is isosceles, and KA = AH; also EK = EA + AK = EA + AH = l. If, therefore, with E as a centre and radius = l, we cut the vertical FK in K, and on the middle point N of HK erect the perpendicular NA, it will intersect the line EK in A, the point where the ring will rest in equilibrio.

The analytical solution of the remaining part of the problem is by no means difficult. Thus, put EF = h, the angle EAH = $2a$, and let the tensions of the cord at the points of suspension E, H, be P′, P″. Then, since P′ must be equal P″, and EAL = LAH, we have (art. 41. cor.) $\text{P} = 2\text{P}' \cos a$. In the right-angled triangle EFK, we have $\sin \text{EKF} = \sin a = \frac{\text{EF}}{\text{KE}} = \frac{h}{l}$. Employing this value in the equation $\text{P} = 2\text{P}' \cos a$, there arises $\text{P} = 2\text{P}' \sqrt{\frac{l^2 - h^2}{l^2}}$, whence $\text{P}' = \frac{\text{P}}{2\sqrt{\frac{l^2 - h^2}{l^2}}} = \frac{\text{P}l}{2\sqrt{(l^2 - h^2)}}$, an equation which gives the tension of the cord at E or H.

Cor. 1. The distances HA and EA may be easily found when required: for, since the relative positions of E and H are given, FH is known as well as EF; and because FK = KE cos a, it is known also: hence NK is known, being $\frac{\text{FK} - \text{FH}}{2}$, and we have, by sim. tri. KF : KE :: KN : KA (= AH) :: NF : AE.

Cor. 2. The locus of all the points A is an ellipse whose foci are E, H, and transverse axis = l.

Cor. 3. When H falls in the horizontal line EG, it will be EA = AH.

196. Prop. *Given the weight* R *attached to the point* A *of a cord which passes over two fixed pulleys in given positions* B *and* C, *the given weights* P, Q, *hanging at the extremities of the cord, to determine the position of* AC *and* AB *when the whole is in equilibrio.* (fig. 5. pl. IX.)

At a convenient distance from B and C, draw the vertical *da* to represent the weight R (fig. 6.), and form the triangle *dae* such that the sides *ae* and *de*, shall be to *da*, as the weights P, and Q, respectively to R. From the point B draw BA parallel to *ea*, and from the point C draw CA parallel to *de*, then will CAB be the position in which the cord will rest in equilibrio. For, if we take any distance AD in the vertical AG to represent R, and complete the parallelogram AEDF, we see, by the similitude of the triangles ADE, *ade*, that DE and DF ought to represent the tensions of AC and AB, and that in this position they are in equilibrio with the weight R.

By means of the known sides of the triangle *ade*, we may calculate the three angles: then, drawing the horizontal line BG, the angle GBC will be given, since the points B and C are; the angle GBA is known also, being the complement of GAB or *dae*; and the angle CBA = GBA − GBC, will be therefore known; thus, in the triangle CAB we know the side CB, and the angles CAB, CBA (and consequently the third angle), from which we may compute, with ease, the sides CA and AB.

COR. 1. When P = Q, the triangle *ade* becomes isosceles, by which means the computation is much simplified.

COR. 2. When, besides having P = Q, we have B and C in the horizontal line, the process is still farther simplified. In this case AC and AB will be equal to each other; the triangle *ade* will be isosceles, and *eia*, similar to BGA. We shall also have $ei = \sqrt{(ae^2 - ai^2)} \propto \sqrt{(P^2 - \frac{1}{4}R^2)}$: so that if BG be put $= \frac{1}{2}d$ and GA $= x$, we shall express $ai : ie :: AG : GB$ thus,

$$\tfrac{1}{2}R : \sqrt{(P^2 - \tfrac{1}{4}R^2)} :: x : \tfrac{1}{2}d, \text{ whence } x = \frac{\frac{1}{4}dR}{\sqrt{(P^2 - \frac{1}{4}R^2)}} = \frac{dR}{2\sqrt{(4P^2 - R^2)}}.$$

197. PROP. *To determine the conditions of equilibrium in the funicular polygon when many forces are acting at different points of the cord, but in the same plane.*

Let $PNN'N''$, &c. (fig. 7. pl. IX.) be the polygon proposed, being kept in equilibrio by the powers P', P'', . . . P^n, acting in the directions PN, $P'N$, $P''N'$, &c. And call t, t', t'', &c. the respective tensions of the parts of the cord PN, NN', $N'N''$, &c. Now, since the equilibrium obtains in the system, it must necessarily have place in each portion of the polygon separately. Hence P, P', and t', must be in equilibrio about the node N; and t'' must be the resultant of the component forces P, P'; the force which acts on the point N' in direction $N'N$, is therefore equivalent to the two forces P, P', acting simultaneously at N; and the node N'' is acted upon as though it were solicited by the four forces P, P', P'', t'', in directions respectively parallel to PN, $P'N$, $P''N'$, $N''N'$. In like manner it may be shewn that the node N'' is kept in equilibrio in the same way as it would be, if subjected to the simultaneous action of the powers P, P', P'', P''', t''', in directions parallel to PN, $P'N$, $P''N'$, &c. And so on throughout. Hence it follows, that *when a funicular polygon is sustained in equilibrio by any number of forces whatever, if we transport these powers parallel to their respective directions, so as all to exert their energies upon one point, it will be kept in equilibrio by their combined action.*

COR. 1. The result just obtained corresponds with that which was deduced in Chap. II. (art. 85.) with regard to forces acting at different points of solid bodies: consequently the other theorems given there may, when required, be applied to the case before us.

Cor. 2. If we neglect the two extreme forces P, P^n, and reason upon the others as in the proposition, it is clear we may conceive all of them applied at the same point (provided they make respectively the same directions with any assumed axe AX), without at all altering the magnitude of their resultant: and hence as that resultant is destroyed by the two powers P, P^n, it must necessarily pass through the point of concourse O of the directions PN, P^nN'''.

Cor. 3. If, therefore, a cord AEB (fig. 10. pl. IX.) fixed at two points A and B, have all its points solicited by any forces whatever in the same plane, it will assume a plane curvature: and the point of intersection O of two tangents will fall upon the direction of the resultant of all the forces applied to the various points of the cord. And if we transfer these forces parallel to their directions, so as to apply them all at the point O, their resultant being resolved into two others acting according to the directions AO and OB, we shall thence obtain the effort exerted upon each of the fixed points.

Cor. 4. The case of the last corol. applies obviously to gravity: for, on the one hand, this force exercises its action on all the points of the cord, and on the other, these efforts may be assimilated to the weight distributed throughout the length of the heavy cord. Hence, the curve thus formed, and known by the names of *funicular curve*, *chainette*, or *catenary*, is a plane curve.

Cor. 5. In the catenary the total effort exerted on the fixed points A and B, is the whole weight of the cord: if, therefore, at the point of concourse O of the two tangents to the curve at A and B, a weight equal to that of the cord were sustained by two threads AO, BO, void of gravity, the points A and B would be acted upon in the same manner as they are by the action of gravity upon the cord AEB; viz. the powers P, P', necessary to retain either the heavy cord AEB, or the equal weight at O, would be the same in both cases. As the resultant is the weight of the cord, if we erect upon O an indefinite perpendicular OE, it will pass through the centre of gravity (art. 106. IV.), and the forces exerted upon A and B will be proportional to the sines of the angles BOE and AOE (art. 48.): thus, if W be the entire weight of the cord, we have

$$W : P : P' :: \sin AOB : \sin EOB : \sin AOE.$$

Cor. 6. The same will obtain, wherever the points A and B are found in the curve, since the state of equilibrium allows us to consider as fixed any two points in the curve. If, therefore, we consider the point P as fixed instead of the point B, the portion AEF of the curve will not change its form: nor will there be any change in the tension P exerted at A.

COR. 7. If several weights W, W′, W″, &c. (fig. 8.) hang upon a cord PC′CP′, the weight of which is inconsiderable with respect to either of those weights, the pressure upon any angle C of the funicular polygon, will be $\propto \frac{\sin dce}{\sin ecc \cdot \sin cco}$, the line cc being a continuation of the vertical $W''c$. For, if a series of parallelograms of forces be described at the angles of the polygon, where the forces $c''o''$, ec, co, $e'c'$, &c. being equal and opposite, destroy each other, we shall have, force $e''c''$: $c''o''$, or ec : : $\sin e''c''c''$ or $\sin c''c''e''$: $\sin P'c''c''$: : $\frac{1}{\sin P'c''c''} : \frac{1}{\sin c''c''e''}$. Again, ec : co or $e'c'$: : $\sin cco$ or $\sin e'c'c'$; $\sin cce$: : $\frac{1}{\sin cce} : \frac{1}{\sin e'cc'}$ or $\frac{1}{\sin cco}$; and so on. Whence it appears in general that any force ec is as $\frac{1}{\sin cce}$. Therefore, because $cc = \frac{\sin cec \times ce}{\sin cce}$, we have $cc = \frac{\sin cec}{\sin cce} \times \frac{1}{\sin ecc} = \frac{\sin dce}{\sin cce \cdot \sin cco}$.

COR. 8. If the number of weights hanging from the cord be increased, and the distances on the cord of the points from which the weights hang be increased indefinitely, or if instead of the weights we conceive pieces of heavy cord, or of chains, to be hung from different points of the cord PCP′, as in fig. 9. our funicular polygon will then become a curve, being indeed a species of catenary. The angle *ecd* will then become the angle of contact formed by the tangent and curve, whose sine is equal to the measure of the angle; and the angles *cce*, *cco*, become equal to the angles *ccd*, *ccd′*, which are supplements to each other. Hence, because the angle of contact is as the curvature of the arch, or reciprocally as the radius of curvature, the weight hanging at any point C, will be reciprocally as the radius of curvature at that point, and the square of the sine of the angle made by the curve (or its tangent) and the vertical.

COR. 9. Lastly, a heavy cord cannot by any force be stretched into a right line, except it be in a vertical position: for, the weight of the cord may be considered as a force applied at its centre of gravity; and then, the cord AEB (fig. 10.) being retained by the two forces P, P′, if W be its weight, we have

(cor. 5.) P : W : : sin EOB : sin AOB;

where, it is obvious, the more the cord is stretched, the greater the angle AOB becomes, and the more nearly the angle EOB approaches to a right angle: so that the cord can only be stretched straight horizontally, when this analogy obtains, P : W : : 1 : 0, that is, when P is infinite. Thus, however small the weight is, it will cause the cord to be curved, unless it be placed vertically: which is, indeed, a circumstance experienced daily.

198. Prop. *To investigate the most useful equations of the simple catenarian curve.*

Let AMC (fig. 11. pl. IX.) be a cord uniformly thick throughout, inextensible, and perfectly flexible, fixed at the two given points A, C, and solicited by gravity in every point. Taking the origin of the rectangular co-ordinates in A, and estimating the x, x, &c. horizontally, and y, y, &c. vertically, we shall have for any point M in the curve $\mathrm{AP}=x$, $\mathrm{PM}=y$, $\mathrm{AM}=z$. The tensions exerted at A and M according to the tangents AD, MD, give, weight of AM : tension at A : : sin ADM, or sin ADF : sin IDM (cor. 5. art. 197.); where, since the tension at A is constant and the weight of the arc AM is proportional to its length, the first ratio in the analogy, is $=z : a$, a being a constant quantity, yet undetermined. Moreover, sin $\mathrm{IDF}=\frac{\dot{x}}{\dot{z}}$, cos $\mathrm{IDF}=\frac{\dot{y}}{\dot{z}}$, and denoting the angle IAD by s, we have sin $\mathrm{ADF}=\sin(\mathrm{IDF}-\mathrm{IDA})=\frac{\dot{x}\sin s-\dot{y}\cos s}{\dot{z}}$: therefore $\dot{z} : a : : \dot{x}\sin s-\dot{y}\cos s : \dot{x}$, whence we deduce the fluxional equation.

$$(1.)\quad \dot{z}\dot{x}=a\dot{x}\sin s-a\dot{y}\cos s.$$

In order to eliminate one of these variables, we regard $\dot{x}$ as constant, and thence obtain $-\ddot{z}\dot{x}=b\ddot{y}$, in which $b=a\cos s$. Hence, substituting $\surd(\dot{x}^2+\dot{y}^2)$ for $\dot{z}$, we get $-\dot{x}\ddot{y}=\frac{b\dot{y}\ddot{y}}{\surd(\dot{x}^2+\dot{y}^2)}$: here $\dot{x}$ being constant, the fluent of the second member is evidently $b\surd(\dot{x}^2+\dot{y}^2)$; that of the first member is $-y\dot{x}$, or its correct fluent $c\dot{x}-y\dot{x}$; we have therefore $(c-y)\,\dot{x}=b\surd(\dot{x}^2+\dot{y}^2.)$ But $\frac{\dot{y}}{\dot{x}}$ being the tangent of the angle formed by the axe AX and the tangent to the curve at every point, if we make $y=0$, we thence have $\frac{\dot{y}}{\dot{x}}-\tan s=\frac{\sin s}{\cos s}$: this condition gives $c=a$, and therefore

$$(2.)\quad \ldots\ldots\quad \frac{\dot{y}}{\dot{x}}=\frac{\surd[(a-y)^2-b^2]}{\pm b}.$$

Now, since $\frac{\dot{y}}{\dot{x}}$ expresses the tangent of the angle formed by AX and the curve or its tangent at any point, in order to find the point where the curve is parallel to AX, or the lowest point of the curve where the tangent is horizontal, we must make

$\frac{\dot{y}}{\dot{x}}=0$, and this will give $y=a-b=$ HB, and from the equa. (1.) it gives $z=a \sin s$. To determine the relations of the curve to BH as an axis, call BL, X, and LN, Y; then, comparing these new co-ordinates with the former, we have $y'=a-b-\mathrm{X}$, and $x=$ AH $-$ Y. Substituting these values in equa. (2.) there arises

$$\frac{\dot{\mathrm{X}}}{\dot{\mathrm{Y}}}=\frac{\sqrt{[(\mathrm{X}+b)^2-b^2]}}{\pm b} : \text{consequently } \dot{\mathrm{Y}}=\frac{\pm b\dot{\mathrm{X}}}{\sqrt{[(\mathrm{X}+b)^2-b^2]}}=$$

$\frac{\pm b\dot{\mathrm{X}}}{\sqrt{(\mathrm{X}^2+2b\mathrm{X})}}$. Taking the fluents of this equation, we have

$$\mathrm{Y}=\pm b \text{ Hyp. log. } [b+\mathrm{X}\sqrt{(\mathrm{X}^2+2b\mathrm{X})}+\text{Hyp. log. } d].$$

Now supposing $\mathrm{Y}=0$, it will give $\mathrm{X}=0$, and Hyp. log. $d=\mp b$ Hyp. log. b. whence we obtain the following correct fluent:

$$(3.) \ldots \mathrm{Y}=\pm b \text{ Hyp. log. } \left(\frac{b+\mathrm{X}+\sqrt{(\mathrm{X}^2+2b\mathrm{X})}}{b}\right)=\pm b. \text{ Hyp. log. } \frac{(z+\sqrt{(b^2+z^2)}}{b}$$

This equation shews that the vertical axe passing through the point where the tangent is horizontal is a diameter, since to each value of X there are two equal and opposite values of Y.

Other algebraic though not finite expressions for the ordinate Y might be easily obtained; but there will no particular advantage arise from pursuing that part of the investigation, we therefore merely exhibit one more equation which flows naturally from the one just given; it is this, $\mathrm{Y}=\pm b$ Hyp. log. $\frac{z+\mathrm{X}}{z-\mathrm{X}}$.

The fluxional equation $\dot{\mathrm{Y}}=\frac{b\dot{\mathrm{X}}}{\sqrt{(\mathrm{X}^2+2b\mathrm{X})}}$ gives $\sqrt{(\dot{\mathrm{X}}^2+\dot{\mathrm{Y}}^2)}=\frac{(\mathrm{X}+b)\dot{\mathrm{X}}}{\sqrt{(2b\mathrm{X}+\mathrm{X}^2)}}$, the fluent of which is BN $=\sqrt{(2b\mathrm{X}+\mathrm{XX})}=z$: from this equation we obtain

$$(4.) \ldots\ldots \mathrm{X}=-b+\sqrt{(b^2+z^2)}.$$

When we would construct the catenary, we must consider as known the points of suspension A and C, and the length AMC of the cord; it will be necessary to determine the position and length of BH, that is, to find AH and BH, or KH and BH, AK and CK being given, and to know the value of the constant quantity b. In order to this we apply the equation (3.) to the point A, substituting for X, BH $-y$, and for Y, AK $-$ AH $-x$, and then introduce this value of X into equation $z=\sqrt{(2b\mathrm{X}+\mathrm{X}^2)}$. Again, in the equation of the curve we put first $x=0$, $y=0$, then $x=$ AK, and $y=$ CK; and in the equation $z=\sqrt{(2b\mathrm{X}+\mathrm{X}^2)}$, we put X $=$ AK, and $z=$ AMC: by these means we obtain three equations comprising only known quantities and the three unknown ones b, AH, BH, whence each of the unknown quantities may be determined.

If the points A, C′, are in the horizontal line AX, we have AH = HC, and arc AB = arc BC′ = ½ABC′. In this case we retain the origin at the point B, and making $z = \frac{1}{2}$ABC′, in the equation $x = -b \pm \sqrt{(b^2+z^2)}$, the value of x thence resulting being substituted in the equation of the curve, in which also y will = ½AC, the equation itself will then contain only one unknown quantity b, the determination of which will become a matter of comparative facility: though it must still be by a method of approximation, because the curve is transcendental.

Cor. 1. Instead of gravity, if any other force exerts its energy in like manner, acting equally upon every point of the flexible line, the same curve will be produced: thus, for example, if the wind be supposed equable, and blow according to right lines parallel to a given line; the cord thus inflated by the wind will assume the shape of the catenary: for, since all things obtain with respect to this other force, as we have supposed in relation to gravity, the results must obviously be similar.

Cor. 2. If the forces acting upon every point of the curve, instead of being exerted in parallel lines, were always exercised in directions perpendicular to the curve, the forces exercised at every point would be inversely as the radii of curvature at those points: if, therefore, the forces were equal to each other throughout, the radius of curvature would then be constant, and the curve would be a circular arc.

Cor. 3. Suppose that, instead of the ends of the cord or chain being fixed by pins at A and C′, the said chain passes over pulleys there, as at C, B, (fig. 5), and hangs down vertically (as CQ, BP) though without any other than its own weight, and that the equilibrium is constituted by the equal and contrary operation of the portions of chain or cords hanging respectively between and beyond the pulleys at A and C′: then, putting BH $= \delta$, the half arc AMB $= \sigma$, and take β so that it shall be to BM, as the tension at B to the weight of the part BM; σ will be to $\beta + \delta$ as the weight of the part AB to the tension at A: this latter tension being equal to the weight of the part hanging vertically beyond A, and the weights of any two portions of the chain being as their length, it follows that $\beta + \delta$ will express the length of each portion of the chain that hangs vertically. Consequently, $2\beta + 2\delta + 2\sigma$ will be equal to the whole length of the chain. Let that be denoted by L, the distance AC′, by 2D, and the number 2·7182818 whose hyperbolic logarithm is 1, by N; then, by the nature of the curve, we shall have $\beta + \delta =$

$\left(N^{\frac{D}{\beta}} + N^{-\frac{D}{\beta}}\right) \frac{1}{2}\beta$, and $\sigma = \left(N^{\frac{D}{\beta}} - N^{-\frac{D}{\beta}}\right) \frac{1}{2}\beta$. Whence $2\beta N^{\frac{D}{\beta}} = L$. Therefore D and L being given, β may be found, and of course δ and σ by the two preceding equations.

COR. 4. D being given and β supposed variable, the fluxion of $\beta N^{\frac{D}{\beta}}$ is $\beta N^{\frac{D}{\beta}} - \frac{1}{\beta} D\beta N^{\frac{D}{\beta}}$, which will be $= 0$, when its fluent is a minimum; in which circumstance it appears that $B = D$. Therefore, substituting for D in the value of L, it appears that the length of the *shortest* chain which can possibly rest suspended in the manner above described is 2ND or 5·4365636 D.

COR. 5. It farther appears, by making the same substitution in the values of $\beta + \delta$ and σ, that when the minimum length of chain is suspended in that manner, the two exterior vertical parts of the chain will each be $\frac{1}{2}D\left(N + \frac{1}{N}\right) = 1{\cdot}5430806D$; semi-arc $AB = BC' = \frac{1}{2}D\left(N - \frac{1}{N}\right) = 1{\cdot}1702012\,D$; and $BH = \frac{1}{2}D\left(N + \frac{1}{N}\right) - D = {\cdot}5430806D$.

COR. 6. Since $\beta N^{\frac{D}{\beta}}$ is a minimum when equal to ND, it is obvious that, L being greater than 2ND, β in the equation $2\beta N^{\frac{D}{\beta}} = L$ will have two real and positive values, one less and the other greater than D; and consequently the chain will then rest in two different positions on the pulleys*.

II. *On Arches and Piers.*

199. The construction of arches is one of the most important and difficult branches of Architecture, particularly when considered in relation to the erection of bridges over broad and rapid rivers: it commonly imposes the double task of blending the handsome forms and the decorations of the ordinary architecture with the firmness and durability which ought always to be found in works destined not merely for the accommodation of the public but in many cases for its safety. The theory of arches, when perused to the extent its importance and utility demands, would itself fill a volume; but all that we shall attempt in this place will be a concise view of the leading particulars, according to the most simple and obvious theory.

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* In the Phil. Trans. No. 231, or New Abridgment, Vol. IV. there is a curious paper on the catenary by Dr. David Gregory, in which is given an elegant construction of the curve by means of the parabola and equilateral hyperbola, besides the quadrature and cubature of various parts of the catenarian space, and its solids of rotation.

Defs. By an arch we here mean an artful disposition of several stones, or bricks, or other suitable materials, the under part of which is in a bow-like form, their weight producing a mutual pressure, so that they not only support each other but may be made to carry the most enormous weights. Other particulars relating to an arch, the defining of which is necessary here, may be soon learnt by turning to fig. 1. pl. X. Thus AS′ESB is the ponderating arch; A or B, the *spring* of the arch; D its *crown;* AB its *span;* CD its *height* or versed sine, or rise; ADB the *intrados,* or the lower surface of the arch (often called the arch); S′ES the *extrados,* being in bridges the superior surface, or the roadway; F′, F, the *flanks* or *hances*; the spaces above these are called the *spandrels;* the portions of wedges which lie in a course contiguous to the intrados are called *voussoirs,* or arch stones; that which is at D is called the *keystone;* the walls or masses PQST, P′Q′S′T′, built to support the arches, and from which they spring as their bases, are either called *piers* or *abutments; piers* when they stand between two neighbouring arches, *abutments* when they support the arches which are contiguous to the shore: the part of the pier from which an arch springs is called the *impost;* the curve formed by the upper sides of the voussoirs the *archivolt;* and the lines FS, F′S′, about the flanks, in which a break is most likely to take place, are called *joints of fracture.* The other terms we shall use will need no explanation.

When we reflect upon the immense quantity of heavy materials suspended in the air in a large arch, and compare it with the small cohesion which the firmest cement can give to such an edifice, we shall be convinced that its parts are not kept together by the force of the cement; the stability of the whole is the result of the just balance and equilibration of all its parts. The principles of this equilibration we shall now exhibit: premising, that they are founded upon the hypothesis of the ponderating matter pressing upon the voussoirs in the vertical direction, and that we here pay no regard to any small pressure in other directions which may be occasioned by filling up the spandrels with rubbles, &c.

200. If we conceive a vault or arch in equilibrio to be composed of a series of very small hard spheres, of polished surfaces, touching one another, and the centres joined by right lines, being in fact an inverted simple catenary; and that the spheres are so connected that they cannot yield to any other impulsion than that of gravity, it is plain the equilibrium will not be disturbed by such inversion of the curve; no part of the curve will be thrust outward or inward by other parts, but the whole will be supported if the feet are firmly fixed: for, since

the situation of the points of the catenary is the same, and their respective inclinations to the horizon, whether in the original or inverted position, the curve being in both instances in a vertical plane, and the action of gravity the same, the relative conditions are the same, and therefore the arch will keep its figure unchanged as well in one situation as the other.

Thus also when arches of other figures are supported, it is, as Dr. David Gregory justly observed, "*because in their thickness some catenaria is included:*" as, for example, if a quantity of heavy but flexible materials fixed only at the points P, P′ (fig. 12. pl. IX.), when left to the sole action of gravity, should arrange itself into the shape of the dotted lines PADBP′ in a vertical plane; then, if the whole were completely inverted, the points P, P′, being still fixed, the equilibrium would be retained notwithstanding the inversion: the arch with the matter above it would stand as in the upper part of the same figure; and the corresponding parts of the hanging and of the standing arch would be similarly situated with regard to the forces from which the equilibrium is derived. Here, too, there would be no point of contrary flexure in the *intrados;* and this we suppose throughout.

201. Hence what was deduced at Cor. 8. art. 197. may be transferred to the present case; that is, *the weight pressing upon any point* C *will be reciprocally as the radius of curvature at that point, and the square of the sine of the angle made by the curve, or its tangent, and the vertical.*

If therefore a weight, as a wall or mass of masonry, be incumbent on the intrados PVCP′ (fig. 12.) in a vertical plane, and all the parts be kept in equilibrio, then *the height* CI *on any point,* C, *is reciprocally as the radius of curvature, and cube of the sine of the angle in which the vertical cuts the curve in that point, or reciprocally as the radius of curvature, and directly as the cube of the secant of the curve's inclination to the horizon.*

For the weight on the indefinitely small portion of the curve ce being inversely as $\mathrm{R} \times \sin^2 d\mathrm{CI}$, or $\mathrm{R} \times \sin^2 d'\mathrm{CI}$ (R being the radius of curvature at C), and the weight of the column $cei\mathrm{I}$, as $\mathrm{IC} \cdot wr = \mathrm{IC} \cdot ce \cdot \sin d'\mathrm{CI} = \mathrm{IC} \cdot \sin d'\mathrm{CI}$; because ce is given: therefore $\mathrm{IC} \cdot \sin d\mathrm{CI} \propto \frac{1}{\mathrm{R} \cdot \sin^2 d\mathrm{CI}}$, and $\mathrm{IC} \propto \frac{1}{\mathrm{R} \cdot \sin^3 d\mathrm{CI}} \propto \frac{\sec^3 k\mathrm{C}d'}{\mathrm{R}}$.

202. This method of deducing the fundamental theorem of equilibrated arches from inverted catenaries either simple or complex, suggests an easy popular mode of ascertaining the shape of a duly balanced arch, when the span, height and shape of the roadway are given; a method which, we believe, was first practised by M. de la Hire. Let it be proposed, for example, to determine the form of an arch which shall have the span PP′,

and the height EV (fig. 12) and which shall have a roadway in the position ADB above it. Let the figure PADBP′ be inverted as represented by the dotted lines: then let a chain of suitable length and uniform thickness be hung at the points P, P′, so that when it assumes the shape of the simple catenary it shall hang a little below the lowest point V. Divide PP into 20, 30, or more equal parts, as in the points 1, 2, 3, &c. and let vertical lines through the points 1, 2, 3, &c. intersect the curve PVP′ in the corresponding points 1, 2, 3, &c. Then take pieces of another uniform chain whose links are tolerably numerous, and hang on at the points 1, 2, 3, &c. of the chain suspended from P, P′. Cut or trim these pieces of chain till their lower ends all coincide with the inverted roadway ADB: the greater lengths which are hung on in the vicinity of P, P′, will draw down that part of the chain, and so cause the part which hung below V, to rise to its assigned position. This process will give us an arch of equilibration, but, as Dr. Robison remarks, some farther modifications may be necessary to make it exactly suit the specified purpose. It is a balanced arch for a bridge which is so loaded that the weight of the arch stones is to the weight of the matter with which the haunches and crown are loaded, as the weight of the chain PVP′, to the sum of the weights of all the little bits of chain, very nearly. But this proportion is not known beforehand; we must, therefore, proceed thus: Adapt to the curve produced in this way a thickness of the voussoirs as great as may be thought sufficient to ensure stability; then compute the weight of the voussoirs and the weight of the gravel, stones, &c. which fill up the haunches, &c. to the roadway. If the ratio of these two weights be the same with that of the corresponding weights of chain, we may rest satisfied with the curve now found: but if different, it may easily be calculated how much must be added to, or taken from, each piece of chain, in order to make these ratios equal; and thus shall we at length ascertain with sufficient accuracy the shape of the curve required, which may be readily transferred from the vertical wall, or framing, near ~~which~~ the chains were suspended, to any other situation.

We might now go on to deduce a few useful propositions in this theory, from the fundamental formula just exhibited: but as it may be more satisfactory to some persons to have this theorem deduced independently of the catenary, we shall demonstrate the same thing, from the consideration that the voussoirs are so many frustums of wedges whose sides are perpendicular to the intrados, each of which being urged by its incumbent weight, endeavours by this force to split the arch.

203. PROP. *The force of a voussoir depending on the magnitude of the angle formed by its sides, the impelling force, and the resistance to be overcome, is on the first account directly as the radius of curvature of the arch at that point, on the second as the square of the sine of the angle included between the tangent of the curve at the given point and the vertical passing through that point, and on the third, as the sine of the same angle.*

1. Let EABF, *eabf*, be two similar concentric curves (fig. 2. pl. X.), and AB, *ab*, two voussoirs similarly situated, whose sides perpendicular to the curve converge to the centre C. The forces of these voussoirs, considered as portions of wedges, are inversely as the sines of the half vertical angles (164, 165. cor. 1.), or, because each wedge occupies an equal portion of its respective arch, directly as the radii of curvature.

2dly, Let H*h* be the invariable breadth of the voussoirs on the arch *eabf*, G*g*H*h* the incumbent weight, which, since GH is supposed given, is as the breadth *hk*, or as the sine of the angle *h*H*k*: by the resolution of the force *g*H into two *h*H, HK, the latter is the force impelling the voussoir to split the arch, which, since *g*H is given, varies as the sine of H*g*K, or *h*H*k*: wherefore, the force impelling the voussoir is as the square of the sine of *h*H*k*.

3dly, The wedge impelled in a direction perpendicular to the curve tends to split the arch, and therefore to move one segment about the fulcrum *e*, the other about the fulcrum *f*. Hence the force of the voussoir acting on the levers H*f*, H*e*, being as either of the perpendiculars *f*P, *e*Q, is as the sine of the angle *f*CP or *h*H*k*.

We have supposed the centre of curvature of the arches at the points A, *a*, *h*H, to be at C: but this is merely to prevent the figure from being too complex, and makes no alteration in the nature of the demonstration.

COR. Hence, if the height of the wall incumbent on any point H of the intrados is inversely as the cube of the sine of *h*H*k* into radius of curvature at that point, or directly as cube of the secant of the angle formed by *h*H and the horizon, and inversely as the radius of curvature, all the voussoirs will tend to split the arch with equal forces, and will be in perfect equilibrium with each other.

204. PROP. *Given the intrados, to find the corresponding extrados.*

In fig. 12. pl. IX. where PVP′ is the intrados and ADB the required extrados, V and D being the respective vertices, put A = DV, $x = k$V the abscissa, $y = k$C the ordinate to any point C, $a =$ CI the requisite altitude at that point, and $z =$ the arc

VC: then, by the last prop. and cor. $a \propto \frac{\sec^3 d'ck}{\mathrm{R}}$, while, by similar triangles, we have $\dot{y} : \dot{z} :: 1$ (radius : $\frac{\dot{z}}{\dot{y}} = \sec d'ck$; and therefore $a \propto \dot{z}^3 \div \mathrm{R}\dot{y}^3$. But in every curve whose ordinate is referred to an axis the radius of curvature is $\mathrm{R} = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$. (Simpson's Fluxions, vol. I. p. 74; Dealtry's Fluxions, art. 102.); whence $a \propto \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3}$, or $a = \mathrm{C} \cdot \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^8}$, where C is a constant quantity, the value of which may be determined by taking the expression for the given line A at the vertex of the curve.

Example. To find the extrados of a circular arch. Let O be the centre of the circle, and the several lines in fig. 3. pl. X. being represented as above, while $\mathrm{PO} = \mathrm{OV} = r$; then, if we make $\dot{y}$ invariable, we have $x = r - \sqrt{(r^2 - y^2)}$, $\dot{x} = \frac{y\dot{y}}{\sqrt{(r^2 - y^2)}}$, and $\ddot{x} = \frac{r^2 y^2}{(x^3 - y^2)^{\frac{3}{2}}}$. Hence CI, or a, or $\mathrm{C} \cdot \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3}$, becomes $\mathrm{C} \cdot \frac{\ddot{x}}{\dot{y}^2}$ $= \frac{\mathrm{C}r^2}{(r^2 - y^2)^{\frac{3}{2}}}$: and this when $y = 0$, or at the vertex where $\mathrm{A} = a$, gives $\mathrm{A} = \frac{\mathrm{C}}{r}$, or $\mathrm{C} = \mathrm{A}r$: and consequently the general value of CI, or $a = \frac{\mathrm{C}r^2}{(r^2 - y^2)^{\frac{3}{2}}}$, becomes $a = \mathrm{A} \cdot \frac{r^3}{(r^2 - y^2)^{\frac{3}{2}}} = \mathrm{DV} \cdot \frac{\mathrm{OV}^3}{\mathrm{O}k^3} = \mathrm{A}$ × cube sec. of elevation.

Hence this simple calculus: to the logarithm of A add thrice the log. sec. of elevation, the sum rejecting 3 (in the index) is the log. of a.

Hence also flows a neat construction: draw the vertical CS intersecting the horizontal diameter in S: join C and the centre O, and on OC let fall from S the perpendicular ST: draw TS parallel to PO, and join ZO: make $\mathrm{C}u = \mathrm{DV}$, and parallel to ZO draw ux; then is $\mathrm{C}x = \mathrm{CI}$. For, by the similar triangles we have

$$\mathrm{CO} : \mathrm{CS} :: \mathrm{CS} : \mathrm{CT} = \frac{\mathrm{CS}^2}{\mathrm{CO}}$$

$$\mathrm{CS} : \mathrm{CT} :: \mathrm{CT} : \mathrm{CZ} = \frac{\mathrm{CS}^4}{\mathrm{CO}^2 \cdot \mathrm{CS}} = \frac{\mathrm{CS}^3}{\mathrm{CO}^2}$$

and $\mathrm{DV} : \mathrm{CI} :: \mathrm{C}u : \mathrm{C}x :: \mathrm{CZ} : \mathrm{CO} :: \frac{\mathrm{CS}^3}{\mathrm{CO}^2} : \mathrm{CO} :: \mathrm{CS}^3 : \mathrm{CO}^3 :: \mathrm{O}k^3 : \mathrm{OV}^3$.

The curve DIA runs up to an infinite height above the spring of the arch, if it be semicircular; and this must evidently be the case with every curve that springs at right angles to the horizontal line. But for a moderate distance on each side the vertex the extrados will assume a shape that may answer tolerably well for a roadway.

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205. When the intrados is elliptic the extrados is of the same kind as for the circle: but when the longer axe is horizontal the ellipsis may take a straight line at top more safely than the circle, because the extrados runs out farther from the vertex, before it takes its rapid rise. In the cycloidal arch of equilibration, the extrados resembles that of the circle and flat ellipse; approaching, however, more nearly to a right line about the vertex, and extending farther from it before it bends upwards. If the intrados is parabolic the extrados is an equal and similar curve, at any given distance from the intrados: but in the hyperbola the extrados approaches continually nearer to the intrados. Hence, in some cases it may be quite convenient to have arches segments near the vertex of very large parabolas. The reader may find the investigations of these particulars in Dr. Hutton's Principles of Bridges, or in the first volume of his "Tracts," lately published: we merely glance at them here.

206. **Prop.** If AEDIB (fig. 4. pl. X.) *be the extrados and* PVP′ *the intrados of an equilibrated arch, and if any number of vertical lines* EF, DV, IC, *&c. be drawn from the one curve to the other, and these lines be divided in a given ratio in the points* a, e, d, i, b, *then the curve drawn through these points of division will also be a proper extrados, the mass contained between it and the intrados being duly balanced, as well as that comprised between* AEDIB *and* PVCP′.

For, since the whole is kept in equilibrio by the vertical pressures of the superincumbent mass on the intrados, the points F, C, V, &c. are sustained in equilibrio by the pressures of the parts EF, DV, IC, &c. bearing upon them: if, therefore, these lines be divided in *e*, *d*, *i*, &c. so that EF : *eF* :: DV : *dv* :: IC : *ic*, &c. all in the same constant ratio, then *aedib* being considered as the extrados, the arch will still be in equilibrium; because the load on the intrados being every where lessened in a constant ratio, its tendency to break the arch will be every where in a constant ratio to its preceding tendency to cause a rupture, and the equilibrium can be no more destroyed in the one case than in the other. And the same kind of reasoning would apply if *aedib* were above AEDIB, or if the weight and density of the materials between the extrados and intrados should be changed *throughout* in any constant ratio.

Cor. Hence we may in many cases give the extrados a pretty regular and practicable form, by diminishing the thickness over the crown: and hence appears one great advantage of iron as a material for bridges, since its requisite thickness at the crown is vastly less than that of stone.

207. **Prop.** *Having given the extrados of an equilibrated arch, to find the intrados.*

Let ADB (fig. 5. pl. X.) be the extrados proposed to which the intrados PVP′ is to be adapted; OV being the common axis of both curves: from *c* and C, corresponding points equidistant from the axis, draw the ordinates *ch*, CH. Put DV (the thickness of the arch at the crown) $= a$, D$h = \dot{\mathrm{X}}$, VH$= x$, the equal ordinates $ch, = \mathrm{CH} = y$, and the arch VC$= z$. Then, by the general form (204,) $\mathrm{C}c \propto \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3} = \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3} \cdot \mathrm{C}$, where C is a constant quantity found by taking the actual value of C*c* in V the vertex of the curve. But it is manifest that C*c*$=$DV$+$VH $-$D$h = a + x - \dot{\mathrm{X}}$; consequently $a + x - \dot{\mathrm{X}} = \mathrm{C} \cdot \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^3} = \frac{\mathrm{C}}{\dot{y}} \times$ flux. of $\frac{\dot{x}}{\dot{y}}$. If, then, we substitute the true value of $\dot{\mathrm{X}}$ in terms of y (which is given because the form of the extrados is known), the equations thence resulting will contain only x and y with their first and second fluxions, and known quantities; and from this the real relation of x and y must be struck out by such means as seem most naturally to apply to any proposed instance.

This, however, in many cases will be a matter of considerable difficulty: we shall here, therefore, solely trace the process in the most useful instance, which, happily, admits of a comparatively simple investigation. We advert to the case in which *the extrados is a straight horizontal line*, which shall be now considered.

Retaining the same notation, we have D$h = \dot{\mathrm{X}} = 0$, and consequently $a + x = \frac{\mathrm{C}}{\dot{y}} \times$ flux. of $\frac{\dot{x}}{\dot{y}}$. Assume $\dot{y} = \frac{\dot{x}}{u}$, whence $u = \frac{\dot{x}}{\dot{y}}$, and $\frac{\mathrm{C}}{\dot{y}} \times$ flux. of $\frac{\dot{x}}{\dot{y}} = \frac{\mathrm{C}u\dot{u}}{\dot{x}}$ that is, $a + x = \frac{\mathrm{C}u\dot{u}}{\dot{x}}$, and of course $a\dot{x} + x\dot{x} = \mathrm{C}u\dot{u}$: taking the fluents of this we have $x^2 + 2ax = \mathrm{C}u^2$, and $u = \sqrt{\frac{x^2 + 2ax}{\mathrm{C}}}$. But because $\dot{y} = \frac{\dot{x}}{u}$ it is also $= \dot{x} \div \sqrt{\frac{x^2 + 2ax}{\mathrm{C}}} = \dot{x}\sqrt{\frac{\mathrm{C}}{x^2 + 2ax}}$. The fluent of this expression is $y = \sqrt{\mathrm{C}} \times$ hyp. log. $(2x^2 + 2ax + 2\sqrt{(x^2 + 2ax)}$, Now at the vertex where $x = 0$, we have $y = \sqrt{\mathrm{C}} \times$ hyp, log. $2a$, so that the corrected fluent is

$$y = \sqrt{\left\{ \mathrm{c} \times \text{hyp.log.} \frac{x + a + \sqrt{(x^2 + 2ax)}}{a}. \right\}}$$

We have yet to ascertain the constant quantity c, in order to which we may proceed thus: when H arrives at O we have $x = vo = h$, and $y = \mathrm{OP} = s$; substituting these in the last equation it becomes $h = \sqrt{\mathrm{c}} \times$ hyp. log. $\frac{s + a + \sqrt{(ss + 2as)}}{a}$ and consequently $\sqrt{\mathrm{c}} = h \div$ hyp. log. $s + a + \frac{1}{a}\sqrt{s^2 + 2as}$. Hence, then, we at length obtain this general value of y, that is, $y = h \times$ hyp. log. $\frac{a + x + \sqrt{(2ax + x^2)}}{a} \div$ hyp. log. $\frac{a + s + \sqrt{(2as + s^2)}}{a}$.

Comparing these equations with equa. (3) art. 198. it will be seen that when $\sqrt{\mathrm{c}} = a$, the intrados PVP′ is the catenary, although the extrados is a horizontal right line: but this will require an immense thickness at the crown; for if $x = 40$, and $y = 50$, we shall have $a = 36{\cdot}88$, which is more than $\frac{1}{3}$ the span of the arch, and more than $\frac{9}{10}$ of its height.

As an example of the use of the preceding formulæ, we subjoin a table calculated by Dr. Hutton, for an arch whose span is 100, height 40, and thickness at the crown 6: which will answer for any other arch whose span, height, and thickness, are related to each other in like manner, by changing all the values of DH and CH in a constant ratio.

HC	DH	HC	DH	HC	DH
0	6·000	21	10·381	36	21·774
2	6·035	22	10·858	37	22·948
4	6·144	23	11·368	38	24·190
6	6·324	24	11·911	39	25·505
8	6·580	25	12·489	40	26·894
10	6·914	26	13·106	41	28·364
12	7·330	27	13·761	42	29·919
13	7·571	28	14·457	43	31·563
14	7·834	29	15·196	44	33·299
15	8·120	30	15·980	45	35·135
16	8·430	31	16·811	46	37·075
17	8·766	32	17·693	47	39·126
18	9·168	33	18·627	48	41·293
19	9·517	34	19·617	49	43·581
20	9·934	35	20·665	50	46·000

208. Prop. *To determine the magnitude of the piers, or abutments, that they may sustain the arch in equilibrio, independently of other arches.*

In order to give a solution to this problem we must assume some particulars as having been determined by adequate experiments and admeasurement: for we do not consider the piers as prisms standing upon their bases and resisting the pressure of the arches, though upon such an hypothesis it would be easy to lay down rules for the determination of the centres of gravity both of the arch and the piers; but, since the stones, &c. of the wall above the voussoirs are bonded in with those of the pier, the pier will by these means become augmented, and the weight of the arch diminished. We, therefore, regard the piers as extending to the *joints of fracture* (art. 199.), and that portion of the arch which is comprised between those joints as a ponderating body resting in a state of equilibrium upon those joints as upon two inclined planes. Let, then, FS, F'S' (fig. 1. pl. X.), be the joints of fracture, G the centre of gravity of the pier QTSFBP, g that of the half arch DEFS, and let go the perpendicular from g upon FS be produced till it meets the horizontal line Q'Q in I: draw GH perpendicular to QQ' and Qh perpendicular to gI: and let the mass of the semi-arch DESF be represented by A, that of the pier by P, and the force of gravity by g: the weight of the former will then be gA, and of the latter gP. The magnitude of the pier is generally computed on the supposition that the pressure of the arch has a tendency to make the pier turn upon Q as a centre or fulcrum; and this hypothesis is often consistent with fact: but when the height CD is small compared with the span, the weight of the arch has a strong tendency to make the pier slide along the horizontal line PI; we shall, therefore, state the conditions of equilibrium on this supposition also. First, supposing the pier solely capable of turning upon Q as a centre of rotation: then will the case be the same as if the body DESF whose weight is gA, by pressing upon the face ES, tended to move the mass FSTQP upon the fulcrum Q. But (art. 156.) the weight gA is to its pressure upon FS, as sine of angle included between ED and FS, to sine of angle ERQ, that is $\sin \text{I} : \text{rad.} :: g\text{A} : \frac{g\text{A}}{\sin \text{I}}$ = pressure of half arch upon the joint of fracture. Now g being the centre of gravity of the half arch, the pressure it occasions is exerted in the direction gI: and G being the centre of gravity of the pier, the force resulting from its weight acts in the vertical direction GH; therefore in the case of equilibrium, we must, by the nature of the lever, have, pressure on

$\text{SF} \times Qh =$ weight of pier $\times$ QH, that is, $\frac{gA}{\sin I} \cdot Qh = gP \cdot QH$, whence we readily obtain

$$\text{(I.)} \quad \ldots \quad \frac{A}{P} = \frac{QH}{Qh} \cdot \sin I.$$

This equation comprises the conditions of the equilibrium of rotation (art. 19.) about the point Q; and we may find by its means any one of the five quantities it contains, when the other four are given.

When the arch springs vertically from rectangular piers, whose height and breadth are H and B respectively, the preceding theorem reduces to

$$\text{(i.)} \quad \ldots \quad HA \cot I = SB + \tfrac{1}{2}HB^2.$$

In the second case, in which we suppose the pier may slide along in the horizontal direction, let f be a force which is exerted horizontally in opposition to the motion of translation: then fP acting in the direction Ik must counterbalance $\frac{gA}{\sin I}$ acting along hI. Here hI being to Ik as radius to cos I, we shall have, rad : cos I $::\frac{gA}{\sin I.}:fP$; whence $gA \cdot \frac{\sin I.}{\cos I.} = fP$, and for an equation including the conditions of the equilibrium of translation we have

$$\text{(II.)} \quad \ldots \quad \frac{A}{P} = \frac{f}{g} \cdot \frac{\sin I.}{\cos I.} = \frac{f}{g} \tan I.$$

As to the position of the joints of rupture, and of the centres of gravity of the semi-arch and pier, they may in most cases be determined with tolerable accuracy, thus: having drawn on pasteboard the arch and proposed pier, upon a pretty large scale, and described the voussoirs of the arch, of the intended thickness, draw from the middle of the key voussoir a tangent to the intrados, and produce it till it again meets the middle of a voussoir, as at F, from which point draw FS perpendicular to the intrados; it will be nearly the position of a joint of fracture. Next, cut the pasteboard through at the several outlines, and find by some of the methods described in art. 106, the centres of gravity of the two parts DESF, STQPF. With regard to the ratio of A to P, it may always be found pretty nearly, either by weighing or measuring the pieces of pasteboard which represent them; and the distances QH, Qh, and angle I, will be ascertained by the construction. If, when these values of A, P, &c. are introduced into the equations, the first members are less than the second, the piers will be large enough to ensure the equilibrium: if otherwise, some of these particulars must be changed until that takes place.

This mode of considering the subject suggests, that to diminish the thrust of the arch, or increase the stability of the pier, the commencement of the flanks ought to be loaded; and that the thickness of the voussoirs near the key ought to be lessened considerably: in short, to make the arch, instead of having a uniform thickness throughout its whole extent, to be very thick at its origin, and at the key to be no thicker than is necessary to resist the pressure of the flanks: for by such a procedure a part of the force which tends to move the pier is thrown upon that which resists being overturned, and the latter will gain a great advantage in point of stability.

208 A. **PROP.** *Given the thickness at the crown of an arch, the relation between the intrados and the extrados, and the height of a rectangular pier; to find the height of the pier, and the height or versed sine,* CD, *of the arch, so that the materials employed shall be a minimum.*

A *general* solution of this problem is not a matter of easy accomplishment. But the principles to be employed may be exemplified, in a comparatively simple case. Let that be taken, then, in which the extrados and the intrados are both parabolas. The quantity which is required to be a *minimum* will be $\mathrm{A}+\mathrm{HB}$. If the intrados be a parabola represented by the equation $y^2 = \frac{b^2}{a}x$, and the extrados another parabola defined in like manner by $'y^2 = \frac{(t+b)^2}{a+t}x'$, the thickness at the crown and at the spring being t: then we shall have

$$\mathrm{A} = \tfrac{2}{3}(b+t)(a+t) - \tfrac{2}{3}ab = \tfrac{2}{3}t(a+b+t).$$

The equation for the minimum, therefore, will be

$$\text{Flux.} \left\{\tfrac{2}{3}t(a+b+t) + \mathrm{HB}\right\} = 0.$$

This being effected, supposing a and B variable, gives

$$-\dot{\mathrm{B}} = \frac{2t}{3\mathrm{H}}\dot{a}.$$

Regarding the arch as a wedge without friction reposing upon the imposts, the equation for the equilibrium of rotation, is, as we have just seen,

$$\mathrm{HA}\cot \mathrm{I} = \mathrm{SB} + \tfrac{1}{2}\mathrm{HB}^2.$$

But we have $\cot \mathrm{I} = \frac{\dot{y}}{\dot{x}} = \frac{b}{2a}$.

Substituting, therefore, the values of cot I and A, and taking the fluxions, there results

$$2bt\mathrm{H}\dot{a} = 4a(at+bt+t^2)\dot{\mathrm{B}} + 4t\mathrm{B}(2a+b+t)\dot{a} + 3\mathrm{HB}^2\dot{a} + 6\mathrm{H}a\mathrm{B}\dot{\mathrm{B}}.$$

Substituting, also, in this equation for $\dot{\mathrm{B}}$ the preceding value of it, we have

$$2bt\mathrm{H} = -\frac{8at^2}{3\mathrm{H}}(a+b+t) + 4t\mathrm{B}(2a+b+t) + 3\mathrm{HB}^2 - 4at\mathrm{B}:$$

and this, combined with the equation of equilibrium, will give the quantities B at a.

It might, perhaps, be imagined that, *cæteris paribus*, there exists a height a which will ensure a *minimum* for B; but it is not so. B diminishes as a augments.

If it be required to determine the dependance of the thrust upon the variation of a, and to ascertain if there be a *minimum*, the fluxion of A cot I must be put $= 0$. In the same example the quantity to throw into fluxions is $\frac{1}{3}bt\left(1+\frac{b}{3a}+\frac{t}{3a}\right)$, a quantity which diminishes in proportion as a increases, and which has for a limit $\frac{1}{3}bt$; such is the *minimum* thrust, which obtains when a is infinite.

If we would know when the tangential thrust, or the pressure on the imposts, is a *minimum*, we shall then have fluxion of $\frac{\mathrm{A}}{\sin \mathrm{I}.} = 0$, or $\left(\dot{\mathrm{A}}\frac{\dot{y}}{}\right)^{\cdot} = 0$, or fluxion of $\frac{2}{3}t\,(a+b+t)\sqrt{\frac{4a^2+b^2}{4a^2}} = 0$; from which there will be found

$$a = \mp\sqrt{\tfrac{1}{4}b^2(b+t)}.$$

III. *Of Domes.*

209. DEF. A *dome* or cupola is a roof of a spherical, spheroidal, or conoidal form, resembling a bell, or an inverted cup. It is a species of arch, or vaulting, the erection of which, like the former, is a scientific art depending upon the principles of equilibrium.

The most ample account of the theory of domes we have yet met with is given by Dr. Robison, in the Supplement to the Encyclopedia Britannica; an abstract of which will be here given, chiefly in the doctor's own words.

PROP. *To determine the thickness of a dome vaulting when the curve is given, or the curve when the thickness is given.*

"Let BbA (fig. 7. pl. X.) be the curve which produces the dome by revolving round the vertical axis AD. We shall suppose this curve to be drawn through the middle of all the arch-stones, and that the coursing or horizontal joints are every where perpendicular to the curve. We shall suppose (as is always the case) that the thickness KL, HI, &c. of the arch-stones is very small, in comparison with the dimensions of the arch. If we consider any portion HAh of the dome, it is plain that it presses on the course, of which HL is an archstone, in a direction bc perpendicular to the joint HI, or in the direction of the next superior element βb of the curve. As we proceed downwards, course after course, we see plainly that this direction must change, because the weight of each course is superadded to that of the portion above it, to complete the pressure on the course below. Through B draw the vertical line BCG, meeting βb, produced in C. We may take bc to express the pressure of all that is above it, propagated in this direction to the joint KL. We may also suppose the weight of the course HL united in b, and acting on the vertical. Let it be represented by bF. If we form the parallelogram bFGC, the diagonal bG will represent the direction and intensity of the whole pressure on the joint KL. Thus it appears that this pressure is continually changing its direction, and that the line, which will always coincide with it, must be a curve concave downward. If this be precisely the curve of the dome, it will be an equilibrated vaulting; but so far from being the strongest form, it is the weakest, and it is the limit to an infinity of others, which are all stronger than it. This will appear evident, if we suppose that bG does not coincide with the curve AbB, but passes without it. As we suppose the arch-stones to be exceedingly thin from inside to outside, it is plain that this dome cannot stand, and that the weight of the upper part will press it down, and spring the vaulting outwards at the joint KL. But let us suppose, on the other hand, that bG falls within the curvilineal element bB. This evidently tends to push the arch-stone inward, toward the axis, and would cause it to slide in, since the joints are supposed perfectly smooth and slipping. But since this takes place equally in every stone of this course, they must all abut on each other in the vertical joints, squeezing them firmly together. Therefore, resolving the thrust bG into two, one of which is perpendicular to the joint KL, and the other parallel to it, we see that this last thrust is withstood by the vertical joints all around, and there remains only the thrust in the direction of the curve. Such a dome must therefore be firmer than an equilibrated dome, and cannot be so easily broken by overloading the upper part. When the curve is concave upwards, as in the lower part of the figure, the line bC always falls below bB, and the point C below B. When the curve is concave downwards, as in the upper part of the figure, b'C$'$ passes above, or without bB. The curvature may be so abrupt, that even b'G$'$ shall pass without b'B$'$, and the point G$'$ is above B$'$. It is also evident that the force which thus binds the stones of a horizontal course together, by pushing them towards the axis, will be greater in flat domes than in those that are more convex; that it will be still greater in a cone; and greater still in a curve whose convexity is turned inwards;

for in this last case the line *b*G will deviate most remarkably from the curve. Such a dome will stand (having polished joints) if the curve springs from the base with any elevation, however small; nay, since the friction of two pieces of stone is not less than half of their mutual pressure, such a dome will stand, although the tangent to the curve at the bottom should be horizontal, provided that the horizontal thrust be double the weight of the dome, which may easily be the case if it do not rise high.

" Thus we see that the stability of a dome depends on very different principles from that of a common arch, and is in general much greater. It differs also in another very important circumstance, viz. that it may be open in the middle: for the uppermost course, by tending equally in every part to slide in towards the axis, presses all together in the vertical joints, and acts on the next course like the key-stone of a common arch. Therefore an arch of equilibration, which is the weakest of all, may be open in the middle, and carry at top another building, such as a lantern, if its weight do not exceed that of the circular segment of the dome that is omitted. A greater load than this would indeed break the dome, by causing it to spring up in some of the lower courses; but this load may be increased if the curve is flatter than the curve of equilibration: and any load whatever, which will not crush the stones to powder, may be set on a truncate cone, or on a dome formed by a curve that is convex toward the axis; provided always that the foundation be effectually prevented from flying out, either by a hoop, or by a sufficient mass of solid pier on which it is set."

" We have seen that if *b*G, the thrust compounded of the thrust *b*C, exerted by all the courses above HILK, and if the force *b*F, or the weight of that course, be every where coincident with *b*B, the element of the curve, we shall have an equilibrated dome; if it falls within it, we have a dome which will bear a greater load; and if it falls without it, the dome will break at the joint. We must endeavour to get analytical expressions of these conditions. Therefore draw the ordinates *b∂b″*, BDB″, *cd*c″. Let the tangents at *b* and *b″* meet the axis in M, and make MO, MP, each equal to *bc*, and complete the parallelogram MONP, and draw OQ perpendicular to the axis, and produce *b*F, cutting the ordinates in E and *e*. It is plain that MN is to MO as the weight of the arch HA*h* to the thrust *bc* which it exerts on the joint KL (this thrust being propagated through the course of HILK); and that MQ, or its equal *be*, or *∂d*, may represent the weight of the half AH.

" Let AD be called x, and DB be called y. Then $be = \dot{x}$, and $ec = \dot{y}$ (because bc is in the direction of the element βb).

It is also plain, that if we make $\dot{y}$ constant, BC is the second fluxion of x, or BC $= \ddot{x}$, and *be* and BE may be considered as equal, and taken indiscriminately for $\dot{x}$. We have also *bc* $= \surd(\dot{x}^2+\dot{y}^2)$. Let d be the depth or thickness HI of the arch-stones. Then, $d\ \surd(\dot{x}^2+\dot{y}^2)$ will represent the trapezium HL; and since the circumference of each course increases in the proportion of the radius y, $dy\ \surd(\dot{x}^2+\dot{y}^2)$ will express the whole course. If $\int$ be taken to represent the sum or aggregate of the quantities annexed to it, the formula will be analogous to the fluent of a fluxion, and $\int dy\ \surd(\dot{x}^2+\dot{y}^2)$ will represent the whole mass, and also the weight of the vaulting, down to the joint HI. Therefore we have this proportion $\int dy\ \surd(\dot{x}^2+\dot{y}^2) : dy\ \surd(\dot{x}^2+\dot{y}^2) = be : b\text{F}, = be : \text{CG}, = \delta d : \text{CG}, \doteq \dot{x} : \text{CG}$. Therefore CG $= \frac{dy\dot{x}\surd(\dot{x}^2+\dot{y}^2).}{\int dy\surd(\dot{x}^2+\dot{y}^2).}$

"If the curvature of the dome be precisely such as puts it in equilibrium, but without any mutual pressure in the vertical joints, this value of OG must be equal to CB, or to $\ddot{x}$, the point G coinciding with B. This condition will be expressed by the equation $\frac{dy\dot{x}\surd(\dot{x}^2+\dot{y}^2)}{\int dy\surd(\dot{x}^2+\dot{y}^2)} = \ddot{x}$, or, more conveniently, by $\frac{dy\surd(\dot{x}^2+\dot{y}^2)}{\int dy\surd(\dot{x}^2+\dot{y}^2)} = \frac{\ddot{x}}{\dot{x}}$. But this form gives only a tottering equilibrium, independent of the friction of the joints and the cohesion of the cement. An equilibrium, accompanied by some firm stability, produced by the mutual pressure of the vertical joints, may be expressed by the formula $\frac{dy\surd(\dot{x}^2+\dot{y}^2)}{\int dy\surd(\dot{x}^2+\dot{y}^2)} > \frac{\ddot{x}}{\dot{x}}$, or by $\frac{dy\surd(\dot{x}^2+\dot{y}^2)}{\int dy\surd(\dot{x}^2+\dot{y}^2)} = \frac{\ddot{x}}{} + \frac{\dot{t}}{t}$, where t is some variable positive quantity, which increases when x increases. This last equation will also express the equilibrated dome, if t be a constant quantity, because in this case $\frac{\dot{t}}{t}$ is $= o$.

"Since a firm stability requires that $\frac{dy\dot{x}\surd(\dot{x}^2+\dot{y}^2)}{\int dy\surd(\dot{x}^2+\dot{y}^2)}$ shall be greater than $\ddot{x}$, and CG must be greater than CB; Hence we learn, that figures of too great curvature, whose sides descend too rapidly, are improper. Also, since stability requires that we have $\frac{dy\dot{x}\surd(\dot{x}^2+\dot{y}^2)}{\ddot{x}}$ greater than $\int dy\surd(\dot{x}^2+\dot{y}^2)$, we learn that the upper part of the dome must not be made very heavy. This, by diminishing the proportion of *b*F to *b*C, diminishes the angle *cbG*, and may set the point G above B, which will infallibly spring the dome in that place. We see here also, that the algebraic analysis expresses that peculiarity of dome-vaulting, that the weight of the upper part may even be suppressed.

"The fluent of the equation $\frac{dy\sqrt{(\dot{x}^2+\dot{y}^2)}}{\int dy\sqrt{(\dot{x}^2+\dot{y}^2)}} = \frac{\ddot{x}}{\dot{x}} + \frac{\dot{t}}{t}$ is most easily found. It is $\text{L} \int dy\sqrt{(\dot{x}^2+\dot{y}^2)} = \text{L}\dot{x} + \text{L}t$, where L is the hyperbolic logarithm of the quantity annexed to it. If we consider $\dot{y}$ as constant, and correct the fluent so as to make it nothing at the vertex, it may be expressed thus, $\text{L} \int dy\sqrt{(\dot{x}^2+\dot{y}^2)} - \text{L}a = \text{L}\dot{x} - \text{L}\dot{y} + \text{L}t$. This gives us $\text{L} \int \frac{dy\sqrt{(\dot{x}^2+\dot{y}^2)}}{a} = \text{L}\,\frac{\dot{x}}{\dot{y}}t$, and therefore $\int \frac{dy\sqrt{(\dot{x}^2+\dot{y}^2)}}{a} = t\,\frac{\dot{x}}{\dot{y}}$.

"This last equation will easily give us the depth of vaulting, or thickness d of the arch, when the curve is given. For its fluxion is $\frac{dy\sqrt{(\dot{x}^2+\dot{y}^2)}}{a} = \frac{\dot{t}\dot{x}+t\ddot{x}}{\dot{y}}$, and $d = \frac{a\dot{t}\dot{x}+at\ddot{x}}{y\dot{y}\sqrt{(\dot{x}^2+\dot{y}^2)}}$, which is all expressed in known quantities; for we may put in place of t any power or function of x or of y, and thus convert the expression into another, which will still be applicable to all sorts of curves.

"Instead of the second member $\frac{\ddot{x}}{\dot{x}} + \frac{\dot{t}}{t}$ we might employ $\frac{p\ddot{x}}{\dot{x}}$, where p is some number greater than unity. This will evidently give a dome having stability; because the original formula $\frac{dy\dot{x}\sqrt{(\dot{x}^2+\dot{y}^2)}}{\int dy\sqrt{\dot{x}^2+\dot{y}^2)}}$ will then be greater than $\ddot{x}$. This will give $d = \frac{pa\dot{x}^{p-1}\ddot{x}}{y\dot{y}^p\sqrt{(\dot{x}^2+\dot{y}^2)}}$. Each of these forms has its advantages when applied to particular cases. Each of them also gives $d = \frac{a\ddot{x}}{y\dot{y}\sqrt{(\dot{x}^2+\dot{y}^2)}}$ when the curvature is such as is in precise equilibrium. And, lastly, if d be constant, that is, if the vaulting be of uniform thickness, we obtain the form of the curve, because then the relation of $\ddot{x}$ to $\dot{x}$ and to $\dot{y}$ is given.

"The chief use of this analysis is to discover what curves are improper for domes, or what portions of given curves may be employed with safety. Domes are generally built for ornament; and we see that there is great room for indulging our fancy in the choice. All curves which are concave outwards will give domes of great firmness: they are also beautiful. The Gothic dome, whose outline is an undulated curve, may be made abundantly firm, especially if the upper part be convex and the lower concave outwards.

" The chief difficulty in the case of this analysis arises from the necessity of expressing the weight of the incumbent part, or $\int dy\sqrt{(\dot{x}^2+\dot{y}^2)}$. This requires the measurement of the conoidal surface, which, in most cases, can be had only by approximation by means of infinite serieses. We cannot expect that the generality of practical builders are familiar with this branch of mathematics, and therefore will not engage on it here; but content ourselves with giving such instances as can be understood by such as have that moderate mathematical knowledge which every man should possess who takes the name of engineer.

" The surface of any circular portion of a sphere is very easily had, being equal to the circle described with a radius equal to the chord of half the arch. This radius is evidently $=\sqrt{(\dot{x}^2+\dot{y}^2)}$.

" In order to discover what portion of a hemisphere may be employed (for it is evident we cannot employ the whole) when the thickness of the vaulting is uniform, we may recur to the equation or formula $\frac{dy\dot{x}\sqrt{(\dot{x}^2-\dot{y}^2)}}{\ddot{x}} = \int dy\sqrt{(\dot{x}^2+\dot{y}^2)}$. Let a be the radius of the hemisphere. We have $\dot{x} = \frac{ay\dot{y}}{\sqrt{(a^2-y^2)}}$, and $\ddot{x}=\frac{a^2-\dot{y}^2}{(a^2-y^2)^{\frac{1}{2}}}$. Substituting these values in the formula, we obtain the equation $y^2\sqrt{(a^2-y^2)}=\int\frac{a^2y\dot{y}}{\sqrt{(a^2-y^2)}}$. We easily obtain the fluent of the second member $=a^3-a^2\sqrt{(a^2-y^2)}$, and $y=a\sqrt{(-\frac{1}{2}+\sqrt{\frac{5}{4}})}$. Therefore if the radius of the sphere be 1, the half breadth of the dome must not exceed $\sqrt{(-\frac{1}{2}+\sqrt{\frac{5}{4}})}$, or 0·786, and the height will be ·618. The arch from the vertex is about 51° 49′. Much more of the hemisphere cannot stand, even though aided by the cement, and by the friction of the coursing joints. This last circumstance, by giving connection to the upper parts, causes the whole to press more vertically on the course below, and thus diminishes the outward thrust; but it at the same time diminishes the mutual abutment of the vertical joints, which is a great cause of firmness in the vaulting. A Gothic dome, of which the upper part is a portion of a sphere not exceeding 45° from the vertex, and the lower part is concave outwards, will be very strong, and not ungraceful.

" Persuaded that what has been said on the subject convinces the reader that a vaulting perfectly equilibrated throughout is by no means the best form, provided that the base is secured from separating, we think it unnecessary to give the investigation of that form, which has a considerable intricacy, and shall merely give its dimensions. The thickness is supposed uniform. The numbers in the first column of the table express the portion of the axis counted from the vertex, and those of the second column are the length of the ordinates.

AD	DB	AD	DB	AD	DB
0·4	100	610·4	1080	2990	1560
3·4	200	744	1140	3442	1600
11·4	300	904	1200	3972	1640
26·6	400	1100	1260	4432	1670
52·4	500	1336	1320	4952	1700
91·4	600	1522	1360	5336	1720
146·8	700	1738	1400	5756	1740
223·4	800	1984	1440	6214	1760
326·6	900	2270	1480	6714	1780
475·4	1000	2602	1520	7260	1800

" The curve formed according to these dimensions will not appear very graceful, because there is an abrupt change in its curvature at a small distance from its vertex; if, however, the middle be occupied by a lantern of equal or of smaller weight than the part whose place it supplies, the whole will be elegant, and free from this defect."

Professor Robison concludes with observing, that " The connexion of the parts arising from cement and from friction has a great effect on dome vaulting. In the same way as in common arches and cylindrical vaulting, it enables an overload on one place to break the dome in a distant place. But the resistance to this effect is much greater in dome vaulting, because it operates all round the overloaded part. Hence it happens that domes are much less shattered by partial violence, such as the falling of a bomb, or the like. Large holes may be broken in them without much affecting the rest; but, on the other hand, it greatly diminishes the strength which should be derived from the mutual pressure in the vertical joints. Friction prevents the sliding in of the arch-stones which produces this mutual pressure in the vertical joints, except in the very highest courses, and even there it greatly diminishes it. These causes make a great change in the form, which gives the greatest strength; and as their laws of action are but very imperfectly understood as yet, it is perhaps impossible, in the present state of our knowledge, to determine this form with tolerable precision. We see plainly, however, that it allows a greater deviation from the best form than the other kind of vaulting; and domes may be made to rise perpendicular to the horizon at the base, although of no great thickness; a thing which must not be attempted in a plane arch. The immense addition of strength which may be derived from hooping, largely compensates for all defects; and there is hardly any bounds to the extent to which a very thin dome-vaulting may be carried, when it is hooped or framed in the direction of the horizontal courses. The roof of the Halle du Bled at Paris is but a foot thick, and its diameter is more than 200, yet it appears to have abundant strength."*

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* They who wish farther to pursue this interesting class of investigations, may advantageously turn to the first vol. of Dr. Hutton's "Tracts," 8vo., the article Arch in the Pantologia, the instructive article Bridge in the Edinburgh Encyclopædia, Bossut "Recherches sur l'Equilibr Voûtes," and Berard's valuable "Statique des Voûtes."

END OF BOOK THE FIRST.

DYNAMICS.

BOOK II.

Introductory Definitions and Remarks.

210. DYNAMICS is that branch of Mechanics which has for its object the action of forces on solid bodies, when the result of that action is motion; and in which, since all motion occupies some portion of time, we introduce time into our investigations. This department of science presents a wide and varied field of discussion, and, when viewed in its full extent, exhibits many questions of considerable difficulty: we shall not attempt to give the whole of these, as such a procedure would draw us far beyond the limits, which must be assigned to this part of the work; but shall select those chiefly which appear necessary as a preparation to the knowledge of the powers and effects of machinery.

211. The sum of the material particles of which a body is composed, is what we denote by the word *Mass*. This mass depends on the volume of the body and that which we call *Density*. We have already observed (art. 10.) that density is directly as the quantity of matter, and inversely as the magnitude of the body: but it will not be improper to deduce concisely the general theorem which comprises this relation. To this end it must be considered that as all bodies are penetrated with a great number of void spaces or pores, their quantity of matter is not proportional to their volume; but under the same volume there will be more or less matter as the particles are nearer or further asunder; and we say that a body has a greater or less Density, according as there subsists a greater or less proximity between its moleculæ. Thus we say a body is more *dense* than another when in an equal volume the former contains more matter than the latter: we say, on the contrary, that it is less *dense* or more *rare* (for *density* and *rarity* are reciprocal qualities) when in an equal volume it comprises less matter. The density serves, therefore, to judge of the number of material particles when the volume is known: thus, we may regard the density as representing the number of equal moleculæ in a de-

terminate volume; as when, for example, we say that gold is 19 times denser than water, we wish to be understood that gold contains 19 times the number of particles that water does in the same space.

Since we represent the density as expressing the number of moleculæ in a determinate volume which we assume as the *unit of magnitude;* it is obvious that to obtain the mass, or the total number of moleculæ, of any body of which the magnitude is known, we must take the rectangle of the density and magnitude. Thus, if we represent generally the body or mass by B, its volume or magnitude by M, and its density by D, we shall have $B = MD$: whence it will be easy to compare the masses, the magnitudes, and the densities of bodies.

In similar bodies the masses are as the densities and cubes of the diameters, or depths, or lengths, or of any like linear dimensions. Hence, if L denote the lineal dimension, we readily deduce these general proportions:

$$B \propto MD \propto DL^3.$$

$$M \propto \frac{B}{D} \propto L^3.$$

$$D \propto \frac{B}{M} \propto \frac{B}{L^3}.$$

$$L \propto \sqrt[3]{\frac{B}{D}} \propto \sqrt[3]{M}.$$

212. *Force* according to our definition (art. 17.) is that which causes a change in the state of a body; or, it is that which either moves or tends to move a body: forces, as we there observed, are no further known to us than by their effects; it is only therefore by the effect any force produces that we can measure it. Now the effect of a force is to give to every material particle of a body a certain velocity: if, therefore, all the parts of a body receive the same velocity (as we suppose here), the effect of the moving cause has for its measure the product of the velocity into the number of moleculæ moved, or the product of the velocity and mass: *A force therefore is proportional to the velocity which it can impress on a known mass, and that mass conjointly.*

Def. *Momentum,* or *Quantity of Motion,* is the rectangle of the mass of a body and its velocity.

Consequently, *forces are measured by the quantities of motion they are capable of producing.*

Thus, if F denote the motive or moving force, B the body moved, and V the velocity imparted to it, we have $F \propto BV$. From this we deduce $V \propto \frac{F}{B}$, and $B \propto \frac{F}{V}$; therefore, 1. *The* *velocity of a body in motion, is as the moving force directly, and the mass inversely.* 2dly, *The body or mass is as the moving force directly, and the velocity inversely.*

If now, two bodies be represented by B, and b, the forces by which they are moved by F, and f, and the velocities imparted to them by V and v, we have $f \propto \mathrm{BV}$, and $f \propto bv$. Let B be supposed equal to b, then will $\mathrm{F} \propto \mathrm{V}$, and $f \propto v$: hence, *when the masses are equal, the moving forces are as the velocities.* Again, supposing $\mathrm{V} = v$, we have $\mathrm{F} \propto \mathrm{B}$ and $f \propto b$: therefore, *when the velocities are equal, the moving forces are as the masses.* Lastly, making $\mathrm{F} = f$, we have $\frac{1}{v} \propto \mathrm{B}$, and $\frac{1}{\mathrm{V}} \propto b$: consequently, *when the moving forces are equal, the velocities are in the inverse ratio of the masses.*

213. The several deductions in the preceding article depend, manifestly, upon our having assumed a just measure of moving force: we say, that forces are measured by the quantities of motion they are capable of producing, and that these quantities of motion are proportional to the products of the masses and their velocities. It follows, therefore, that when these products are equal, the quantities of motion, or the equivalent forces, are equal: but equal forces acting in opposite directions produce an equilibrium (art. 35.); so that if bodies which move with equal quantities of motion (according to our definition) in opposite directions, are in equilibrio after they meet, such a fact being proved will at the same time evince the truth of the foregoing deductions. In order, then, to shew the truth of this principle, we extract a paragraph from M. Laplace's *Exposition du Systême du Monde.* L. III. Ch. 3.

"The most simple case of the equilibrium of many bodies, is that of two physical points which rencounter with velocities equal and directly contrary. Their mutual impenetrability, that property of matter in virtue of which two bodies cannot occupy the same place at the same instant, evidently annihilates their velocities, and reduces them to the state of rest. But, *if two bodies of different masses strike each other with opposite velocities, what is the relation of the velocities to the masses in the case of equilibrium?* To resolve this problem, imagine a system of contiguous physical points, ranged on the same right line, and animated with a common velocity in the direction of that line; conceive, in like manner, a second system of contiguous material points, disposed on the same right line, and animated with a common velocity and contrary to the preceding, in such a manner that the two systems striking mutually shall be in equilibrio. It is obvious that if the first system were composed of only one material point, each point of the second system would extinguish in the point struck a part of its velocity equal to the velocity of the system; the velocity of the point struck ought therefore in the case of equilibrium to be equal to the product of the velocity of the second system by the number of its points, and we may substitute for the first system one point singly incited to a velocity equal to this product. We may likewise substitute for the second system a material point animated with a velocity equal to the product of the first system by the number of its points. Thus, instead of these two systems we have two points which bring themselves to an equilibrium by striking with contrary velocities, of which the one will be the product of the velocity of the first system by the number of its points, and the other will be the product of the velocity of the points of the second system, by their number: these products, therefore, must be equal in the case of equilibrium."—"This product of the mass by the velocity we name *quantity of motion;* it is this also which we mean by the *force of bodies:* for the equilibrium of two bodies or of two systems of material points which strike each other in contrary directions, the quantities of motion, or the forces opposed, must be equal, and of consequence the velocities must be reciprocals to the masses."

214. We must not omit observing, that about a century ago there was a warm dispute among the mathematicians, in order to determine whether we ought to consider the force of bodies in motion proportional to the velocity or to the square of the velocity: it is easy, from what has preceded, to reduce this question to a simple enunciation which will remove all difficulty. The word *force* denoting any cause of which the nature is unknown, and of which the effects are the only things we can measure, it is evident that by the term *measure of force,* we can only mean that of its effects: now the effects may be considered under different aspects, each comporting with a species of measure particular and conformable to its nature. If we consider the effect of the force as consisting in the destruction of a certain sum of obstacles or of quantities of motion, this sum must, as above shewn, be expressed by $BV + bv +$ &c. that is, it is proportional to the velocity simply. But if we consider the effect of the force, not with relation to the sum of the obstacles, but to their number, this number will depend upon the space passed over, that is, upon the time and the velocity at each instant, and as will appear further on, will be represented by $\frac{1}{2}BV^2$, or will be proportional to the square of the velocity when all the obstacles are equal, as when equal solicitations of gravity furnish constant obstacles to a rising body. The same may be represented rather differently, thus: let a certain force ϕ, such for instance as would propel a body B with a velocity U, be capable by its instantaneous action of raising a mass M whose

weight is W to a certain height H; and let g denote the force of gravity, while $\dot{t}$ is an evanescent element of time. Then that which has been employed to raise W to the height H will be equivalent to WH, this being the effect produced. But H being a space run over, may be expressed by the product of a velocity V and a time T; and, on the other hand, we have $W = gM, = \frac{g\dot{t}M}{\dot{t}}$, where $g\dot{t}$ is manifestly the velocity V' which would be generated by gravity in the element of time $\dot{t}$. Consequently, $WH = \frac{V'M}{\dot{t}} \times VT = VV'M\frac{T}{\dot{t}} = u^2M\frac{T}{\dot{t}}$, u being the mean proportional between the velocities V and V': and since T and $\dot{t}$ are homogeneous quantities, we shall have $WH \propto Mu^2$, the original force being thus resolved into the product of a mass by the square of a velocity, conformably to the notion attached by most foreigners to the term *vis viva*. This force is, notwithstanding, measured by the product BU above: so that the warm discussions on the question whether the forces of bodies in motion ought to be estimated by the product of a mass into its velocity, or by the product of the mass into the square of its velocity, are reduced to a dispute about words. Provided we always reason conformably to either definition once adopted, the adoption being regulated by the nature of the individual enquiry, the conclusions will always be the same; the different measures being reducible to the same origin. *

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* The following observations of an able writer deserve the attention of the student: "There is a consideration, which, if we mistake not, both in science and in art, will very much decide to what measure of force the mechanist must have recourse. The nature of the propositions on which those measures are founded, must be consulted; from whence it will appear whether the one or the other is most easily applicable to a given case.

"The first method of measuring the force of percussion, is founded on this principle, that if the pressure or accelerating force, that acts uniformly during any interval of time, be multiplied into that time, and if the sum of all the products so formed be taken, that sum will be proportional to the simple power of the velocity communicated. Now, this theorem, in order that it may be used readily, requires that the relation between the forces and the time should be known; or, in other words, that we should be able readily to express the force in terms of the time, or the time in terms of the force; in either case, the determination of the velocity is reduced to a problem in the summation of series, or in the quadrature of curves, more or less difficult as the relation between the time and the force is more or less complicated.

"Again, the second method of finding the velocity communicated by the successive impulses of an accelerating force, is by multiplying each force into the length of the line over which the body has moved while that force acted on it; and the sum of all these products will be proportional to the square of the velocity, and, of course, the square root of the said sum, to the velocity itself.

"Now it is obvious, that in order to apply this theorem to any case, we must be able to express the forces in terms of the distances at which they act; for then the sum of the products described in the theorem will either be found by the summation of series, or the quadrature of curves; so that the thing wanted will be determined. The circumstance, therefore, which distinguishes the one of these kinds of dynamical problems from the other, is, *whether the forces that produce the motion can be most easily expressed in terms of the time reckoned from a given instant, or in terms of the distance reckoned from a given point.* Instances of both cases are easy to be given. Suppose it required to determine the velocity of a body accelerated or retarded by the action of a constant force, as heavy bodies are in their descent or ascent at the surface of the earth;—In this case, either of the two methods may be employed indifferently. The force being given, if it be multiplied into the time during which it acts, the product will be proportional to the velocity, according to the first proposition. And in the same way, if the given force be multiplied into the distance passed over, the square root of the product will be proportional to the velocity; and thus, in either way, may the velocity, with nearly equal facility, be determined. It must be determined in both ways to make the investigation complete; and it is a matter of indifference with which we begin.

"But it is not so if the accelerating force is variable, and expressed by some function of the distance from a given point, (as gravitation really is when we take in a considerable range): the first step in the inquiry must be made by help of the second proposition, that is, by multiplying the force into the fluxion of the distance from the said point, and making the fluent (which will easily be found) equal to the square of the velocity. The velocity being thus expressed in terms of the distance, the time required for moving over a given distance will next be found. It is in this way that Newton has resolved the very problem here proposed, in the 39th proposition of the first book of the Principia. It is therefore according as the *data* in any problem furnish means for integrating one or other of the formulas derived from the propositions above mentioned, that the one or the other must be employed in the solution of that problem." (*Edinburgh Review*, vol. xii. p. 127.)

CHAPTER I

On Motion, uniform and variable.

215. The motion of a physical point, or of a body, is *uniform* (art. 15.) when it moves always in the same manner, or when it passes over equal spaces in any equal intervals of time whatever. This motion, the most simple of all, and the most easy to conceive, probably exists in no part of nature, but is only a pure abstraction of the mind: it is, notwithstanding, important to consider it, because motions, were it not for obstructions, would in general be uniform (art. 21.), and because it conduces greatly to the analysis of all other motions.

That affection of motion which we call *velocity* being measured by the space uniformly described in a given time (art. 15.) is in fact a measure of motion itself, and is that which characterises each species of uniform motion: it is customary in mathematical discussions to fix upon a small period of time, a second, for example, as a unit, and to call *the velocity of a moving body at any instant the space which the body would describe uniformly during a unit of time.*

Hence it follows, that in the uniform motion of a body, *the spaces run over are proportional to the times employed.* For if it describe V feet in one second, it will describe $2V$ feet in two seconds, $3V$ feet in three seconds, and TV feet in T seconds, T being any number whole or fractional. This being granted, we may now state a proposition from which the whole doctrine of uniform motions will readily flow,

216. Prop. *When bodies have different uniform motions, the spaces described are proportional to the times and velocities jointly.*

Let V and v be the velocities of the two bodies B and b, T and t the times of their motions, S and s the spaces described, likewise let s' be the space described by b in the time T:

Then $S : s' :: V : v$

$s' : s :: T : t$

And, comp. $S : s :: TV : tv$. That is $S \propto TV$.

Cor. 1. The velocity is as the space divided by the time: for the preceding expression gives $V \propto \frac{S}{T}$. Or, since the same will hold in any corresponding indefinitely minute portions of the space and time, we shall have $V = \frac{S}{T}$, Cor. 2. *The velocities of two bodies moving uniformly are directly as the spaces and inversely as the times:* for we have $V : v :: \frac{S}{T} : \frac{s}{t}$.

Cor. 3. *In equal times the velocities are proportional to the spaces run over:* for $T = t$, gives $V : v :: S : s$.

Cor. 4. *If the velocities are equal, the spaces passed over are proportional to the times:* for $V = v$, gives $St = sT$, or $S : s :: T : t$.

Cor. 5. *If the spaces passed over are equal, the velocities are reciprocally as the times:* for when $S = s$, we have $V : v :: \frac{1}{T} : \frac{1}{t} :: t : T$.

Cor. 6. Since the areas of rectangles are in the ratio compounded of the ratios of their sides, if the bases represent the velocities of two motions, and altitudes the times, the areas will represent the spaces described.

Cor. 7. Since it has been shewn that the forces which give motion to bodies are proportional to their quantities of motion, and these to the rectangles of the masses and velocities (art. 212.), that is, $F \propto Q \propto BV$; we may, by combining this with the present proposition, have the following formulæ of relation of the six quantities, force F, momentum or quantity of motion Q, mass or quantity of matter B, time T, space S, and velocity V; the forces being supposed instantaneous or impulsive, and the motions uniform:

$$F \propto Q \propto BV \propto \frac{BS}{T}.$$

$$Q \propto F \propto BV \propto \frac{BS}{T}.$$

$$B \propto \frac{F}{V} \propto \frac{Q}{V} \propto \frac{FT}{S} \propto \frac{QT}{S}.$$

$$T \propto \frac{S}{V} \propto \frac{BS}{F} \propto \frac{BS}{Q}.$$

$$S \propto TV \propto \frac{TF}{B} \propto \frac{TQ}{B}.$$

$$V \propto \frac{S}{T} \propto \frac{F}{B} \propto \frac{Q}{B}.$$

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217. We have before said that we know nothing more of forces than by their effects in moving bodies: we call those equal forces, however different they may be in their nature, which give to bodies equal momenta, or which, when the bodies are equal, give to them equal velocities; and we say that forces

are greater or less, when they give to bodies greater or less momenta, or when they impress upon equal bodies greater or less velocities. Therefore, when different forces act upon equal bodies, *the forces are*, cæteris paribus, *proportional to the velocities imparted:* the velocity, then, being proportional to the force, these two quantities may be represented the one by the other, and all which we have established in Chap. I. of the first Book on the composition and resolution of forces may be applied to the composition and resolution of velocities. Hence it might seem unnecessary to give here the demonstration of the most extensive proposition, as it relates to velocities and directions; but, since it may be proved satisfactorily in small compass, and admits of two or three useful deductions, we are unwilling to omit it.

PROP. *The simultaneous action of two impulsive forces* P, P′, *on a body* A, *which would impress upon it separately the velocities* V, V′, *in the directions* AC, AC′, *will cause that body to move uniformly over the diagonal of the parallelogram whose sides are in the directions of those forces.*

Imagine that the body A (fig. 6. pl. X.) is placed on a plane ACC′ which moves uniformly in the direction AC′ with such a velocity as in each unit of time will carry it over a space equal to the line AC′: it is certain that this body, considered with relation to the plane on which it is placed, has no motion; yet if a spectator fixed immoveably out of that plane observe the body A, he will attribute to it a motion equal and parallel to that of the plane. Now, if we conceive that any impulsive force whatever, P, acts upon the body A in the direction PAC, and impresses upon it such a velocity that in a unit of time it would pass over a space equal to AC, there can be no doubt that if the body were acted upon by this force only it would be found at the point C at the termination of the unit of time. But since, in consequence of the motion of the plane, the line AC advances towards C′B by a motion uniform and parallel, so that it would really coincide with C′B at the end of a unit of time, it is obvious that the point C will then coincide with the point B, and that, of consequence, the body A which partakes of the motion of the plane ought to be found in B at the end of the first unit of time. We may prove, in like manner, that at the end of any part or multiple whatever, T, of this unit, the body A, animated with the same velocity AC, ought to run over a proportional space A*c* = T × AC, while the common motion constrains the line A*c* to pass parallel to itself over a distance A*c*′ = T × AC′. This line coincides, therefore, with *c*′*b*, and consequently *b* is the place of the body A at the end of the time T. And it is manifest that all the points *b*, *b*, that may be determined by the same reasoning, are found on the same diagonal AB, since AC : *cb* :: AC : CB. The body A, therefore, actually describes the diagonal AB. But, besides this, its motion along this line must be uniform: for A*b* : AB :: AC : AC :: T × AC : AC :: T : 1; that is to say, A*b* is to AB as the time employed in passing through A*b* to that occupied in passing over AB. Consequently the motion of the body A along the diagonal AB is uniform. Since a body at rest on a moveable plane has the same motion as the plane, it is clear that if the plane were at rest, but that the body moved uniformly according to the right line P′AC′, with the velocity AC′ equal to that which would be impressed upon it by the force P′, and received at the point A from the force P a velocity AC in the direction PAC, it would describe uniformly the diagonal AB of a parallelogram formed upon the sides AC, AC′, which represent the velocities of the body in those respective directions, while the diagonal AB represents its new velocity. Q. E. D.

218. We may likewise shew that, *if a body be acted on by two similar variable forces (for the same time) whose directions and magnitudes are expressed by the adjacent sides of a parallelogram concurring in the body, it will describe the diagonal of the parallelogram.*

Let the forces act by impulses, at the beginning of equal particles of time, and let A*c*′, *c*′*c*′, *c*′*c*′, and A*c*, *cc*, *cc*, be the relative magnitudes of corresponding impulses. Then, by the action of the two first impulses the body will, by the preceding article, describe the diagonal A*b*; and by the next two the diagonal *b*B, of the parallelogram *dd*′ whose sides *bd*′, *bd*, are equal and parallel to the representatives of those new impulses; but the forces are similar, therefore the parallelograms *cc*′, *dd*′, are similar; and, having parallel sides and a common point *b*, they exist about the same diagonal AB. The same may be shewn for a third pair of impulses: and so on, *ad libitum*. Let now the particles of time be evanescent and the forces incessant, and the same demonstration will obtain.

Cor. If the forces by which the body is urged in the directions AC, AC′, be not similar, it will move in some curve line, whose nature will depend on the relation of the forces. Of this many instances will occur as we proceed.

219. There remains another general theorem, which it will not be amiss to exhibit in this place; viz. *If a number of bodies be moving in any manner whatever, and an equal force act on each particle of matter, in the same or parallel directions, their relative motions will not be affected.*

The motion of any body A (fig. 8. pl. X.) with respect to another moving body B, is compounded of the real motion of A and the opposite to the real motion of B: for, let A move uniformly from A to C, while B moves uniformly from B to D; draw AE equal and parallel to BD, join AB, EC, DC, and ED. The motion of A with regard to B consists in its change of position and distance. If while A had described AE, B described BD, there would be no change of relative place or distance: but A is now at C, and DC is its new direction and distance: therefore, the relative (art. 12.) or apparent motion of A is EC. Let the parallelogram ACFE be completed: then it is evident that the motion EC is compounded of EF, which is equal and parallel to AC, the real motion of A; and of EA, the equal and opposite to BD, the real motion of B.

Now, let the motions of A and B sustain the same change; let the equal and parallel motions AG, BH, be compounded with the motions AC and BD: or, suppose forces to act at once on A and B, in the parallel directions AG, BH, and with equal intensities: on either supposition the resulting motions will be AC′, BD′, the diagonals of the parallelograms AGC′C, and BHD′D. Then, constructing the figure as before, we see that the relative motion is E′C′, and that it is equivalent to EC both with respect to magnitude and direction.

Here we may again remark the constant analogy between the composition of motions and that of forces. In the former the relative motions of things are not changed, whatever common motion be compounded with them all: and in the latter the relative motions and actions are not changed by any external force, however considerable, when equally exerted on all the moleculæ in parallel directions.

By means of this it is that we account for the circumstance of the evolutions of a fleet in a uniform current being the same, with relation to the several ships, and produced by the same means, as in still water: also, that the motions and operations in a ship, sailing smoothly and regularly along, are performed in the same manner as though the vessel were at rest: and again, universally, of all bodies included in a given space (for instance, those on the surface of the earth,) their motions amongst themselves will be the same, their congress the same, the force of their percussion the same, and all their mutual operations, whether the space they occupy is at rest, or whether it moves in a trajectory compounded of the diurnal and annual motions about the centre of the solar system; or lastly, whether these are combined with a motion about some far more distant centre of force *.

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* Those who are desirous of pursuing further the theory of apparent and relative motions, particularly as it regards the phenomena of the motions of the planets, are referred to Chap. IX. of my *Treatise on Astronomy*.

220. Prop. *Given the velocities* V, V', *with which two bodies move in the same line and towards the same point, and the distance* S' *of one from the other at the commencement of their motion, to find the time and place of their rencounter.*

If any point A be assumed as a fixed point in a line, from which we would estimate any distances upon that line through which a body passes; then, if s' be the distance from this fixed point to the point where a body commences its motion, or, as we may call it, *the initial space*, the general equation for uniform motions will become $\mathrm{S}' = s' + \mathrm{T}'\mathrm{V}'$; let this be the equation for the body which commences its motion at the distance s' before the other body; let also $\mathrm{S} = s + \mathrm{TV}$, be the equation for that other body, in which we have $s = 0$, and consequently $\mathrm{S} = \mathrm{TV}$. Here the spaces being both estimated from the same point A, and the times of moving being equal, we have $\mathrm{S} = \mathrm{S}'$, and $\mathrm{T} = \mathrm{T}'$: we have therefore

$$(\mathrm{I.}) \;\ldots\ldots\; \mathrm{T} = \mathrm{T}' = \frac{s'}{\mathrm{V} - \mathrm{V}'}, \text{ and } \mathrm{S} = \mathrm{S}' = \frac{s'\mathrm{V}}{\mathrm{V} - \mathrm{V}'}.$$

These equations manifestly solve the problem.

221. Prop. *Let it be proposed to find at what time the two moving bodies will be at any given distance* D *from each other.*

In this case it will be necessary to have $\mathrm{S} - \mathrm{S}' = \pm \mathrm{D}$, where we put the double sign ±, because the bodies may be at the distance D from each other either before or after their rencounter: that is, we may either have $\mathrm{S} > \mathrm{S}'$, or $\mathrm{S} < \mathrm{S}'$. Now from this equation of condition conjointly with the preceding ones we obtain the following:

$$(\mathrm{II.}) \;\ldots\; \mathrm{T} = \frac{s' \pm \mathrm{D}}{\mathrm{V} - \mathrm{V}'}, \;\ldots\; \mathrm{S} = \mathrm{V} \cdot \frac{s' \pm \mathrm{D}}{\mathrm{V} - \mathrm{V}'}, \;\ldots\; s' = \frac{\mathrm{V}s' \pm \mathrm{V}'\mathrm{D}}{\mathrm{V} - \mathrm{V}'}.$$

222. Prop. *Having two bodies moving uniformly in the perimeter of any complete curve, to find their points of rencounter.*

Supposing the curve to be rectified, it is obvious that the solution to this problem will be comprised in the equations I. art. 220. But besides the point of meeting which may be found thus, the bodies may have several others, since they may continue to run along the curve and pass again and again over the same points, and in the course of this motion have many different points of meeting: now, in this state, the first point of rencounter is taken as a new point of departure, and we may consider the two bodies as distant from each other the whole perimeter p of the curve. Here p will obviously correspond with s' in equa. I. above, and the instant of the second rencounter will be distant from that of the first an interval $\mathrm{T}' = \dfrac{p}{\mathrm{V} - \mathrm{V}'}$;

we have, therefore, estimating the time from the origin of the motion to that of the second rencounter, $T = \frac{s'}{V-V'} + \frac{p}{V-V'} = \frac{s'+p}{V-V'}$. Reasoning in the same manner, for the third rencounter we shall have $T = \frac{s'+2p}{V-V'}$; and in general the nth rencounter will happen at the end of a time $T = \frac{s'+(n-1)p}{V-V'}$.

In like manner we obtain an equation for the time employed by the second body in reaching the nth point of rencounter: and, comparing the two equations, we find

$$\text{(III.)} \quad \ldots\ldots \quad s = V \,.\, \frac{s'+(n-1)p}{V-V'}, \quad \ldots\ldots \quad s' = \frac{Vs+V'p(n-1)}{V-V'}$$

If several bodies move uniformly over the same curve, we have similar equations for each of them, which must be compared two by two; and the nth rencounter of the first and third body will be given by the equation

$$T = \frac{s''+(n'-1)p}{V-V''}.$$

Thus, to find the point where three bodies will meet, it is necessary to make this value of T equal to the former, which furnishes this equation

$$\frac{\frac{1}{p}s''+n'-1}{V-V''} = \frac{\frac{1}{p}s'+n-1}{V-V'}, \text{ or } \frac{\frac{1}{p}s''+n'-1}{\frac{1}{p}s'+n-1} = \frac{V-V''}{V-V'}.$$

The problem in this case obviously becomes indeterminate: but as the values of n and n' must be whole numbers, the number of solutions is less considerable than might at first be supposed. The matter, however, need not be pursued in this place.

223. To give an example of the use of some of these equations: *Suppose that we have a clock shewing the hours, minutes, and seconds:* we may conceive the extremities of the three hands as three moveable points moving over the same circumference; the determination of the rencounter two and two, or of all three of the hands, will not be attended with any difficulty. For putting V, V', V'', the respective velocities of the second, minute, and hour hands; taking also the minute for the unit of time, and the perimeter for the unit of space, we thence have $p = 1$, $V = 1$, $V' = \frac{1}{60}$, $V'' = \frac{1}{720}$; and the preceding formulæ give

For the rencounter 2 and 2 of the . . . $\left\{ \begin{array}{ll} \text{second and minute hands} & T = \frac{60\,(s'+n-1)}{59}. \\ \text{hour and second hands} & T = \frac{720\,(s''+n'-1)}{719}. \end{array} \right.$

Rencounter of all three hands . . . . $\left\{ \frac{60\,(s'+n-1)}{59} = \frac{720\,(s''+n'-1)}{719}. \right.$

If therefore we put for s' and s'' their values, which are the distances of the hands at the commencement of the time, we shall resolve the problem completely, if we satisfy these equations by means of any whole numbers n, n', whatever.

II. *On Motions uniformly varied.*

224. A body which has received only a single impulsion will, according to the first law of motion, persevere in its motion with the same velocity and in the same direction it had at the first instant: but if it receives a new impulsion, either in the same direction or in a direction contrary to the first, it will then move with a velocity equal to either the sum or the difference of the two velocities which it received successively. If, therefore, we conceive that at successive intervals of time the body receives new impressions, either in the same or contrary directions, it will be transferred to different parts of space with a *varied* or unequable motion; its velocity will be different at the commencement of each interval of time. In variable motions the velocity undergoing repeated changes, it is usual to estimate it at any time whatever by the space it is capable of passing over during a unit of time, if its motion for that interval continued the same as at the instant where we would consider the velocity. Or in variable motions, the velocity of a body at any determinate instant is the space which it would run over in every unit of time, if at that instant the action of the power ceased, and the motion became uniform.

Defs. We call in general any force which acts on a body so as to make it vary its motion an *accelerating force:* when, in equal intervals of time, it acts equably, or the velocity undergoes equal mutations, we call it *a constant or uniform accelerating force, or a constant retarding force,* according as it tends to augment or diminish the actual velocity of the moving body.

When a single body is acted upon by a constant force, there are four quantities which become the objects of mechanical consideration, viz. the *space* described, the *time* of description, the *velocity* acquired, and the *force* which produces it; any three of which being given, the other may be ascertained. But when different forces act upon bodies of different masses, these are two additional quantities for consideration, making in the whole six kinds of magnitudes which affect the discussion.

225. PROP. *The velocities generated in equal bodies by the action of constant forces are in the compound ratio of the forces and times of acting.*

For, when the times are the same the velocities generated each instant are as the forces of acceleration, and consequently the velocities generated at the end of equal times are as those forces; and if the forces are the same the velocities generated are as the times wherein the forces act; because, when the force is given equal velocities are generated in equal times, and consequently the whole velocities acquired are as the times wherein the given force acts: wherefore, both times and accelerating forces being different, the velocities generated will be as the forces and times of action, jointly.

COR. 1. *The momenta generated in unequal bodies are also conjointly as the forces and their times of action.* This is evident, because momenta in unequal bodies may be substituted for proportional velocities in equal bodies, throughout the whole reasoning.

COR. 2. *The momenta lost or destroyed in any times are likewise conjointly as the retarding forces and their times of action.* For, whatever momenta any force generates in a given time would an equal force destroy in an equal time, by acting in a contrary direction.

And the same is true of the increase or decrease of motion, by forces that either conspire with, or oppose, the motions of bodies.

COR. 3. *The velocities generated or destroyed in any times are directly as the forces and times, and reciprocally as the bodies or masses.* For, since the compound ratios of the bodies and their velocities are as those of the forces and times, the velocities are as the forces and times divided by the bodies.

226. PROP. *In motions uniformly accelerated, when the force and body are given, the space described during a certain time is the half of that which the body, moving uniformly with the last acquired velocity, would describe in an equal time.*

Since the velocities are as the times of description, when the body and force are given, the velocities which a given body is found to have successively for the duration of each consecutive interval form an arithmetical progression, g, $2g$, $3g$, &c. of which the last term is gt or v, the number of terms being t, that is to say, being marked by the number of solicitations of the accelerating force. And since each of the velocities is nothing else than the space which the body would describe uniformly during the corresponding interval, the total space described during the time t will therefore be the sum of the terms of this arithmetical progression; which, because g and v

are the extremes and t the number of terms, will be expressed by $\frac{1}{2}t\ (g+v)$. Or if s be the total space described by the body, then will $s = \frac{1}{2}t\ (g+v)$. Conceive now that the accelerating force acts (as by hyp.) without intermission, or, which comes to the same, imagine that the time t is divided into an indefinite number of infinitely small parts, or instants, and that at the beginning of each instant the accelerating force gives an impulsion to the body. Then g being infinitely minute in relation to v, which is the velocity acquired during the indefinite number of instants denoted by t, must be omitted in the equation $s = \frac{1}{2}t\ (g+v)$, which will become simply $s = \frac{1}{2}tv$, the space actually described.

This granted, imagine that at the end of the time t the accelerating force ceases to act; then, by the first axiom, the body will persevere in its motion with the velocity v it has acquired: but in uniform motions the spaces described are as the times and velocities jointly (art. 216.), therefore the body moving with the velocity v, during the time t, will describe a space $s' = tv$; which is evidently double the space $\frac{1}{2}tv$ described by the body in an equal time, by the constant action of the accelerating force. Q.E.D.

227. Prop. *The spaces described by a body uniformly accelerated are as the squares of the times.*

Since the velocities acquired increase as the time expired, if φ be the velocity at the end of one second, then the velocity acquired after a number t of seconds will be φt; thus we have $v = \varphi t$. The equation $s = \frac{1}{2}vt$, found in the preceding article, becomes therefore $s = \frac{1}{2}\varphi t^2$. If, in like manner, we represent another space by S, which is described by uniform acceleration during the time T, we shall have $\text{S} = \frac{1}{2}\varphi \text{T}^2$. Hence we see that $s : \text{S} :: \frac{1}{2}\varphi t^2 : \frac{1}{2}\varphi \text{T}^2 :: t^2 : \text{T}^2$. Q.E.D.

Cor. 1. Because the velocities acquired are as the times, we have also *the spaces described as the squares of the velocities.*

Cor. 2. *Therefore either the velocities or the times are as the square roots of the spaces described from the commencement of the motion.*

Cor. 3. All that has been shewn here applies equally to motions *uniformly retarded;* provided that by the times we mean those which are to elapse before the extinction of the velocity, and by the spaces those which remain to be described until the body is brought to rest. Similar propositions have, therefore, been applied to the motions of balls resisted by banks of earths, blocks of wood, &c. See *Dr. Hutton's Select Exercises,* and *Atwood on Motion.*

Cor. 4. *The velocity φ which will be acquired at the end of* *a second is that which the accelerating force can generate in a second chosen as a unit of time; it is therefore a measure of the accelerating force, and may of course be safely substituted for that force in any of our subsequent investigations.*

SCHOLIUM.

228. We may now exhibit general theorems for resolving all problems relating to motions uniformly accelerated or retarded. In order to this, put B = any body or mass, F = the force acting constantly upon it, T = the time of its action, V = the velocity generated (or lost) in the time T, S = the whole space described, Q = the momentum or quantity of motion, at the end (or beginning) of the time: then the fundamental relations are $S \propto TV$, $Q \propto BV$, $Q \propto FT$, from which are the propositions and corollaries just laid down, we have—

$$B \propto \frac{Q}{V} \propto \frac{FT}{V} \propto \frac{QT}{S} \propto \frac{FT^2}{S} \propto \frac{FT^3}{QS} \propto \frac{Q^2}{FS} \propto \frac{Q^2}{FTV} \propto \frac{FS}{V^2}.$$

$$Q \propto BV \propto FT \propto \frac{BS}{T} \propto \frac{FS}{V} \propto \frac{FT^2V}{S} \propto \sqrt{BFS} \propto \sqrt{BFTV}.$$

$$F \propto \frac{Q}{T} \propto \frac{BV}{T} \propto \frac{QV}{S} \propto \frac{QS}{T^2V} \propto \frac{Q^2}{BTV} \propto \frac{BV^2}{S} \propto \frac{BS}{T^2}$$

$$V \propto \frac{S}{T} \propto \frac{FT}{B} \propto \frac{Q}{B} \propto \frac{QS}{FT^2} \propto \frac{FS}{Q} \propto \frac{Q^2}{BFT} \propto \sqrt{\frac{FS}{B}} \propto \sqrt{\frac{F^2ST}{Q^2}}$$

$$S \propto TV \propto \frac{FT^2}{B} \propto \frac{QT}{B} \propto \frac{FT^2V}{Q} \propto \frac{QV}{F} \propto \frac{Q^2}{BF} \propto \frac{Q^2V}{F^2T} \propto \frac{BV^2}{F}$$

$$T \propto \frac{S}{V} \propto \frac{Q}{F} \propto \frac{BV}{F} \propto \frac{BS}{Q} \propto \sqrt{\frac{BS}{F}} \propto \sqrt{\frac{QS}{FV}} \propto \frac{Q^2}{BVF}.$$

When any quantities are given, or their relations to some fixed quantities of the same kind known, they are to be left out in the general theorems: thus, if the body be proportional to the force, we shall have $S \propto TV \propto F T^2 \propto \frac{V^2}{F^1}$, where $F^1 = \frac{F}{B}$.

Hence, in general, when the circumstances produced by the operation of any kind of accelerating or retarding force, as that of gravity, are computed or otherwise ascertained, the analogous circumstances produced by any other constant force may be readily inferred, even without these formulæ; it will simply be requisite to compute the effects of gravity by the theorems in art. 227, applied to some decisive experiment; and then, by some equally decisive experiment assign the relation of the other accelerating force to that of gravity, and take all the other circumstances in the same ratio. Instances of this may be seen in arts. 267, 366, 369, &c.

229. As it is sometimes necessary to consider the effect of accelerating forces upon bodies already in motion, it will be worth while to deduce a general formula for that purpose. To

this end let g represent, as in art. 226., the velocity due to the acceleration during each unit of time, gt will then be the total velocity acquired at the end of the time t: then, if we denote by v the velocity which the body has at the commencement of the time, and by V the velocity at the end of the time t, we have $\text{V} = v + gt$. But $\text{V} = \frac{\dot{s}}{\dot{t}}$, both in uniform and variable motions, the force or corresponding velocity being supposed constant for the indefinitely small time $\dot{t}$. Therefore $\frac{\dot{s}}{\dot{t}} = v + gt$, and taking the correct fluents, we have

$$\text{(IV.)} \quad \text{S} = s + vt + \tfrac{1}{2}gt^2.$$

Here the constant quantity s evidently represents the initial space, viz. the distance between the point of departure and that in relation to which we consider the several positions of the moving body: for $t = 0$, gives $\text{S} = s$.

230. The general equation given in the last article may be readily constructed: for it is plain, from the theory of conic sections, that its locus is a common parabola. For, changing S into $x + a$, and t into $y + b$, we have

$$x + a = \text{S} + vy + vb + \tfrac{1}{2}g\,(y + b)^2$$

Then, determining the constant quantities a and b, by the equations

$$v + bg = o, \text{ and } a = s + vb + \tfrac{1}{2}gb^2$$

We thence find $b = -\frac{v}{g}$, $a = s - \frac{v^2}{2g}$ and the equation (IV.) in the last article will become $y^2 = \frac{2}{g}x$, which is an equation to a parabola whose parameter is $\frac{2}{g}$.

Now the nature of the motion being supposed given, and the constant quantities, s, v, g, being known, if AE (fig. 1. pl. XI.) is the line passed over by the moveable body, and we take $\text{AB} = s$, the initial space B will be the point of departure. Here it results from the preceding values of a and b, that if we make $\text{CA} = -\frac{v}{g}$, $\text{CD} = s - \frac{v^2}{2g}$, and then construct on DF as an axis a parabola DMmm', whose vertex is D, and parameter $\frac{2}{g}$, it will be the curve required; or that in which the ordinates AP, Ap, &c. will represent the times, and the corresponding abscissas PM, pm, &c. the spaces.

If we draw a tangent BR to the point B of the parabola, it will form with AT an angle RBN, of which the tangent will be $= v$. Moreover, if we set off from any point P in AT, the axis of the times, two consecutive seconds, or units of time, Pp, pp', and draw to the corresponding points M, m, of the locus, the tangents Mr, mr', we shall have $nr = v + gt$, $n'r' = v + g\ (t + 1)$, and consequently $n'r' - nr = g$. Or, we may find the length of g by another method: for, since PM $= s + vt + \frac{1}{2}gt^2$, and $pm = s + v\,(t + 1) + \frac{1}{2}g\ (t + 1)^2$, therefore $mn = v + gt + \frac{1}{2}g$; but $nr = v + gt$, whence $mr = \frac{1}{2}g$, and $2mr = g$.

COR. When $s = 0$, A coincides with B: and when $v = 0$, B coincides with D, and RN vanishes.

III. *Variable Motions in general.*

231. When a moving body is subjected to the energy of a force which acts on it without interruption, but in a different manner at each instant, the motion is called in general, *variable motion*. We have instances of variable motions in the unbending of springs: although the velocity continues to be augmented, yet the degrees by which the augmentation proceeds are diminishing. It is the same with regard to the degrees by which the motion of a ship arrives at uniformity: the action of the wind on the sails diminishes in proportion as the vessel acquires greater velocity, because the action of the wind varies as the difference between its velocity and that of the sail on which it acts.

The different natures of constant and variable accelerating forces, and their corresponding motions, have been illustrated by Dr. Hutton in the following manner. "Let two weights, W, w, be connected by a thread passing over a pulley at A, B, or C (figs. 2, 3, 4, pl. XI.); and let the weight W descend perpendicularly down, while it draws the smaller weight w up the line AD, or BE, or CF, the first being a straight inclined plane, and the other two curves, the one convex, and the other concave to the perpendicular. Then the small weight w will always make some certain resistance to the free descent of the large weight W, and that resistance will be constantly the same in every part of the plane AD, the difficulty to draw it up being the same in every point of it, because every part of it has the same inclination to the horizon, or to the perpendicular; and consequently the accessions to the velocity of the descending weight W will be always equal in equal times; that is, in this case W descends by a uniformly accelerating force. But in the two curves BE, CF, the resistance or opposition of the small weight w will be constantly altering as it is drawn up the curves, because every part of them has a different inclination to the horizon, or to the perpendicular: in the former curve the direction becomes more and more upright, or nearer perpendicular, as the small weight w ascends, and the opposition it makes to the descent of W becomes more and more, and consequently the accessions to the velocity of w will be always less and less in equal times; that is, w descends by a decreasing accelerating force; but in the latter curve CF, as w ascends, the direction of the curve becomes less and less upright, and the opposition it makes to the descent of w becomes always less and less; and consequently the accessions to the velocity of w will be always more and more in equal times; that is, w descends by an increasing accelerating force. So that although the velocity continually increases in all these cases, yet whilst it increases in a constant ratio to the times of motion, in the plane AD; the velocity increases in a less ratio than the time it ascended up BE, and in a greater ratio than the time increases in the other curve CF." *Hutton's Math. Dict.* art. *Acceleration.*

The principles necessary for the determination of the circumstances of variable motions are easily deducible from what has been done with respect to uniform motions, and those which are uniformly accelerated or retarded, as will be seen in the next proposition.

232. Prop. *To find the fundamental equations which apply to variable motions.*

In whatever manner any motion is varied, if we consider it with relation to evanescent instants, we may conceive its velocity to be invariable during any such indefinitely minute interval. But when the motion is uniform the velocity is expressed by the quotient of the space s, described during the interval of time t, divided by that time (216.): Therefore, when the velocity is only uniform for the evanescent instant $\dot{t}$, the velocity must be expressed by the indefinitely small space $\dot{s}$, described during this instant, divided by the instant itself. We have, therefore,

$$(\text{I.}) \ldots v = \frac{\dot{s}}{\dot{t}} \text{ or } \dot{s} = v\dot{t}.$$

The equation $v = \varphi t$ (art. 227.) which expresses the relation of the velocities to the times, in motions uniformly accelerated, gives $\varphi = \frac{v}{t}$; that is to say, when the accelerating force, or, rather the quantity φ by which it is measured (227. cor. 4.), is constant, it has for its expression the quotient of the velocity v, which it generates during a certain time t, divided by that time: therefore, if the accelerating force φ acts differently at each instant, we imagine it to be constant only for the evanescent instant $\dot{t}$, in which it would generate the velocity $\dot{v}$, and consequently,

$$\text{(II.)} \ldots \varphi = \frac{\dot{v}}{\dot{t}}, \text{ or } \dot{v} = \varphi\dot{t}.$$

In the equation $v = \varphi t$ we understand φ to denote the velocity that the accelerating force generates in the moving body during a determinate unit of time, as a second, by an action continued and always equal. In the equation $\dot{v} = \varphi\dot{t}$, we ought to understand the same thing. But it is necessary to observe, that the accelerating force being supposed variable, the quantity φ which represents the velocity which it would be capable of generating if it acted as a constant accelerating force during a second is different for every instant of its motion. Thus we easily conceive that when the accelerating force becomes smaller, the velocity which it will be capable of generating in a second, by its action repeated uniformly during each instant of this second, must be smaller, and *vicè versâ*.

The two preceding equations readily furnish a third, which may often be advantageously adopted: for, from the equa. $\dot{s} = v\dot{t}$, we deduce $\dot{t} = \frac{\dot{s}}{v}$: substituting this value of $\dot{t}$ in the equation $\dot{v} = \varphi\dot{t}$, we readily find

$$\text{(III.)} \ldots \varphi\dot{s} = v\dot{v}, \text{ or } \varphi = \frac{v\dot{v}}{\dot{s}}.$$

Again employing the same equations, since $\dot{v} = \varphi\dot{t}$, and $v\dot{t} = \dot{s}$, we have by multiplication $v\dot{t}\dot{v} = \varphi\dot{t}\dot{s}$; whence, striking out $\dot{t}$, there remains $v\dot{v} = \varphi\dot{s}$. But $v\dot{v} = \frac{1}{2}(vv)^{\cdot}$, consequently,

$$\text{(IV.)} \quad \varphi\dot{s} = \tfrac{1}{2}(vv)^{\cdot}.$$

In the reasoning by which we found the equations $\dot{v} = \varphi\dot{t}$, we have considered the velocity as increasing. If, therefore, cases arise in which the velocity diminishes, its fluxion will become negative, and the equations $\dot{v} = \varphi\dot{t}$, and $\varphi\dot{s} = v\dot{v}$, in order to accommodate them to all cases which may arise, must be written with the double sign: viz. $\pm\dot{v} = \varphi\dot{t}$, and $\varphi\dot{s} = \pm v\dot{v}$, the superior sign obtaining when the motion is accelerated, and the lower one when it is retarded.

The equation $\dot{s} = v\dot{t}$, or $v = \frac{\dot{s}}{\dot{t}}$, being fluxed gives $\dot{v} = \left(\frac{\dot{s}}{\dot{t}}\right)^{\cdot}$: if this value be substituted for $\dot{v}$ in the equation $\varphi\dot{t} = \pm\dot{v}$, it will become

$$(\text{V.}) \ldots\ldots \varphi\dot{t} = \pm\left(\frac{\dot{s}}{\dot{t}}\right)^{\cdot}$$

And this equation must be employed when $\dot{t}$ is supposed variable: but if we imagine, as it is often right to do, that $\dot{t}$ is constant, we have $\varphi\dot{t} = \pm\frac{\ddot{s}}{\dot{t}}$; wherefore

$$(\text{VI.}) \ldots \varphi\dot{t}^2 = \pm\ddot{s}, \text{ or } \varphi = \frac{\ddot{s}}{\dot{t}^2}.$$

SCHOLIUM.

233. According to whatever law the motions of bodies may be varied, we may construct curves as loci of the equations which comprise the relations of the times and spaces: but since there will be as many kinds of curves as there may arise equations comprising the law of the variations, it will be impossible to attend to them minutely here. All, therefore, that will be remarked in this place is, that when the motion is accelerated, the corresponding curve will present its convexity to the axis of the time; while, if the motion is retarded, the concavity of the curve will be presented to that axis; and if in any instant whatever the motion becomes uniform, the curve will then degenerate into a right line, which will be a tangent to that point of the curve which corresponds with the instant of time in which the uniformity of the motion commences.

234. As the formula $\varphi = \frac{\ddot{s}}{\dot{t}^2}$ is of the utmost importance in the theory of varied motions, and as the manner in which we have deduced it above has been sometimes objected to; we shall here present a more rigorous demonstration of the same, which was first given by the celebrated D'Alembert, and is deduced from the known theory of curves.

Let AP, AP′, &c. represent the times, and PM, P′M′, &c. the spaces described (fig. 6. pl. XI.): call AP, t, PM, s; and imagine the three ordinates PM, P′M′, P″M″, to be infinitely near to each other; make $\dot{t}$ constant, or PP′ = P′P″, and P′M′ = $\dot{s}$. This done, it is evident that KM′ = $\dot{s}$ will be passed over in the time PP′ = $\dot{t}$, and that QM″ = $\dot{s}$, will be run over in the time P′P″ = $\dot{t}$: now there are two hypotheses respecting the acceleration which has place in the spaces $\dot{s}$, $\dot{s}$, which we ought to distinguish with great care. 1. The augmentation of the velocity which obtains in these spaces may be gradually acquired during the instant $\dot{t}$, viz. the motion may be continually accelerated from P to P', and from P' to P''. 2dly, The augmentation of the velocity which takes place in the space KM may be given it all at once in M', in such a manner that KM will be described by a uniform motion; in like manner, the augmentation which obtains in QM'' may be reckoned to be acquired all at once in M'', so that QM'' be described by a uniform motion, &c. In this second hypothesis, the motion is made as it were by small leaps, which take place at the end of each instant $\dot{t}$, the motion retaining its uniformity during that instant.

The effect produced at the termination of $\dot{t}$ is the same on either hypothesis: the effect consisting in running over the spaces $\dot{s}$, $\dot{s}$, &c. But it is not the same with regard to what takes place during the existence of each instant; for, according to the one or the other hypothesis the elements MM', $M'M''$, &c. are different. In the first, where the acceleration of the motion is conceived to be always taking place throughout the instant $\dot{t}$, the corresponding element of the curve is a real curve MgM', $M'nM''$, &c. different from the cords MM' $M'M''$, &c. In the second, the motion being considered as uniform during the instant $\dot{t}$, the elements MgM', $M'nM''$, &c. become rectilinear, and differ not from the cords MM', $M'M''$, &c. Let us first shew what results from the former case.

Draw to the point M' of the curve the rigorous tangent $M'R$, then will the space QR be that which the body would run over during the instant $\dot{t}$, if the velocity acquired at the point M' were continued uniformly; the space RM'' will be that run over in virtue of the acceleration which obtains between P' and P'', and the space nr will be that passed over, in consequence during the time $P'd$; we will examine the relation which exists between nr and $M''R$ with regard to the times $P'd$, $P'P''$. The arc $M'M''$, being infinitely small, may be considered as appertaining to any curve whatever, and of consequence to a parabola; where, by the property of this curve, $M'R$ being a tangent, we have the proportion $nr : M''R :: P'd^2 : P'P''^2$; for, by Hutton's Conics, Prop. ix. Parab. $nr : M''R :: M'r^2 : M'R^2$, and it is obvious that $M'r^2 : M'R^2 :: P'd^2 : P'P''^2$. Therefore the spaces nr, $M''R$, are as the squares of the times $P'd$, $P'P''$, employed in describing them; therefore during the instant $P'P'' = \dot{t}$, the motion is uniformly accelerated. Whence we have $M''R = \Psi\dot{t}^2$, Ψ being a constant quantity during the instant P′P″; but it is different in the succeeding instants.

If we prolong the cord MM′ to X, we have $\mathrm{QX} = \mathrm{KM}' = \dot{s}$, and $\mathrm{XM}'' = \mathrm{QM}'' - \mathrm{QX} = \dot{S} - \dot{s} = \ddot{s}$: but, by a well-known general property of curves, $\mathrm{XM}'' = 2\mathrm{M}''\mathrm{R}$ (see Emerson on Curve Lines, book ii. pr. 2.), therefore $\mathrm{M}''\mathrm{R} = \frac{\ddot{s}}{2}$; and, substituting this in the equation $\mathrm{M}''\mathrm{R} = \Psi\dot{t}^2$, we have $\Psi = \frac{\ddot{s}}{2\dot{t}^2}$. This expression denotes that the space which will be described in a unit of time, in virtue of the uniform acceleration which has place during the instant $\dot{t}$, is equal to $\frac{\ddot{s}}{2\dot{t}^2}$. Thus we see, on the first hypothesis, the means of measuring the acceleration at each instant.

In the second hypothesis the motion being uniform during the instant PP′, and the element of the curve MgM′ coinciding with the side MM′ of the inscribed polygon, if the body continues to be moved with the velocity acquired in M′, the space which it will describe during the following instant will be QX, since, in this case, the tangent ought to be the prolongation of the infinitely little side MM′. Hence it follows, that the space passed over during the instant P′P″ in virtue of the acceleration which has place at the point M, is XM″, which, as we have seen, is $= 2\mathrm{M}''\mathrm{R} = \ddot{s} = 2\Psi\dot{t}^2$. If we make $2\Psi = \varphi$, we have, for the measure of the acceleration, the equation $\varphi = \frac{\ddot{s}}{\dot{t}^2}$, φ expressing here the double of the space which would be run over in a unit of time, in virtue of the acceleration of the first hypothesis. Now the double of that space is precisely the velocity acquired by a like acceleration (art. 226); therefore φ expresses the velocity that the moving body would acquire in a unit of time, if the motion continued to be uniformly accelerated by the quantity with which it was augmented during the instant P′P″ in the hypothesis of the rigorous curve; therefore, the formulæ obtained by the two methods are identical.

We see, therefore, that whether, supposing the curve rigorous, the acceleration is measured by $\frac{\ddot{s}}{2\dot{t}^2}$, or, supposing it polygonal, it is measured by $\frac{\ddot{s}}{\dot{t}^2}$; either of these measures is indifferent, provided that we always estimate by the same the different effects which we would compare. The space described by the moving body is the same on either hypothesis, that is to say, QM'' during the instant $P'P''$; only in the first QR is run over in virtue of the motion previously acquired, and RM'' in virtue of the acceleration; while in the second these are QX and XM''. It is, otherwise, easy to assign a direct reason why the elementary space run over, in the hypothesis of the polygonal curve, is the double of that described on the supposition that the curve is rigorous: namely this;—in the first case the body acquires all at once the increment of its velocity, while it is obtained in the second by a uniform acceleration; it ought, therefore, during the same time $\dot t$, to describe a double space.

235. Having now deduced the chief formulæ in variable motions, it remains for us to present an example or two of their use and application.

I. Suppose that a material point, or very small globe, placed at A (fig. 7. pl. XI.) is solicited by two forces; the one tending to make it move from A towards B, with a motion uniformly varied; the other tending, on the contrary, to push it back from A towards D: the circumstances of the motion of the globule are required, on the supposition that the repulsive power impresses upon it an accelerating force varying inversely as the distance from the point B. Let $AB = a$, $AN = s =$ the space passed over at the end of the time t; the accelerating force which arises from the repulsion from A towards D will be $= \frac{m}{a+s}$, m being a constant quantity depending upon the law according to which the repulsive force acts. Lastly, let $g =$ the constant accelerating force which arises from the impulsion of the moving point from A towards D. The force accelerating the motion which we consider as being the difference of these two forces, we have by the equation (232. VI.) which gives

$$\frac{\ddot s}{\dot t^2} = \frac{m}{a+s} - g.$$

To find the fluent of this equation, we must multiply by $\dot s$, whence will arise $\frac{\dot s}{\dot t} \times \left(\frac{\dot s}{\dot t}\right)^{\cdot} = v\dot v = \frac{m\dot s}{a+s} - g\dot s$; and consequently, by a well-known form, $\frac{1}{2}v^2 = m\,\text{H}\cdot\text{L}\cdot(a+s) - gs + \text{C}$. Where, since at the point A we have $v = 0$, and $s = 0$, we conclude that $\text{C} = -m\,\text{H}\cdot\text{L}\cdot a$. Therefore

$$v = \pm\sqrt{(2m\,\text{H}\cdot\text{L}\cdot\frac{a+s}{a} - 2gs)}.$$

This equation determines the velocity that the moving body has when it has run over the space s: here if we put for v its value $\frac{\dot{s}}{\dot{t}}$ (art. 232. i.) and find the corresponding fluent, we shall obtain the time t in terms of the space: but this is sometimes attended with considerable difficulty.

The problem just resolved finds its application in a case which we shall now state: if a heavy body, as a piston, is forced into a cylinder or vertical tube BD, open only at the extremity D, which the piston closely fits, and if the part AB is full of a compressed elastic fluid, or of an expansive vapour; then, not considering the friction of the piston against the sides of the tube, it is obvious that this piston will be subjected to the action of gravity which tends to make it descend and impresses a constant accelerating force g, and at the same time to the repulsive force of the elastic fluid: but this fluid having less spring as it is less compressed, viz. as the piston is farther distant from the extremity B, the accelerating force thence arising varies inversely as the distance of the moveable piston from the bottom of the tube.

We have an example of this species of motion in the balls of guns, and pieces of cannon, driven by the inflammation of the powder: this produces instantaneously a great quantity of an aeriform fluid, of which the repulsive force is inversely as the space in which it is contained. We here neglect the consideration of the weight of the ball, since it has but little effect upon the velocity up to the mouth of the piece, the weight being nothing in theory when the axis of the piece is horizontal. We therefore make $g = 0$, or, which amounts to the same, we consider at the commencement of the calculation the accelerating force as $= \frac{m}{a+s}$: consequently,

$$v = \sqrt{\left(2m\text{H}\cdot\text{L} \cdot \frac{a+s}{a}\right)}.$$

Making s to equal the distance of the point A from the orifice, this equation gives us the velocity with which the ball issues from the piece.

If the weight of the powder, and of the ball, be taken into the computation, it will of course become more intricate: the general principle, however, is still the same. These particulars, with other minutiæ affecting the investigation, are considered by Dr. Hutton, in an excellent solution which may be seen in the 3d volume of his 8vo. Tracts.

236. II. Let there be at D (fig. 7.) a material point, or globule, solicited by an accelerating force varying inversely as the square of the distance of the moveable from the point B; it is required to find the equation of its motion.

Put BD $= a$, DN $= s =$ the space passed over at the end of the time t: when the globule has arrived at N, its distance from B will be $a - s$, and the accelerating force is $\frac{m}{(a-s)^2}$, m being again a constant quantity which depends upon the nature of this force; viz. its magnitude at a unit of distance from the centre of attraction. Here, then, we have from the equation at 232. vi.

$$\frac{\ddot{s}}{\dot{t}^2} = \frac{m}{(a-s)^2}$$

Multiplying, as before, by $\dot{s}$, we obtain $\frac{\dot{s}}{\dot{t}} \times \left(\frac{\dot{s}}{\dot{t}}\right)^{\cdot} = v\dot{v} = \frac{m\dot{s}}{(a-s)^2}$; whence $v^2 = \frac{2m}{a-s} + \text{C}$. Supposing that at the origin D the globule were not animated with any velocity, we should have at the same time $s = 0$, and $v = 0$; therefore $\text{C} = -\frac{2m}{a}$. Substituting this, and reducing, we have

$$v = \sqrt{\frac{2m}{a}} \times \sqrt{\frac{s}{a-s}}.$$

Now, to obtain from this equation of the relations between v and s that which obtains between s and t, we must substitute $\frac{\dot{s}}{\dot{t}}$ for v: then taking the reciprocal of the expression, multiplying by $\dot{s}$, and the quantity affected with the radical by $a - s$, we have $\dot{t} = \sqrt{\frac{a}{2m}} \times \frac{a-s}{\sqrt{(as - s^2)}}\,\dot{s}$, where the last factor is equivalent to $\frac{\frac{1}{2}a - s}{\sqrt{(as - s^2)}}\,\dot{s} + \frac{1}{2}a \times \frac{\dot{s}}{\sqrt{(as - s^2)}}$. The first term has for its fluent $\sqrt{(as - s^2)}$: that of the second is found by transforming s into $\frac{1}{2}a - z$; it is then $\frac{1}{2}a \times \text{arc}\left(\cos = \frac{2z}{a}\right)$: consequently

$$t = \sqrt{\frac{a}{2m}} \times \left\{ \sqrt{(as - s^2)} + \tfrac{1}{2}a.\ \text{arc}\left(\cos = \frac{a - 2s}{a}\right)\right\}$$

This requires no correction, because s ought to be nothing at the same time that t is.

The preceding values of v and t resolve the problem proposed in the most general manner, comprising all the particular circumstances of the motion: we remark, for example, that $s = a$, gives $v = \infty$, and $t = \sqrt{\frac{a}{2m}} \times \frac{1}{2}a\pi$, where π is the circumference of the circle whose diameter is unity: the first of these expressions indicates that the velocity of the moving point is infinite at the centre of attraction; which is natural to conceive, because the force is so much the more intense as the moveable is nearer the centre. The second expression is proportional to $a\sqrt{a}$, or $a^{\frac{3}{2}}$; whence it follows, that *the times employed by two bodies in descending from repose to the centre of attraction are respectively as the square roots of the cubes of the heights fallen from.*

In the case where bodies fall by their own gravity towards the earth, the attracting body being considered as a point with regard to the distance s, we shall have $m = 32\frac{1}{6}$ (art. 242.) and $t = \cdot 785398\, a\sqrt{\frac{2a}{m}}$.

237. III. It is required to determine the circumstances of velocity, time, and space, with relation to a body, which moves from quiescence in consequence of an attracting force which varies directly as the distance from the centre of force.

Let the point from which the body commences its motion be P (fig. 5. pl. XI.), and let PC $= a$, its distance from C the centre of force; let $v =$ the velocity at any variable distance AC $= s$, and at any distance d from c let f be the force compared with that unit of force whose representative is m. Then, by the nature of the problem, it will be $d : s :: f : \frac{fs}{d}$, the force at the distance s, compared with unity, or $\frac{mfs}{d}$ will be that force with respect to the measure m, corresponding with φ in our equation 232. iii. Hence, since v increases as s decreases, we shall have

$$v\dot{v} = -\frac{mfs}{d}\dot{s}.$$

This equation gives us $v^2 = -\frac{mf}{d}s^2 + \text{C}$. Here when $v = 0$, $s = a$, and $0 = -\frac{mf}{d}a^2 + \text{C}$; therefore $\text{C} = \frac{mf}{d}a^2$; and, consequently $v^2 = \frac{mf}{d} \times (a^2 - s^2)$, and $v = \sqrt{\frac{mf}{d}} \times \sqrt{(a^2 - s^2)}$. Hence then, if with centre C and radius CP, the quadrant PDB be described, and at the point A whose distance from C is $= s$, the ordinate AD be drawn, because $\text{AD} = \sqrt{(\text{CD}^2 - \text{CA}^2)} = \sqrt{(a^2 - s^2)}$, we shall have $v = \text{AD}\sqrt{\frac{mf}{d}}$. So that the velocity corresponding to any space PA moved over, is as the *sine* AD of the circle answering to the versed sine PA, the radius being PC.

In order to find t, we must adopt the equation (1) or $\dot{t}v = -\dot{s}$, whence arises $\dot{t} = -\frac{\dot{s}}{v} = \sqrt{\frac{d}{mf}} \times \frac{-\dot{s}}{\sqrt{(a^2 - s^2)}}$. Now, if z = arc PD, we have $\dot{z} : -\dot{s} :: a : \sqrt{(a^2 - s^2)}$; therefore $\frac{-\dot{s}}{\sqrt{(a^2 - s^2)}} = \frac{\dot{z}}{a}$, and consequently $\dot{t} = \frac{\dot{z}}{a}\sqrt{\frac{d}{mf}}$: the fluent of this expression is $t = \frac{z}{a}\sqrt{\frac{d}{mf}} = \frac{\mathrm{PD}}{\mathrm{CP}}\sqrt{\frac{d}{mf}}$; which wants no correction, because when $t = 0$, $z = 0$. So that, while *the velocity at any point* A *is as the corresponding sine* AD, *the time of descent to that point is as the arc* PD. When A arrives at C we have $t = \frac{\mathrm{PDB}}{\mathrm{PC}}\sqrt{\frac{d}{mf}} = \frac{1}{2}\pi\sqrt{\frac{d}{mf}}$, for the time of falling to the centre. Hence, from whatever altitude CP the body begins to fall towards the centre, its whole time of descent will be the same, $\frac{\mathrm{PDB}}{\mathrm{PC}}$ being in all cases $= \frac{1}{2}\pi = 1{\cdot}570796$, a constant quantity.

COR. If a body be acted upon by a force which is every where as the distance from C, the time of its descent to that centre from any point P is to the time in which it would descend through that same space, if impelled by half the first force uniformly continued, as the circumference of a circle to four diameters. For, on the first supposition the time is as the quadrantal arc PB; and on the latter the time is as 2PC; and PB : 2PC :: circumf.: 8PC or 4 diameters.

238. On the supposition that the earth were a homogeneous sphere, the force of attraction to which any body below its surface would be subjected varies as the distance from the centre: if, therefore, a perforation were made in a right line from the surface to the centre, the circumstances of a body falling from the surface will be determined from the preceding investigation. In this case d, the distance at which the effects of the force are known would be = 20935200 feet, and f the effect of gravity compared with unity, or mf, compared with the assumed measure m, would be represented by $32\frac{1}{6}$ feet, the velocity acquired by a falling body after one second. Hence, when the body has fallen to C, we shall have $v = \mathrm{CB}\sqrt{\frac{mf}{d}} = d\sqrt{\frac{mf}{d}} = \sqrt{mfd} = 25950$ feet, or 4·9148 miles per second, for its velocity there: and $t = 1{\cdot}570796\sqrt{\frac{d}{mf}} = 1267\frac{1}{4}'' = 21^{m}7\frac{1}{4}''$, for the time of falling to the centre.

239. We close the subject of variable motions with observing that if the velocity v be as any power n of s the space described, the time may be found by the method of fluxions; for,

since v is as s^n, and $\dot{t}$ as $\frac{\dot{s}}{v}$, it will also be as $\frac{\dot{s}}{s^n}$; consequently $t = \frac{s^{1-n}}{1-n} + c$, the correction. The application of this theorem to any cases which the student wishes to pursue must be left to his own industry. It has been our object to exhibit here only two or three of the most useful examples, to shew the utility of the doctrine of variable forces: it is now time to advert to other topics.

CHAPTER II.

On the Descent and Ascent of Heavy Bodies, in vertical Lines; the Motion of Projectiles; Descents along inclined Planes, and Curves; the Vibrations of Pendulums, &c.

240. The motion of heavy bodies at or near the surface of the earth occupies an extensive portion of the theory of mechanics; and its numerous applications to the various purposes of life renders it highly worthy the attention of the student. At the beginning of Chap. III. of our first Book we made a few such observations on the nature of gravity as were requisite in discussing the subject of the centre of inertia: in addition to what was there stated, we present a remark or two, more immediately connected with the business before us. Gravity being that force which solicits all bodies to descend in vertical lines, or those which are perpendicular to the surface of the earth, it would follow that, if that surface, as composed of land and sea, were perfectly spherical, the directions of gravity would all concur at its centre. The earth, however, is not perfectly spherical; yet is its variation from that shape so trifling, that with respect to the objects we now mean to consider, it need not be regarded. We observed in art. 106, that, in most mechanical enquiries, the directions of gravity may be considered as parallel: that it may be seen to what extent this remark may be applied, let it be considered that a circle whose radius is 20935200 feet (art. 238.) will have more than 6000 feet for the measure of a minute of a degree, and upwards of 100 feet for that of a second; so that *the directions of gravity at two places on the earth's surface, a mile asunder, will not vary one minute from parallelism.*

As to the magnitude of the gravitating force, strictly speaking, it is different at different distances from the equator, and at different distances from the centre of the earth: but the quantities of these differences, so far as they depend upon the variety of situation on the earth's surface, are very small, (286) and need not yet be attended to; and the differences resulting from different distances from the centre of the earth will not be sensible in any cases we shall have to consider here with regard to falling bodies. Thus, if two bodies were situated, the one at the surface of the earth, and the other at a mile above it, the difference in the attractions to which they would be subjected would be 3966^2-3965^2, or 7931 compared with 3965^2, or nearly one part in 2000 of the whole attraction: so that as a mile is greater than any altitude, or any depth from the surface, with regard to which we shall have to trace the effects of falling or rising bodies, we may consider the force of gravity as constant. We consider therefore this force as acting incessantly, and acting equally at each instant upon every particle of matter. Now it is clear that, if every particle of a body receive the same velocity, the aggregate of the body will move with the same velocity as would have been impressed upon a single molecule: consequently the velocity which gravity impresses upon any mass whatever does not depend upon the magnitude of that mass; but is the same with respect to the smallest mass as the greatest. It is true, we do not, when bodies of different masses and densities descend through the air, observe them all to fall from the same heights in equal times; but this is occasioned by the resistance of the medium, and when that is taken away, as in the receiver of an air-pump, the most dense and the most rare bodies fall through equal spaces in equal times. These premises being admitted, we may readily establish the ensuing proposition.

241. Prop. *The chief properties of motion delivered in arts. 226...230, with respect to constant forces, have place in the motions of bodies descending freely in consequence of the action of gravity.*

This is evident from the preceding remarks, granting the assumption that gravity is a constant force: for it is no more than saying that the laws which are shewn to obtain with regard to constant forces in general, apply directly to any individual constant force proposed.

Or, if we adopt the universally received hypothesis, that the gravitating force towards the earth varies inversely as the square of the distance from its centre, we may deduce the truth of this proposition from what was done in art. 236. in relation to that species of variable motion. Retaining the notation of that article, we have $v=\sqrt{\frac{2m}{a}}\times\sqrt{\frac{s}{a-s}}$, and $\dot{t}=\sqrt{\frac{a}{2m}}\times\frac{a-s}{\sqrt{(as-s^2)}}\dot{s}$: in which equations, in order that they may apply to the case of heavy bodies falling at or near the surface of the earth, we suppose a to represent the radius of the earth, and that s, the space described, is indefinitely small compared with a; the first equation then becomes $v = \frac{\sqrt{2ms}}{a}$, and consequently $\dot{t} = \frac{a\dot{s}}{\sqrt{2ms}}$. Now if g be the value, or representative, of the accelerating force at the distance a from the centre of the earth, we have $m : g :: a^2 : 1$, whence $m = ga^2$. This value of m put for it in the last given expression for v makes it become $v = \sqrt{2gs}$: hence $t = \frac{1}{\sqrt{2g}} \times \int \frac{\dot{s}}{\sqrt{s}} = \frac{2\sqrt{s}}{\sqrt{2g}} = \sqrt{\frac{2s}{g}}$, and consequently $s = \frac{1}{2}gt^2$. This value of s agrees exactly with that at art. 227. the general representative ϕ of the accelerating force in that expression being here supplied by g, the particular representative of the force of gravity at the earth's surface. Gravity may, therefore, be regarded as impressing on falling bodies a constant accelerating force; and the proposition is true.

242. We must now ascertain the real value of g, the measure of the force of gravity on the earth's surface: this, as we have already remarked, varies at different distances from the earth's equator; but it will be sufficient at present if we determine it for some one place, London, for example. Now, it is found by means of accurate experiments with the pendulum (art. 271.), and by other means which need not here be described, that a heavy body in the latitude of London falls nearly 16 $\frac{1}{12}$ feet in the first second of time from its quiescent state, and has then (art. 226.) acquired a velocity which, if uniformly continued, would carry it over twice 16 $\frac{1}{12}$, or 32 $\frac{1}{6}$ feet in the next second; but the action of gravity upon the body continuing, the motion will be such that at the end of the next second of time the body will altogether have passed over 2^2, or four times 16 $\frac{1}{12}$ feet (art. 227.), that is, 64 $\frac{1}{3}$ feet; and will have acquired a velocity (art. 225.) of twice 32 $\frac{1}{6}$, or 64 $\frac{1}{3}$ feet per second: and in a similar manner will the force of gravity operate in the succeeding seconds; so that

If the times in seconds be	1,	2,	3,	4,	&c.
The vel. acq. in feet will be	$32\frac{1}{6}$,	$64\frac{1}{3}$,	$96\frac{1}{2}$,	$128\frac{2}{3}$,	&c.
The spaces in the whole times	$16\frac{1}{12}$,	$64\frac{1}{3}$,	$144\frac{3}{4}$,	$257\frac{1}{3}$,	&c.
The spaces for each second	$16\frac{1}{12}$,	$48\frac{1}{4}$,	$80\frac{5}{12}$,	$112\frac{7}{12}$,	&c.

Thus, then, the action of gravity continues to change the state of the body: at the end of the first second from quiescence it had impressed upon it a velocity which, if continued uniformly, would carry it through 32 $\frac{1}{6}$ feet in the next second: but the solicitations of gravity being still exerted on the body, it actually describes $48\frac{1}{4}$ feet, and acquires a velocity of $64\frac{1}{3}$ feet; at the end of the third second it has acquired a velocity of $96\frac{1}{2}$; and at the end of the fourth a velocity of $128\frac{2}{3}$ feet. Now $128\frac{2}{3} - 96\frac{1}{2} = 96\frac{1}{2} - 64\frac{1}{3} = 64\frac{1}{3} - 32\frac{1}{6} = 32\frac{1}{6}$, is the invariable difference of the determinations to motion at the end of each successive second of time: consequently $32\frac{1}{6}$ feet, being a constant quantity naturally arising from the free motions occasioned by gravity, is a just indication and proper measure of its constant intensity upon bodies subjected to its operation; that is, g, in our theorems, is $= 32\frac{1}{6}$ feet. The space passed over in the first second may, it is true, be used, because it is the half of $32\frac{1}{6}$, and because halves have the proportion of the wholes: but it should not be forgotten that this is only *half* the true measure of gravitating force.

243. Hence, if bodies simply fall from a quiescent state, not being projected downwards by any additional force, the velocities acquired will be as the times, and the whole spaces described as the squares of either; in such manner, that

If the times be as the Nos.	1,	2,	3,	4,	5,	6,	&c.
The velocities acq. will be, as	2,	4,	6,	8,	10,	12,	&c.
The whole spaces, as	1,	4,	9,	16,	25,	36,	&c.
The space for each time, as	1,	3,	5,	7,	9,	11,	&c.
Their constant differences,		2,	2,	2,	2,	2,	&c.

The latter answering to $32\frac{1}{6}$, the value of g. And if, while this value of g is retained, we put v for the velocity acquired at the end of any time t, and s for the space described from quiescence during that time, we shall, by comparing the latter equations in art. 241. with the general formula in art. 228., have the following general equations for the free descent of heavy bodies; viz.

$$s = \tfrac{1}{2}gt^2 = \frac{v^2}{2g} = \tfrac{1}{2}tv.$$

$$t = \frac{v}{g} = \sqrt{\frac{2s}{g}} = \frac{2s}{v}.$$

$$v = \sqrt{2gs} = \frac{2s}{t} = gt.$$

$$g = \frac{v}{t} = \frac{2s}{t^2} = \frac{v^2}{2s}.$$

To these theorems reference must be made in all cases where great accuracy is required; but in many practical instances the fraction $\frac{1}{6}$ may be dropped, and the computation much facilitated by taking $g = 32$ feet: the theorems may then take this form:

$$s=16t^2=\frac{v^2}{64}=\tfrac{1}{2}tv.$$

$$t=\frac{v}{32}=\tfrac{1}{4}\sqrt{s}=\frac{2s}{v}.$$

$$v=8\sqrt{s}=\frac{2s}{t}=32t.$$

244. DEFS. *The height due to a given velocity* is the height from which a body must fall freely to acquire that velocity; and *the velocity due to a given height* is the velocity which a body will have acquired after descending freely through that height.

It will now be easy to solve the common questions relating to the descent of heavy bodies. Thus, if it were required to find the velocity due to a given height 400 feet, and the time of falling through it; we should have $v=8\sqrt{s}=8\times 20=160$ feet: and $t=\frac{v}{32}=\frac{160}{32}=5$ seconds, the time of descent.

245. But if the body, instead of falling from quiescence be impelled vertically with any initial velocity v, we shall then have the equation IV. art. 229. for that comprehending the circumstances of its motion, in which, if we reckon the initial space $=0$, we shall have

$$s=\mathrm{v}t+\tfrac{1}{2}gt^2,\text{ or } s=\mathrm{v}t-\tfrac{1}{2}gt^2$$

according as the velocity is impressed downwards or upwards, s being estimated in the direction of that velocity. In the second case we have $v=\mathrm{v}-gt$, as is evident, because the velocity impressed by gravity downwards is gt at the end of the time t. From this expression it appears that the body continues to rise so long as v exceeds gt; and at the point where $\mathrm{v}=gt$ the body will have attained its greatest elevation: lastly, when v becomes less than gt, v will be negative; that is, the body is descending again, because the initial velocity has been extinguished by gravity. The maximum of elevation is $\frac{\mathrm{v}^2}{2g}$, and the time employed to attain it is $\frac{\mathrm{v}}{g}$. The body will descend from the state of rest at its greatest height according to the preceding laws, and the equation of its motion will be $s=\tfrac{1}{2}gt^2$, estimating the spaces downwards. At the end of the time $t=\frac{\mathrm{v}}{g}$ the body will have run over the space $\frac{\mathrm{v}^2}{2g}$; thus it will employ to re-descend to the point of departure the same extent of time as it took to mount to its greatest height; and finally, at the termination of its fall it will have acquired the velocity v of projection.

Hence we see,

1st. That a heavy body thrown upwards vertically with any velocity will lose equal velocities in equal times.

2d. That the heights to which bodies projected upwards will rise, before their motions are extinguished, are as the squares of the initial velocities, or as the squares of the times.

3dly. That those heights are the halves of the spaces which they would uniformly describe, in the same time, with the initial velocities.

4thly. If a body be projected upwards with the velocity it acquired in any time by descending freely, it will lose all its velocity in an equal time; it will ascend to the height from whence it fell, and will describe equal spaces in equal times both in rising and falling, but in an inverse order; it will also have equal velocities at any one and the same point of the line described, both in ascending and descending.

We may, therefore, find to what elevation a body thrown vertically has arisen, when we know the time elapsed between the commencement of its motion and its return to the point from which it was projected. For example, a body thrown vertically upwards returns at the end of 18 seconds; it therefore occupied 9'' in ascending; consequently $s = 16t^2 = 16 \times 81 =$ 1296 feet nearly the height to which it ascended; and $v = 32t =$ $32 \times 9 = 288$ feet per second, the initial velocity.

So, again, it will be easy from the same principles, to tell, when a body is projected vertically upwards with a given velocity and continues to be operated upon freely by the force of gravity for a given time, whether it be above or below the point of projection at the end of that time. 1. Suppose the initial velocity to be 50 feet per second, and the time four seconds. Then $vt - \frac{1}{2}gt^2 = 200 - 257\frac{1}{3} = -57\frac{1}{3}$; consequently, at the end of the proposed time the body would be $57\frac{1}{3}$ feet *below* the point from which it was projected. 2. But if the time had been 3 seconds, we should have had

$$vt - \tfrac{1}{2}gt^2 = 200 - 144\tfrac{3}{4} = 55\tfrac{1}{4};$$

where, since vt is the greatest, the difference shows the height at which the body is *above* the point of projection at the end of the given time.

Lastly: Suppose that at the same moment a body begins to fall from quiescence from the point D (fig. 7. pl. XI.), another body is projected upwards from B with the velocity due to the height BC; it is required to find the point N at which the two bodies would meet. Let CB be denoted by a, DB by d, and DN by x. Then will CD $= a - d$, and CN $= a - d + x$. Time of *descending* through DN $= \sqrt{\frac{2x}{g}}$. Time of *ascending* through BN ($=$ time down CB $-$ time down CN) $= \sqrt{\frac{2a}{g}} - \sqrt{\frac{2(a-d+x)}{g}}$

These, by the nature of the problem, must be equal; that is,

$$\sqrt{\frac{2x}{g}}=\sqrt{\frac{2a}{g}}-\sqrt{\frac{2(a-d+x)}{g}}$$

$$\text{or } \sqrt{x}=\sqrt{a}-\sqrt{(a-d+x)}$$

$$\therefore \sqrt{(a-d+x)}=\sqrt{a}-\sqrt{x}$$

$$\text{and } a-d+x=a+x-2\sqrt{ax}.$$

Whence $2\sqrt{ax}=d$, and $x=\frac{d^2}{4a}$.

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246. The equation expressing the conditions of falling bodies has for its locus the common parabola; being in fact the same as the locus for constant forces, the construction of which we have already explained in art. 230.

The true theory of falling and rising bodies was first given by Galileo, who may, indeed, be looked upon as the father of the science of Dynamics: his discoveries on these topics were published in a work entitled *Dialoghi delle Scienze nuove,* &c. printed for the first time, at Leyden, in 1637. His method of investigation (excepting that he represents the spaces described by the motions of bodies to be areas, which is rather unnatural) is very elegant: and as he deduces the laws of their motions from two theorems only, we shall, for the satisfaction of the student, insert them here.

THEOR. I. *The time in which any space is passed over by a moving body, with a motion uniformly accelerated from rest, is equal to the time in which the same space would be passed over by the same moveable, carried with a uniform velocity which is half the greatest and ultimate velocity of the former uniformly accelerated motion.*

Let the line AB (fig. 8. pl. XI.) represent the time in which the space CD is described by a moveable, with a uniformly accelerated motion from quiescence at C, and let the velocity acquired at the end B of the time AB be represented by the line EB, drawn at pleasure to AB, and join AE. Divide AB into any number of equal parts, and through the points of division draw as many lines parallel to EB; these will represent the increasing degrees of velocity after the first instant A. Bisect BE in F, and draw FG, AG, parallel to BA and BF respectively: then will the parallelogram AGFB be equal to the triangle AEB, and the sides GF, AE, bisect each other in I. For if the parallels in the triangle AEB be extended to GF, the aggregate of all those in the quadrilateral will be equal to the aggregate of those in the triangle AEB; those in the triangle IEF being equal to those in

GAI, and those in the trapezoid AIFB being common. Now, since all and each of the instants of the time AB answer to all and each of the points of the line AB, and since parallels drawn from these points comprehended in the triangle AEB represent the increasing degrees of augmented velocity, while the parallels comprised within the parallelogram represent in like manner as many degrees of equable velocity, it appears that as many momenta or effects of velocity were produced in the accelerated motion according to the increasing parallels of the triangle AEB, as in the uniform motion according to the parallels of the quadrilateral GB: the deficiency in the first half of the motion, represented by the triangle AGI, being made up by the supernumerary triangle IEF in the latter half of the time. It is manifest, therefore, that those spaces will be equal which will be passed over in the same time by two moveables, one of which moves with a velocity uniformly accelerated from rest, but the other with an equable motion according to a velocity which is half the greatest velocity of the accelerated motion.

Theor. II. *If a moveable body descend from rest with a uniformly accelerated motion, the spaces it passes over in any times whatsoever are to each other as the squares of those times.*

Let the line AB (fig. 9. pl. XI.) represent an interval of time from any first instant A, in which take any two times AD and AE; and let HI be a line in which the moveable descends from rest at the point H with a uniformly accelerated motion; the space HL being passed over in the first time AD, and the space HM in the time AE: then is the space HM to the space HL in the duplicate proportion of the time AE to the time AD. Draw the line AC, making any angle with AB, and from the points D and E draw the parallels DO, EP; the former representing the velocity acquired at the end of the time AD, and the latter the velocity acquired at the end of the time AE: then, it is manifest, from the last theorem, that the spaces MH and LH are the same which with equable motions whose velocities would be as the halves of PE, and OD, would be passed over in the times EA and DA. Now it has been demonstrated (art. 216.) that of bodies carried with an equable motion the spaces passed over are in proportion to each other in the ratio compounded of that of the velocities and that of the times: but here the ratio of the velocities is the same as that of the times (for that proportion which the half of PE has to the half of OD, the same has PE to OD, and the same has AE to AD), therefore the spaces are in a duplicate ratio of the times, or $HM : HL :: AE^2 : AD^2$.

Hence it is manifest that the ratio of the spaces is likewise the duplicate of the ratio of the ultimate velocities; that is, $HM : HL :: PE^2 : OD^2$.

COR. 1. Hence, if there were how many soever equal times taken successively from the first instant of motion, as, for example, AD, DE, EF, FG, in which are passed over the spaces HL, LM, MN, NI; those spaces will be to one another as the uneven numbers, viz. 1, 3, 5, 7: for this is the ratio of the excesses of the squares of the lines exceeding one another equally, the excess of which lines is equal to the least of them: whilst, therefore, the velocities are increased according to the simple series of numbers in equal times, the spaces run through in the same times increase according to the series of the uneven numbers.

COR. 2. Hence again, it is inferred, that, if from the beginning of the motion any two spaces are taken, passed through in any times, those times shall be to each other as either of the said spaces is to a mean proportional between them.

247. Galileo likewise shows that the same laws of acceleration obtain in the motion of bodies along inclined planes; and he illustrates the conformity of his theory with nature by a relation of some of his experiments. "We took," says he, "a prism of wood about twelve yards long, half a yard wide, and about three inches thick; in which thickness we made a very straight groove a little more than an inch wide, and to render it very smooth and sleek we glued within it a piece of vellum polished as much as possible. In this groove we let fall a ball made of the hardest brass, round, and well polished. Then, elevating one end of this prism at pleasure a yard or two above the plane of the horizon, we let the ball descend along the groove, observing in the manner I shall tell you presently the time spent in its fall from top to bottom.

"We repeated this often, in order to be certain of the quantity of time spent in the descent; and in these times we never found any difference worth mentioning, not even the tenth part of a second: this being thoroughly established, we let the same ball descend but a fourth part of the length of the groove, and measuring the time of the descent we found it to be exactly the half of the other: and then making trial of the other parts, by comparing the time of its descent through the whole length with the times in which it ran through ½, ⅔, ¾, or, in a word, with the time of its running through any part of its length, by experiments nearly a hundred times repeated, we always found the spaces run through to be to one another as the squares of the times; and this in all inclinations of the plane, *i. e.* of the groove in which the ball was made to descend. We also observed the times of the descents along different inclinations to retain the same proportion to one another which we shall find demonstrated and assigned to them.

"Now, as to the measuring of time, we hung on high a large vessel of water, which, by a very small hole through the bottom, emitted a small thread of water, which was received by a small cup, all the time the ball was falling through the several parts of the groove: then the small portions of water thus collected were weighed every time in a very exact pair of scales, and the differences and proportions of the weight exhibited to us the differences and proportions of the times, and that, as I said before, so accurately, that those trials often repeated never differed any thing worth speaking of." *Dial.* III.

II. *On the Motion of Projectiles, in Vacuo.*

248. It is not our intention to enter at all largely into the subject of gunnery: but merely to present a brief view of the theory of the motion of bodies any way projected from the surface of the earth in an unresisting medium, and then to add a few remarks on the disparity between this theory and the circumstances attending bodies actually projected into the atmosphere. In our discussions on this subject we shall pay no regard to the motions of the earth, nor to the variation either in the direction or magnitude of the force of gravity: for we are to determine the path of the projectile with respect to the surface of the earth; and the force of gravity upon any body that can be projected by human contrivances is at all times so nearly equal and in parallel directions (art. 240.) that it would be a useless refinement to attend to the deviations from equality and parallelism. When bodies are projected either directly *upwards* or directly *downwards* the circumstances of their motion may be ascertained by means of the theorems in art. 245. We shall now consider the circumstances resulting from their being projected *obliquely;* observing that the theorems we shall deduce are only so far of consequence as that they shew what would be the nature of the motion of balls and shells fired from pieces of ordnance, in free space.

249. Prop. *The line described by a heavy body thrown in any direction not perpendicular to the horizon is a parabola.*

Let the body be projected from the point B in the direction BH (fig. 11. pl. XI.) with the velocity it would have acquired by falling from A to B, then will the continual action of gravity cause the body to be continually deflected from the line BH so as to describe a curve line concave towards the horizon. Produce AB both ways as to K and T; and through any two points V and G of the curve draw VE, GK, parallel to BH, and VC, GH, parallel to AB the direction of gravity. Now it follows from the composition of motions, that the body will arrive at the point V of the curve in the same time that it would have described the right line BC by the uniform projectile motion alone, or the vertical BE by the force of gravity alone; and in like manner the body will have arrived at G in such time as it would have uniformly described the line BH with the original velocity at B, or the vertical BK by falling freely. But the motion along BH being uniform, we have BC : BH : : $t \cdot$ BC : $t \cdot$ BH (the letter t denoting the time of describing BC, BH, &c.) : : $t \cdot$ BE : $t \cdot$ BK : and, because the motion along BK is uniformly accelerated, we have BE : BK : : $t^2 .$ BE : $t^2 .$ BK : : BC^2 : BH^2 : : EV^2 : KG^2. Therefore, the curve BVG is such that the abscissæ BE, BK, are as the squares of the corresponding ordinates EV, KG; or that the parts VC, GH, of the parallel lines are as the squares BC, BH of the intercepted parts of the line BH : and consequently the curve is a parabola to which BC is a tangent.

COR. 1. *The horizontal line* ADd *drawn through the point* A *is the directrix of the parabola:* for, let BE be taken = AB. Then $t \cdot \text{BE} = t \cdot \text{AB}$; but BC is described in the same time with the velocity acquired by falling through AB; therefore (arts. 226. 242.) $\text{BC} = 2\text{AB}$, and $\text{EV} = 2\text{BE}$: and hence $\text{EV}^2 = 4\text{BE}^2 = 4 \times \text{BE} \times \text{BA} = \text{BE} \times 4\text{BA}$: so that 4AB is the parameter of the parabola BVG, and consequently ADd is the directrix, because AB is one-fourth of the parameter of the diameter BK.

COR. 2. *The times of describing the different portions* BV, VG, *of the curve, are as the corresponding parts* BC, CH, *of the tangent at* B, *or the intercepted parts* AD, Dd, *of the directrix.* For $t \cdot \text{BV} = t \cdot \text{BC}$, and $t \cdot \text{VG} = t \cdot \text{CH}$, by the demon. of the prop. and because Dd cuts the three parallels BT, VC, GH, BC is to CH, as AD to Dd.

COR. 3 *The velocity estimated horizontally is uniform.*

COR. 4 *The velocity of the projectile at any point* G *of the curve, estimated in the direction of the tangent to the parabola at that point, is as the secant of the angle contained between that tangent and the horizon.* For the motion in the horizontal direction Ad is uniform; and Ad is to TG as radius to the cosecant of HGT, or secant of the angle included between the horizon and GT.

COR. 5. *The velocity at the point* G *in the direction* TG *is equal to that which a heavy body would acquire by falling freely from the directrix through* dG. For, let the points a, f, be equidistant from A and d, but indefinitely near them, and draw the verticals ab, fg: conceive these to approach towards

AB, *d*G, and ultimately to coincide with them; then will B*b* be ultimately to *g*G as the velocity in the curve at B to the velocity at G. But B*b* : G*g* : : BH : TG : : vel. at B : vel. at G; and, by the nature of the parabola $BH^2 : TG^2$: : AB : *d*G : also AB is to *d*G as the square of the velocity acquired by falling through AB to the square of the velocity acquired by falling through *d*G (art. 227. cor. 1.): but the velocity in BH or in the point B of the parabola is = the velocity acquired by falling through AB (by hyp.); therefore the velocity in TG, or in the point G of the curve, is = the velocity acquired by falling through *d*G.

Cor. 6. *The velocity in the vertical direction at any point* G *of the curve is to the original projectile velocity at* B *as* 2HG *to* BH. For the times in BH and HG being equal, and the velocity acquired by falling freely through HG being such as would carry the body uniformly over 2HG in an equal time; we have (art. 216. cor. 3.) BH to 2HG, as the projectile velocity at B to the vertical velocity at G.

250. Defs. The height due to the original velocity of the projectile, as AB (fig. 11.) or CA (fig. 12.), is called the *Impetus*. In our general theorems it will be denoted by I.

The distance AB (fig. 12.) between the point of projection and the point where the body falls, measured on the plane AB, is called the *Amplitude*, *Random*, or *Range*: it will be represented by R.

Some authors restrict the word *Amplitude* to the range on a horizontal plane.

The angle DAB made by the axis of the gun, or the tangent to the curve at A, and the direction of the object at B, is called the angle of *Elevation* above the plane AB: it will be denoted by E.

The angle DBA included between the vertical DB (passing through the object) and the plane AB, is called its *angle of position*: we represent it by P.

The angle ZAD made by the vertical ZA (passing through the point of projection) and the direction of the piece, is called the *zenith distance*: it will be denoted by Z.

251. Prop. *To give a general view of the relative situation and properties of such lines as are most useful in considering the motions of projectiles.*—

Let a body be projected from B (fig. 13. pl. XI.) in any direction BC with the velocity acquired by falling through AB: this body by art. 249. will describe a parabola BVPM. Produce AB both ways till OA = AB = BK; and with centres A, B, and radius AB, describe the semicircle, ODB, AHK: then with axis AB and semiaxis GE = AB = AD, describe the semi-ellipse AEB;

and, with focus B, vertex A, diameter AB, and tangent AD, parallel to the horizon, draw the semiparabola APS. This done, we observe

I. *The horizontal line* ADL *is the directrix of every parabola which can be described by a body projected from* B *with the velocity acquired by falling through* AB. This is evident from the foregoing proposition, and the usual definitions of the directrix.

II. *The semicircle* AHK *is the locus of all the foci of these parabolas.* For BA, the distance of a point B of any parabola from the directrix AD, is equal to BF, its distance from the focus F of that parabola: consequently, the foci of all the parabolas which pass through B, and have AD for their directrix, must be in the circumference of the circle whose centre is B and radius AB.

III. *If* BC, *the line of direction, cut the upper semicircle in* C, *and the vertical* CF *be drawn intersecting the lower semicircle in* F, *then is* F *the focus of the parabola, which is described by the body whose impetus is* AB, *and initial direction* BC. For, if AC, BF, be joined, then is ACFB evidently a rhombus, and the angle ABF is bisected by BC: consequently the focus is somewhere upon the line BF. But it is also upon the circumference of AHK; and must, therefore, be at F the point where they intersect.

When C is in the superior quadrant of ODB, F is in the superior quadrant of AFK: and when *c* is in the inferior quadrant of ODB, as when B*c* is the initial direction, then the focus *f* of the corresponding parabola B*v*M is in the inferior quadrant of AHK.

IV. *The semiellipse* AEB *is the locus of the vertices of all the parabolas, and the vertex* V *of any one of them* BVPM *is in the point where this ellipse intersects the vertical* CF. For, let this vertical intersect the horizontal lines AD, GE, BN, in θ, λ, N: then, it is manifest that $\lambda\theta = \frac{1}{2}N\theta = N\lambda$, and that $\lambda V = \frac{1}{2}\theta C = \frac{1}{2}NF$; therefore, $\lambda\theta - \lambda V = \frac{1}{2}(N\theta - NF)$, or $\theta V = \frac{1}{2}\theta F$; consequently $\theta V = \frac{1}{2}VF$, and V is the vertex of the parabola.

The vertex is at V or *v*, in the upper or lower quadrant of the ellipse, according as the focus is in the upper or lower quadrant of the semicircle AHK.

V. *If from the point of projection* B *the line* BFP *be drawn through the focus of any one of the parabolas, as of* BVM, *cutting the parabola* APS *in* P, *then will the curve* BVM *touch the curve* APS *in* P. For, if we draw Pδz parallel to AB, cutting the directrix Oz of the parabola APS in z, and the directrix AD of the parabola BVM in δ, then will $PB = Pz$, by the nature of the parabola: but $BF = BA = AO = \delta z$; therefore, $PB - BF = Pz - \delta z$, or $PF = P\delta$, and the point P is in the parabola BVM. And since the tangents to both parabolas in P bisect the angle SPB, they coincide; consequently the two parabolas, having a common tangent in P, touch each other at that point.

COR. All the parabolas which can be described by a body projected from B, with the velocity acquired by falling from A to B, will touch the concavity of the parabola APS, and lie wholly within it.

VI. P *is the most distant point of the line PB which can be struck by a body projected from B with the velocity due to the height* AB. For, if the direction be more elevated than BC, the focus of the parabola which the body would describe will lie between B and A, and the parabola will touch APS in some point between A and P; so that, being wholly within the parabola APS, it must intersect the line BP in some point nearer to B than P is. If, on the contrary, the direction is less elevated than BC, the parabola described by the projectile will touch APS in some point below P, and will, therefore, cut BP in some point not so far distant from B as P.

VII. *The parabola* APS *is the locus of the greatest ranges on any planes* BP, BS, *&c. and no point lying without this parabola can be struck while the initial velocity remains unchanged.*

VIII *To produce the greatest range on any plane* BP, *the line of direction* BC *must bisect the angle* OBP *formed by that plane and the vertical:* for in this case the parabola described by the body touches APS in P, and its focus is in the line BP; consequently the tangent BC bisects the angle OBP.

COR. *On a horizontal plane the greatest range is made when the elevation is 45° or half a right angle. And that greatest range is 4 times the height of the parabola which produces it. Because, the parameter is 4 times the distance from the focus to the vertex.*

IX. *A point* M *in any plane* BS, *lying between* B *and* S, *may be struck with two directions* BC *and* Bc; *and these directions are equidistant from the direction* Bt *which gives the greatest random on that plane.* For, with centre M and radius ML (the nearest distance to the directrix) let a circle LFf be described; it will manifestly either touch or cut the circle AHK in two points F and *f*, which are the foci of two parabolas BVM, BUM, having the directrix AL, and diameter ABK. Here the intersection of the circle ODB with the verticals FC, *fc*, determine the directions BC, B*c*, of the tangents. Now, draw A*t* parallel to BS, and join *t*B, *tc*, F*f*; then is OB*t* = ½OBS, and B*t* the direction which produces the greatest range on the plane BS: but F*f* being a chord of the circles described about the centres B and M, is perpendicular to BM; as is likewise C*c* to A*t*; and arch C*t* = arch *ct*; and therefore the angle CB*t* = angle *c*B*t*.

COR. On a horizontal plane, the ranges are equal at angles of direction equidistant from 45°; as at 40° and 60°, 35° and 65°, &c. the impetus being given.

X. *Every velocity of projection has a corresponding set of parabolas, with their directions and ranges, and,* cæteris paribus, *the ranges on any plane are as the squares of the initial velocities* For, every magnitude of velocity has an impetus AB corresponding to it: and, as the height due to any velocity increases or diminishes in the duplicate ratio of that velocity, it is obvious that all the ranges with a given direction will vary in the same ratio.

252. PROP. *To exhibit the relations between all the circumstances of velocity, elevation, position, range, greatest height, and time.*

Suppose a ball or shell projected from A (fig. 12. pl. XI.) with the velocity due to the height CA, in order to strike the mark B situated in the given line AB. Make AZ = 4AC, and draw BD parallel to it, or perpendicular to the horizon. On ZA describe a circular segment ZAD containing an angle ZDA = ∠ DBA, and draw AD to the intersection of this circle with BD: then will a body projected from A, in the direction AD, with the velocity due to the height CA, strike the object B. For, produce CA downwards till BF drawn parallel to DA meets it in F; and join ZD: then, it is manifest from the construction that the angles ADZ and DBA are equal, as likewise AZD and DAB; that AB touches the circle in A, and that the triangles ZAD, ADB, are similar.

$$\text{Hence } BD : DA :: DA : AZ$$

$$\text{And } DA^2 = BD \cdot AZ$$

$$\text{Consequently } BF^2 = AF \cdot AZ = AF \cdot 4AC:$$

Therefore a parabola which has AF for a diameter, and AZ its parameter, will pass through B, and this parabola will be the path of the projectile.

But when BD cuts the circle ZDA, it cuts in two points D, *d*; so that there are two directions which will solve the problem. If B′D′ only touch the circle in D′ there is but one direction, and AB′ is the maximum range with this velocity. If the vertical through B does not meet the circle, the problem is impossible; the initial velocity being too small. When B′D′ touches the circle, the two directions AD′, A*d*′, coalesce into one, producing the greatest range, and bisecting the angle ZAB; and the other two directions AD, A*d*, producing the same range AB, are equidistant from AD′; agreeably to Nos. viii. ix. of the preceding article.

Now, to deduce the relations acquired, let us confine our attention to the direction AD in the figure; where we see

$$\text{That } \mathrm{AZ} : \mathrm{AD} :: \sin \mathrm{ADZ} : \sin \mathrm{AZD} :: \sin \mathrm{DBA} : \sin \mathrm{DAB} :: \sin \mathrm{P} : \sin \mathrm{E}$$

$$\mathrm{AD} : \mathrm{DB} :: \sin \mathrm{DBA} : \sin \mathrm{DAB} :: \ldots\ldots\ldots\ldots \sin \mathrm{P} : \sin \mathrm{E}$$

$$\mathrm{DB} : \mathrm{AB} :: \sin \mathrm{DAB} : \sin \mathrm{ADB} :: \ldots\ldots\ldots\ldots \sin \mathrm{E} : \sin \mathrm{Z}$$

$$\text{Therefore comp. } \mathrm{AZ} : \mathrm{AB} :: \sin^2 \mathrm{P} \cdot \sin \mathrm{E} : \sin^2 \mathrm{E} \cdot \sin \mathrm{Z} :: \sin^2 \mathrm{P} : \sin \mathrm{E} \cdot \sin \mathrm{Z}.$$

$$\text{Or } 4\mathrm{I} : \mathrm{R} :: \sin^2 \mathrm{P} : \sin \mathrm{E} \cdot \sin \mathrm{Z},$$

$$\text{whence } 4\mathrm{I} \cdot \sin \mathrm{E} \cdot \sin \mathrm{Z} = \mathrm{R} \cdot \sin^2 \mathrm{P}.$$

This equation obviously comprises the relations of E, I, P, R, and Z.

Let us next determine the time of flight; which is equal to the time of describing AD uniformly, or of falling through DB. Now, since AB : DB : : sin ADB : sin DAB : sin Z : : sin E, we have $\mathrm{DB} = \frac{\mathrm{R} \cdot \sin \mathrm{E}}{\sin \mathrm{Z}}$. Here DB is equal to s in the general equation $s = \frac{1}{2}gt^2$ (art. 243.): therefore $\frac{\mathrm{R} \cdot \sin \mathrm{E}}{\sin \mathrm{Z}} = \frac{1}{2}gt^2$; and consequently $t = \sqrt{\frac{\mathrm{R} \cdot \sin \mathrm{E}}{\frac{1}{2}g \cdot \sin \mathrm{Z}}}$.

Lastly, to find the greatest height (H) of the projectile above the plane; draw from Q the middle point of AB, the line Q T t parallel to B D d: then, by a well known property of the parabola, QP = PT, or QT = 2QP; and by similar triangles AQ : QT : : AB (= 2AQ) : BD = 2QT = 4QP = 4H. But AB : BD : : sin Z : sin E : : R : 4H; consequently 4H · sin Z = R · sin E, and $\mathrm{H} = \frac{\mathrm{R} \cdot \sin \mathrm{E}}{4 \sin \mathrm{Z}}$.

Let us now take from the general equations, art. 243. the value of $v = \sqrt{2gs}$, from which, because $s = \mathrm{I}$ in the present case, and $v = \mathrm{V}$, we shall have $\mathrm{V} = \sqrt{2g\mathrm{I}}$; and comparing this with the preceding equations in this article, we may throw together the general theorems relating to projectiles, thus:

Projectiles, on Oblique Planes.

$$\mathrm{R} = \frac{\sin \mathrm{E} \sin \mathrm{Z}}{\sin^2 \mathrm{P}} \; \mathrm{I} = \frac{\sin \mathrm{E} \sin \mathrm{Z}}{\frac{1}{2}g \sin^2 \mathrm{P}} \mathrm{V}^2 = \frac{g \sin \mathrm{Z}}{2 \sin \mathrm{E}} \mathrm{T}^2 = \frac{4 \sin \mathrm{Z}}{\sin \mathrm{E}} \mathrm{H}.$$

$$\mathrm{V} = \sqrt{2g\mathrm{I}} = \sin \mathrm{P} \sqrt{\frac{g\mathrm{R}}{2 \sin \mathrm{E} \sin \mathrm{Z}}} = \frac{g \sin \mathrm{P}}{2 \sin \mathrm{E}} \mathrm{T} = \frac{2 \sin \mathrm{P}}{\sin \mathrm{E}} \sqrt{\tfrac{1}{2}g\mathrm{H}}.$$

$$\mathrm{T} = \frac{2 \sin \mathrm{E}}{\sin \mathrm{P}} \sqrt{\frac{2\mathrm{I}}{g}} = \frac{2 \sin \mathrm{E}}{g \sin \mathrm{P}} \mathrm{V} = \sqrt{\frac{2 \sin \mathrm{E}\mathrm{R}}{g \sin \mathrm{Z}}} = \sqrt{\frac{2\mathrm{H}}{g}}.$$

$$\mathrm{H} = \frac{\sin \mathrm{E}}{4 \sin \mathrm{Z}} \mathrm{R} = \frac{\sin^2 \mathrm{E}}{\sin^2 \mathrm{P}} \mathrm{I} = \frac{\sin^2 \mathrm{E}}{2g \sin^2 \mathrm{P}} \mathrm{V}^2 = \tfrac{1}{8}g\mathrm{T}^2.$$

$$\mathrm{I} = \frac{\sin^2 \mathrm{P}}{4 \sin \mathrm{E} \sin \mathrm{Z}} \mathrm{R} = \frac{\mathrm{V}^2}{2g} = \frac{g \sin^2 \mathrm{P}}{8 \sin^2 \mathrm{E}} \mathrm{T}^2 = \frac{\sin^2 \mathrm{P}}{\sin^2 \mathrm{E}} \mathrm{H}.$$

And from many of these the angle of elevation may readily be found *.

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* The case in which the plane on which the ball falls does not pass through the point of projection, though not considered above, is very easily

263. When the plane AB is horizontal, the angle $DBA = P = 90°$, and its sine radius; also angle $ZAD = Z$, becomes $=$ comp. E; therefore, since $\sin E \cos E = 2 \sin 2E$, $\sin^2 E = 2\,\text{vers}\sin 2E$, and $\sin \div \cos = \tan$, the preceding theorems will assume the following shape when applied to

Projectiles, on Horizontal Planes.

$$R = 4I \sin E \cos E = 2I \sin 2E = \frac{\sin 2E}{g} V^2 = \frac{gT^2}{2\tan E} = \frac{4H}{\tan E}.$$

$$V = \sqrt{2gI} = \sqrt{\frac{gR}{\sin 2E}} = \frac{gT}{2\sin E} = \frac{2}{\sin E}\sqrt{\tfrac{1}{2}gH}.$$

$$T = 2\sin E\sqrt{\frac{2I}{g}} = \sqrt{\frac{2R\tan E}{g}} = \frac{2\sin E}{g}V = 2\sqrt{\frac{2H}{g}}.$$

$$H = \tfrac{1}{4}\tan E R = I \sin^2 E = \frac{\text{vers}\sin 2E}{4g}V^2 = \tfrac{1}{8}gT^2.$$

$$I = \frac{R}{8\sin 2E} = \frac{V^2}{2g} = \frac{g}{16\,\text{vers}\sin 2E}T^2 = \frac{H}{\sin^2 E} = \frac{H}{2\,\text{vers}\sin 2E}.$$

And from many of these, again, the angle of elevation may soon be found. Here too, as in the instance of falling bodies, g may be taken $= 32$, without leading to any important error.

COR. 1. In horizontal ranges, if $E = 45°$, we have $R = 4H = 2I$: consequently, *the maximum range on a horizontal plane is equal to double the impetus, or to four times the greatest height of the projectile above the plane*; the impetus being, in that case, double the greatest height. Vide No. viii. art. 250.

COR. 2. When $E = 45°$, its sine is $= \sqrt{\frac{1}{2}}$; therefore $T = 2\sqrt{\frac{I}{g}} = \frac{1}{2}\sqrt{\frac{1}{2}I}$ nearly $= \cdot 35356\sqrt{I}$ nearly.

constructed. Let B be the point from which the ball is projected, CB the impetus (the figure may be readily conceived and drawn), GH the plane, whether horizontal or inclined (and not passing through B) on which the ball is to fall, and BD the direction in which it is projected from B. Draw BF making the angle $DBF = DBC$, and upon it take $BF = BC$; then is F evidently the focus of the parabolic trajectory of the ball. Draw from F a line FI to meet GH perpendicularly in I, and produce it till $If = IF$. Through C draw CR an indefinite perpendicular to BC, it will be the directrix of the parabola. Produce IF upwards to meet the directrix in E; and on CR set off from E towards R, EK a mean proportional between EF and Ef. From K let fall KP perpendicularly to CR, it will meet GH, in P, the point where the parabolic curve will intersect that line, as required. For, because by construc. $EK^2 = EF \cdot Ef$, it follows that EK is a tangent and Ef a secant to the circle that passes through the points K, F, f: consequently, since FP, and fP, are equal, it follows that both GH and KP pass through the centre of the circle; and thence that P is that centre, that $KP = PF$, and P a point in the parabola. The formulæ that are deducible from this construction, simple as it is, are rather too complex for insertion in the present note: though the computation requisite for any particular example may be readily enough drawn from the diagram.

Cor. 3. *When the initial velocity is given in feet, and* $E = 45°$, *we have* T *nearly* $= \frac{\sqrt{\frac{1}{2}}}{16}v = \cdot 044194\ v$.

The expressions in this and the former corollary may be useful to the bombardier in cutting the fuzes of his shells to proper *lengths, so that they may just* burst at the very instant they come *to the object fired at.*

Cor. 4. *When* the elevation is 15°, we have $\sin 2E = \sin 30° = \frac{1}{2}$: consequently $R = \sin 2E\ 2I = \frac{1}{2} \times 2I = I$. That is, *at an elevation of* 15°, *the range on a horizontal plane is equal to the impetus.*

Cor. 5. *When the elevation is* 75° *the horizontal range is equal to the impetus.* For 15° and 75° are equidistant from 45°, and therefore the ranges at those elevations are equal.

Cor. 6. In many cases, where in appearance a body has not received any impulsion, but is abandoned to the sole action of gravity, the body, notwithstanding, describes a parabola like a projectile. This, for example, is the case with a body let fall from the top of a ship's mast when the vessel is in motion. If we notice the point on the deck where the body falls, we shall find it no further distant from the mast than it would have been had it fallen from the top when the ship was at anchor; so that with regard to the mast the body has described a right line parallel to it: but with respect to a spectator situated at rest out of the vessel, it has really described a parabola, abstracting, as in the case of projectiles, from the resistance of the air. For, when the body is abandoned to the action of gravity, it has the same velocity as the vessel has, since they were previously moving on together: its actual motion is, therefore, the same as though the vessel were immoveable, and the body were projected horizontally with a velocity equal to that with which the vessel moves, and in the same direction.

Other corollaries which flow naturally from the preceding propositions and theorems may be deduced by the student at his leisure.

On the Ricochet.

254. Def. The *Ricochet* is a motion by which a projectile, after having struck any obstacle whatever, is so reflected as to recommence a motion similar to that which it had at first. The word signifies duck-and-drake, or rebounding; because when a ball has this motion it goes bounding along, striking the ground several times, like the bounding of a flat stone along the surface of water, when thrown almost horizontally. In *Ricochet* firing the ball is projected with small charges, and at elevations generally between 3 and 7 degrees. What little we shall advance on this species of projectile motion will be chiefly illustrative.

I. The less the angle of elevation at which the ball is projected, the more (*cæteris paribus*) is it in the state for the ricochet: for then the force of projection being exercised more entirely in the horizontal direction, will take much more time before it is extinguished by the resistance of the air and other obstacles. If the projectile were entirely void of elasticity, and the surface on which it falls were horizontal, and inflexible, there would then be no ricochet: because, the velocity of the projectile when it arrives at C (fig. 1. pl. XII.) according to any direction MC, may be decomposed into two others: of which the one QC perpendicular to the surface will be simply destroyed without any restitution, the body having no springiness; while the other velocity PC subsists alone, (abstracting from friction and the air's resistance), and the body, instead of bounding, will slide along CZ.

II. But if there should be an eminence CE (fig. 2. pl. XII.) at the point C where the body strikes the surface, the motion according to MC will then be decomposed into a motion QC perpendicular to the surface CE of this obstacle, and another PC in the direction of that surface, by which the moveable will advance in the direction PE, and may describe after quitting the point E a new curve of the same nature with that which it would describe if it were projected from the gun at E with the same velocity; and this in such manner, that after it is elevated to a certain point it may again descend and strike the earth in a new point I, where it may recommence a similar motion if the circumstances are similar: and so on.

III. If the obstacle is flexible or moveable, as the earth, water, &c. there may be a ricochet even when the surface is perfectly horizontal. For the moveable tends by its vertical velocity QC (fig. 3. pl. XII.) to plunge itself more or less, according to the nature of the obstacle; while with the velocity represented by PC it ploughs its way, and forms a furrow of which the depth continues to increase till the vertical velocity QC is extinguished. Then, by virtue of the velocity remaining in the horizontal direction, it thrusts before it the matter which opposes it, and becomes gradually turned towards the side where it has the least resistance: thus, in its return from its greatest depth, the cavity of the furrow operates, with respect to the body, in the same manner as the obstacle CE (fig. 2.) in the preceding case. Now as the facility of the escape from the earth or other obstacle is the greater (*cæt. par.*) as the total depth of the furrow is less, and as this depth depends on the vertical velocity QC, which will be always smaller as the angle MCP is smaller, or as the angle of projection is smaller, we see how the facility of the ricochet depends upon the angle of projection being small.

IV. The ricochet likewise depends much upon the figure of the projectile. If we want, for example, a ricochet on the water, and the projectile were globular; in order to this the velocity MC must be such that the vertical velocity QC may be entirely consumed before the vertical diameter of the sphere is completely immersed: for, if once it is covered with water, the resistance of the fluid will act equally on all sides the direction of the projectile, in such a manner that it can only be turned in consequence of the action of gravity, which will, therefore, be a direct impediment to the ricochet.

V. As the plunging can only be made successively or gradually, it is manifest that during the time in which it is going on the centre of the ball will describe a curve line; for the direction in which the resistance is made changes continually. If, for example, when the centre C (fig. 10. pl. XI.), after having described any track whatever PC, tends to move according to the prolongation CI of its actual direction, we conceive two tangents BR, DS, parallel to that direction; it is evident that only the part BVL is subjected to the resistance; and that, the body being spherical, the resultant CK of all the resistances offered to the different points of the surface BVL has a direction which tends to elevate the body above CI; in such manner that CIEK may be the parallelogram of forces, and CE will be the path the body will take, instead of CI, abstracting from the effect of gravity.

VI. Lastly, if both the projectile and the obstacle are flexible, or both springy, these circumstances will still further contribute to the facility of the ricochet. To take a very simple instance—Suppose that the projectile only is flexible, and that it is perfectly elastic, or restores itself with a force equal to that of compression; still abstracting from the consideration of gravity. At the instant in which the body moving in the direction AC (fig. 4. pl. XII.) touches the surface, its velocity is resolved into a horizontal velocity QC, which subsists always the same (disregarding friction), and the resistance of the medium in which the body moves: but the vertical velocity PC tends gradually to compress the body itself, becoming gradually extinguished, while that represented by QC subsists unchanged; so that it is clear that the centre C approaches the plane HZ by degrees which are continually diminishing, while the degrees by which it advances parallel to that plane remain constant: if, therefore, we conceive at every instant a parallelogram of forces constructed, of which the horizontal side is to the vertical side as the horizontal velocity to that which remains in the vertical direction, the diagonal of this parallelogram, which will indicate for each instant the route of the centre C, will be differently situated, so that the centre will approach to ZH by describing a curve line CR during the time of the compression; when the compression ceases the centre C will move for an instant on a tangent parallel to HZ; after which the elastic force of the body counteracting that of compression restores to the body gradually the velocity by which it moves from the plane, in the inverse order of the degrees by which it approached the plane; and so causes its centre to describe the second branch RO of the curve, equal and similar to CR. Finally, when it has arrived at the point O, whose distance from the plane HZ is equal to the radius CI, it will move on according to the tangent OT situated in the same manner as AC. Hence, then, the oblique stroke of an elastic body upon an inflexible and immoveable plane is made (abstracting from gravity) in such manner that the angle of reflexion is equal to the angle of incidence, these two angles being measured by those contained between the horizontal plane and the tangents at the extremities C and O of the curve described by the centre during the compression and the restitution of the spring; and this curve is always the smaller as the compression and restitution approach the more nearly to being instantaneous.

VII. If we pay regard to gravity, and BD is the line according to which the body is projected, it will describe the portion DC of the parabola whose tangent is BD just when the body touches the plane; and when the compression, &c. has ceased and the centre arrives at O, it will describe another portion SO of a parabola, equal to the former, and similarly situated.

VIII. Friction likewise contributes to the facility of the Ricochet; because it gives to the moveable a rotation which enables it to surmount more easily the different obstacles it meets with. Such are the causes and the principal circumstances of the ricochet: we have attended to them rather minutely, though popularly, because the principles here applied may often be directed by the student to other motions which frequently occur.

SCHOLIUM.

255. The theory of the motion of projectiles just exhibited depends upon three suppositions, all of which are inaccurate. 1st. That the force of gravity is the same in every point of the curve. 2dly. That it acts in parallel lines. 3dly. That the body moves in a non-resisting medium. Of these suppositions, however, the two first, as we observed in art. 248. produce no

error which deserves notice: but the third is a source of considerable difference between this theory and experiments, particularly when the initial velocity of the projectile is great. The resistance of the air is variable, according to the different velocities and magnitudes of the projectiles: on this account the trajectory of the projectile is not a parabola, nor any known and regular curve; its vertex is not in the middle, but more remote from the point of projection than from the other extremity; and the part of the trajectory through which the body descends is less curved than that through which it ascends. These circumstances are very perceptible to the sight in the motion of stones, arrows, balls, and shells; and even in a jet of water or mercury we may trace the same particulars, unless the velocity be small, when the path nearly coincides with a parabola. Besides this, a body projected with any considerable velocity is not only deflected from a parabolic path in a vertical direction, but is made to deviate laterally, and change the plane of motion: in some experiments indeed this deviation has been equal to $\frac{1}{6}$ or $\frac{1}{7}$ of the actual range. Such material discrepances between the theory and practice have induced several philosophers, at different times, to institute courses of experiments, in order to improve the theory by a comparison with their results: the most extensive and important of these are the experiments by Messrs. Robins, Thompson, and Dr. Hutton; for accounts of them the reader may advantageously consult Robins's Gunnery, or Brown's Translation of Euler's Gunnery, Phil. Trans. for 1778 and 1781, and Hutton's Tracts. Dr. Hutton has, likewise, touched upon this subject in the second volume of his Course of Mathematics; and as what he has there given contains much practical information in a small compass, we shall extract an article or two, as peculiarly appropriate to the subject before us.

256. " Before the propositions can be applied in resolving the several cases in the practice of gunnery, it is necessary that some more data be laid down, as derived from good experiments made with balls or shells discharged from cannon or mortars, by gunpowder, under different circumstances. For without these those theorems can be of very little use in real practice, on account of the imperfections and irregularities in the firing of gunpowder, and the expulsion of balls from guns, but more especially on account of the enormous resistance of the air to all projectiles that are made with any considerable velocities. As to the cases in which projectiles move with small velocities, or such as do not exceed 200, or 300, or 400 feet per second, they may be resolved tolerably near the truth by the parabolic theory

laid down above. But in cases of great projectile velocities it is quite inadequate, without the help of several data drawn from many and good experiments. For so great is the effect of the resistance of the air to projectiles of considerable velocity, that some of those which in the air range only between 2 and 3 miles at the most would in vacuo range about ten times as far, or between 20 and 30 miles.

" The effects of this resistance are also various, according to the velocity, the diameter, and the weight of the projectile. So that the experiments made with one size of ball or shell will not serve for another size, though the velocity should be the same; nor will the experiments with one velocity serve for other velocities, though the ball be the same. And therefore it is plain that, to form rules for practical gunnery, we ought to have good experiments made with each size of mortar, and with every variety of charge, from the least to the greatest. And not only so, but these ought also to be repeated at many different angles of elevation, namely, for every single degree between 30° and 60° elevation, and at intervals of 5° above 60° and below 30°, from the vertical direction to point-blank. By such a course of experiments it will be found that the greatest range, instead of being constantly that for an elevation of 45°, as in the parabolic theory, will be at all intermediate degrees between 45° and 30°, being more or less both according to the velocity and the weight of the projectile; the smaller velocities and larger shells ranging farthest when projected almost at an elevation of 45°; while the greatest velocities, especially with the smaller shells, range farthest with an elevation of about 30°.

257. " There have, at different times, been made certain small parts of such a course of experiments as is hinted at above. Such as the experiments or practice carried on in the year 1773, on Woolwich Common; in which all the sizes of mortars were used, and a variety of small charges of powder. But they were all at the elevation of 45°; and consequently these are defective in the higher charges, and in all the other angles of elevation.

" Other experiments were also carried on in the same place in the years 1784 and 1786, with various angles of elevation indeed, but with only one size of mortar, and one only charge of powder, and that but a small one too: so that all those nearly agree with the parabolic theory. Other experiments have also been carried on with the ballistic pendulum, at different times; from which have been obtained some of the laws for the quantity of powder, the weight and velocity of the ball, the length of the gun, &c. Namely, that the velocity of the ball varies as the square root of the charge directly, and as the square root of the weight of ball reciprocally; and that some rounds being fired with a medium length of one-pounder gun, at 15° and 45° elevation, and with 2, 4, 8, and 12 ounces of powder, gave nearly the velocities, ranges, and times of flight, as they are here set down in the following Table.

Powder.	Elevation of gun.	Velocity of ball.	Range.	Time of flight.
oz.		feet.	feet.	
2	15°	860	4100	9″
4	15	1230	5100	12
8	15	1640	6000	14½
12	15	1680	6700	15½
2	45	860	5100	21

"But as we are not yet provided with a sufficient number and variety of experiments on which to establish true rules for practical gunnery, independent of the parabolic theory, we must content ourselves with the data of some one certain experimented range and time of flight at a given angle of elevation; and then by help of these and the rules of the parabolic theory, determine the like circumstances for other elevations that are not greatly different from the former."

A copious and most interesting account of the whole of Dr. Hutton's experiments, may now be seen in the 2d and 3d vols. of his 8vo Tracts, recently published. From these we venture to extract, farther, the following summary of important deductions.

"1. From a general inspection of this second course of these experiments it appears, that all the deductions and observations made on the former course, are here corroborated and strengthened, respecting the velocities and weights of the balls, and charges of powder, &c. It further appears also, that the velocity of the ball increases with the increase of charge only to a certain point, which is peculiar to each gun, where it is greatest; and that by further increasing the charge, the velocity gradually diminishes, till the bore is quite full of powder. That this charge for the greatest velocity is greater as the gun is longer, but yet not greater in so high a proportion as the length of the gun is; so that the part of the bore filled with powder, bears a less proportion to the whole bore in the long guns, than it does in the shorter ones; the part which is filled being indeed nearly in the inverse ratio of the square root of the empty part.

"2. It appears that the velocity, with equal charges, always increases as the gun is longer; though the increase in velocity is but very small in comparison to the increase in length; the velocities being in a ratio somewhat less than that of the square roots of the length of the bore, but greater than that of the cube roots of the same, and is indeed nearly in the middle ratio between the two.

"3. It appears, from the table of ranges, that the range increases in a much lower ratio than the velocity, the gun and elevation being the same. And when this is compared with the proportion of the velocity and length of gun in the last paragraph, it is evident that we gain extremely little in the range by a great increase in the length of the gun, with the same charge of powder. In fact, the range is nearly as the 5th root of the length of the bore; which is so small an increase, as to amount only to about a 7th part more range for a double length of gun.—From the same table it also appears, that the time of the ball's flight is nearly as the range; the gun and elevation being the same.

"4. It has been found, by these experiments, that no difference is caused in the velocity, or range, by varying the weight of the gun, nor by the use of wads, nor by different degrees of ramming, nor by firing the charge of powder in different parts of it. But that a very great difference in the velocity arises from a small degree in the windage: indeed with the usual established windage only, viz. about $\frac{1}{26}$ of the calibre, no less than between $\frac{1}{3}$ and $\frac{1}{4}$ of the powder escapes and is lost: and as the balls are often smaller than the regulated size, it frequently happens that half the powder is lost by unnecessary windage."

The valuable work from which the above quotation is taken will tend more than any other which has yet appeared to promote the true theory of gunnery. Yet, it must be acknowledged, that Dr. Hutton's experiments and investigations, excellent as they are, tend by no means so completely as might be wished to the determination of the actual path of a military projectile. But it is hoped that much additional light will be thrown upon this important branch of enquiry, in the course of the new experiments on gunnery recently devised, and now carrying on at Woolwich.

III. *Descents on Inclined Planes, and Curve Surfaces.*

258. **Prop.** *The force which accelerates or retards the motion of a body upon an inclined plane is to the force of gravity, as the height of the plane to its length, or as the sine of the plane's elevation to radius.*

Let AD (fig. 5. pl. XII.) be the plane, of which the height is BD, and the length AD; and let C be the place of a body upon the plane. From C draw the vertical CG to represent g the accelerating or retarding force of gravity, according as the body is supposed to be moving downwards or upwards, and complete the parallelogram CEGF: now the force CG may obviously be resolved into the two CF, CE, of which the former being in the direction perpendicular to the plane is extinguished by its reaction, but the latter being parallel to the plane has its entire effect, and is that which either directly co-operates with or opposes the motion of the body along the plane. But the triangles ADB, GCE, are similar, the sides of the latter being respectively perpendicular to the sides of the former: therefore, GC : CE : : DA : DB : : rad : sin DAB. And, if we suppose the body at any other point C′, the resolution of the forces will again furnish the same result. Whence the proposition is manifest.

Cor. 1. Since the accelerating or retarding force, on the same plane, is in a given ratio to that of gravity, which is a uniform force, it also is a uniform force.

Cor. 2. Hence the laws before laid down for accelerated and retarded motions hold good for those on inclined planes: thus, in descending along the same plane, or planes equally inclined, the velocities acquired are as the spaces descended from quiescence; the spaces descended are as the squares of the velocities, or as the squares of the times; and, if a body be thrown up an inclined plane, with the velocity it acquired in descending, it will lose all its motion, and will ascend to the height from which it descended in an equal time, and will repass any point of the plane with the same velocity as it passed it in descending.

Cor. 3. If h represent the height and l the length of the plane, the accelerating or retarding force will be represented by $\frac{h}{l} g$; or, if the angle DAB be denoted by e, that force will be represented by $g \sin e$. Hence if we substitute this for g, in the values of s and v, in art. 243. they will become $s = \sin e \, \frac{1}{2}gt^2$, and $v = g \sin et = \surd(2gs \sin e)$: and the equations for t will undergo similar modifications. These will enable us to deduce some useful theorems as further corollaries.

COR. 4. *If two bodies descend from the highest point* D *of the plane at the same instant, one descending along the plane* DA, *the other through the vertical* DB, *their contemporeous positions* C *and* I *will always be shewn by the intersections of a line as* CI *with the two planes, this line being constantly perpendicular to* DA. This will appear by comparing the equation $s = \frac{1}{2} g \sin e t^2$, for the motion down the plane, with the corresponding equation $s' = \frac{1}{2} g t'^2$ expressing the vertical motion: for, if we divide one of these equations by the other, we have $\frac{s}{s'} = \frac{g \sin e \frac{1}{2} t^2}{\frac{1}{2} g t'^2}$ which because $t = t'$, gives $s = s' \sin e$; whence $s' : s :: 1 : \sin e$, that is, DI : DC : : rad ; sin DAB, which indicates that DI is the hypothenuse of a right-angled triangle, whose angle I is equal to the elevation of the plane.

COR. 5. *If the diameter of a circle be perpendicular to the horizon, and chords be drawn from either extremity, the times of descent down all the chords will be equal, and each equal to the time of free descent through the vertical diameter.* Thus in fig. 6. pl. XII. the angle at C being a right angle, the time of running through AC on the inclined plane is (by the preceding corol.) equal to the time of falling through AB; and, since D and E are right angles, the same will be true of the descents through AD, and AE: also, if AI be drawn parallel to CB, the time of descent through AI will, in like manner, be equal to that through AB; and because AI = CB, and have equal elevations, they will be described in equal times: and the same may be shewn of DB, EB, &c.

COR. 6. If, from any point A in a vertical plane, any number of lines AC, AD, AE, AF, AG, &c. be drawn in that plane, and bodies be let fall from quiescence at A along these lines, then at the end of any equal times whatever these bodies will all be found in the circumference of a circle passing through A (fig. 7. pl. XII.) Thus, when the body falling vertically is at E, the others will be at C, D, F, G, respectively: again, when this body is at *e*, the others will be at *c*, *d*, *f*, *g*, in the circumference of a circle touching the former in A; and, when this body is at B, the others will be at *c'*, *d'*, *f'*, *g'*, in a third circle touching the other two in A: and so on. All which is manifest from Cor. 5.

COR. 7. And if the lines diverging from A are not confined to one plane, but directed all around, then bodies descending along these lines will at every separate instant of time be all found in the surface of some one of a series of spheres, which touch one another in the superior point A.

COR. 8. *If any line* CH (fig. 5. pl. XII.) *be drawn parallel to the horizon, and if two bodies move from quiescence at* D, *the one* *along the inclined plane* DA, *the other through the vertical* DB, *the velocity acquired by the one at* C *will be equal to that acquired by the other at* H; *those velocities being estimated by the directions of their respective motions.* For the equation comprising the conditions of velocity of the body moving along DA is $v = \sqrt{(2gs \sin e)}$ (Cor. 3.), and the corresponding equation for the vertical descent is $v' = \sqrt{(2gs')}$. But in the present instance, $v = v'$, therefore $2gs \sin e = 2gs'$, or $s \sin e = s'$; consequently $s' : s :: \sin e : 1$, that is, DH : DC : sin DAB : rad.

Hence CH is parallel to the horizontal line AB.

Cor. 9. Hence, since the same might be shewn of the corresponding point C′ of any other plane AD, *if several bodies are let fall from one point* D, *without any impulsion, and run over different planes, they will all have acquired the same velocity when they arrive at any horizontal plane.*

Cor. 10. *If two or more bodies have equal velocities at any equal altitudes* C, H, C′, *their velocities will be equal at all other equal altitudes,* A, B, A′.

Cor. 11. *Hence, also, the velocities acquired by descending down any planes are as the square roots of their heights* DB. Thus, the velocity at A, being the same as the velocity at B, is as $\sqrt{}$DB.

Cor. 12. *The time of descent along* DA *is to the time of descent through the vertical* DB *as* DA *to* DB. Here DA $= s$, DB $= s'$, and the equations are $s = \frac{1}{2}g \sin e t^2$, and $s' = \frac{1}{2}gt'^2$; whence, if we convert these into an analogy, we have $t^2 : t'^2 :: \frac{s}{\sin e} : s' :: \frac{DA}{\sin e} : DB$. But, in the triangle DAB, $\frac{DB}{DA} = \sin e$; therefore, $t^2 : t'^2 :: DA^2 : DB^2$, or $t : t' :: DA : DB$.

Cor. 13. As the above proportion obtains whatever is the inclination of the plane, it results, that *the times employed in passing over various inclined planes of the same height are respectively as the lengths of the planes.*

259. Prop. *If a body descend along any number of contiguous planes, it will ultimately acquire the same velocity as would have been acquired by falling perpendicularly through the height of the system; supposing no velocity lost on passing from one plane to another.*

Let AB, BC, CD, (fig. 8. pl. XII.) be the contiguous planes: produce DC, CB, the two lower till they meet AF parallel to the horizon in F and E respectively. Then, by Cor. 9. of the preceding prop., the velocity at B is the same whether the body descend through AB or EB: and therefore the velocity at C will be the same whether the body fall through ABC or EC, which is also, by the same corol. equal to the velocity acquired by falling through FC. Consequently the velocity will be the same at D, on this hypothesis, whether the body passed over the system ABCD, or through the single plane FD, or, lastly, through the vertical FG.

COR. Hence, comparing this prop. with corol. 2. and 11. of the preceding, it follows that the velocities acquired in falling down any systems of planes are as the square roots of their vertical altitudes; and that, if the body be projected from D up the planes with the velocity acquired, it will ascend through this or any other system of planes (no part of which is *reclining*) to the same perpendicular height from D.

260. PROP. *When a body moves over a system of planes, the velocity lost in passing from any plane to the succeeding one is to the velocity it had then acquired, as the versed sine of the angle made by the planes to radius.*

Conceive that the body after moving over the plane AB (fig. 9.) has such a velocity as would carry it over the space BF in the next second, or unit of time, if the motion were not obstructed: from F draw the line FD perpendicular to the second plane BC, and complete the parallelogram DE, then will the velocity BF be resolvable into the two BE, BD; of which the former is evidently destroyed by the resistance of the plane, and of consequence the velocity BF will be reduced to the velocity BD on the plane BC. Now, with centre B and radius BF describe the arc FI, so shall DI = BF − BD = the velocity lost at the angle B; and it is obvious that DI : BF : : vers. sin DBF : rad. As in the proposition.

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261. When a body falls freely by the continual solicitation of gravity, every particle in it is equally accelerated, or every particle descends towards the horizon with the same velocity, and, therefore, no rotation will be given to the body. The same may be said when a body descends along a perfectly smooth inclined plane, if that part of the force which is in a direction perpendicular to the plane be supported; that is, if a perpendicular to the plane drawn from the centre of gravity of the body pierce the plane in a point which is in contact with the body. But if this part of the force be not sustained by the plane, the body will partly roll and partly slide, till this force is sustained; and then the body will slide wholly. When the lateral motion is entirely prevented by the adhesion of the plane, the body will keep at rest if the vertical from its centre of gravity fall within its base; but if it fall below that base the body will roll. If the adhesion be not sufficient to prevent all lateral

motion, the body will partly roll and partly slide. In all these cases the space described, the time of motion, &c. must be ascertained by means of other principles; the principal, &c. of which will be explained in the chapter on Rotation (art. 316.).

262. Prop. *The times of describing two systems of inclined planes* ABCD, *abcd* (fig. 8. pl. XII.), *whose number, inclinations, and ratio of their lengths, are the same, are to each other as the square roots of the lengths of the systems.*

Because the planes are equally inclined to the direction of the force, the time of running through AB is to that through *ab* as $\sqrt{AB}$ to $\sqrt{ab}$. (art. 25[illegible]. cor. 11.) And if bodies fall down EC, *ec*, then will time in EC : time in *ec* : : $\sqrt{EC} : \sqrt{ec} : : \sqrt{AB} : \sqrt{ab}$; also, time in EB : time in *eb* : : $\sqrt{EB} : \sqrt{eb} : : \sqrt{AB} : \sqrt{ab}$; therefore, time in EC : time in *ec* : : time in EB : time in *eb*; and hence, the remainder, or time in BC : remainder, or time in *bc* : : $\sqrt{AB} : \sqrt{ab}$, by Euc. V. 19. And since (by hyp.) no motion is lost in passing from one plane to another, the times of passing over BC, and *bc*, are the same, whether the bodies begin to move at A and *a*, or at E and *e*: so that when the bodies descend along ABC, *abc*, it will be time in BC : time in *bc* : : $\sqrt{AB} : \sqrt{ab}$. In the same manner it may be shewn that time in CD : time in *cd* : : $\sqrt{AB} : \sqrt{ab}$. Hence, time in AB : time in *ab* : : time in BC : time in *bc* : : time in CD : time in *cd*; and consequently the time in AB + BC + CD : time in *ab* + *bc* + *cd* : : time in AB : time in *ab* : : $\sqrt{AB} : \sqrt{ab} : : \sqrt{(AB+BC+CD)} : \sqrt{(ab+bc+cd)}$.

Cor. 1. If the lengths of the planes, and their angles of inclination ABE, BCF, be diminished indefinitely, the limits, to which these systems approximate, are similar curves, similarly posited, in which (as will be seen in the next prop.) no velocity is lost: hence *the whole times of descent through these curves will be as the square roots of their lengths.*

Cor. 2. *The times of descents along similar circular arcs, similarly situated, are as the square roots of the arcs, or as the square roots of the radii of their respective circles.*

263. Prop. *If a body fall from quiescence down a curve surface which is perfectly smooth, the velocity acquired is equal to that which would be acquired in falling through the same perpendicular height.*

For, by art. 260. when a body passes from one plane AB (fig. 9.) to another BC, the whole velocity : the velocity lost : : radius : versed sine of DBF. Consequently, when the angle FBD is diminished *sine limite*, the velocity lost is diminished *sine limite*: and if the lengths of the planes as well as their angles of inclinations be indefinitely diminished, the system approximates to a curve as its limit, in which there is no velocity lost:

wherefore, the whole velocity acquired will be equal to that which a body would acquire in falling through the same vertical altitude. Art. 259.

Or, if we apply the fluxional method to the known properties of curves in general, the same thing may be shewn thus. Suppose the body moves from quiescence at C, (fig. 10. pl. XII.) and that at the end of the time t the constant action of gravity has brought it to the point M, in the same vertical plane YAX. On the horizontal axis AX let AP $=$ X, and on the vertical AY let AB $=h$, the right ordinate to the point M, that is, PM $=$ Y, and the space described CM $=s$; then will the velocity of the moveable in the direction of the element of the curve at M be $\frac{\dot{s}}{\dot{t}}$. Gravity acting upon the body at M in the direction MP, its force must be decomposed into two others, the one perpendicular to the surface in M will be annihilated by its reaction, the other will be in the direction of the tangent at M: and as the cosine of the angle which this tangent makes with AX is $\frac{\dot{y}}{\dot{s}}$ or rather $-\frac{\dot{y}}{\dot{s}}$, since s increases as y decreases, the component of the gravitating force in the direction of the tangent will be $-g\frac{\dot{y}}{\dot{s}}$; thus we have $\frac{\dot{v}}{\dot{t}}=-g\frac{\dot{y}}{\dot{s}}$. From this we find $\frac{\dot{s}}{\dot{t}}\dot{v}=-g\dot{y}$, or $v\dot{v}=-g\dot{y}$, because $v=\frac{\dot{s}}{\dot{t}}$ (art. 232. I.). Taking the fluent we have $\frac{1}{2}v^2=-gy+\text{C}$. Now, at the point C, $v=0$, and $y=h$, therefore $\text{C}=gh$, and consequently $v^2=2gh-2gy=2g\,(h-y)$ $=2g\times$ BN. Whence it follows, since BN here $=s$ in art. 243. that the body has in M the same velocity in the direction of the tangent that it would have had at N after falling freely through BN. Thus also it appears that the theorem exhibited in cor. 9. art. 258. is only a particular case of the principle just deduced.

COR. 1. *If a body be projected up a curve (having no point of contrary flexure) the perpendicular height to which it will rise is equal to that through which it must fall to acquire the velocity of projection.* For the body will always have its ascending motion retarded by a force whose intensity is equal to that which would accelerate the motion when descending.

COR. 2. If Mzm (fig. 10. pl. XII.) be a curve, whose lowest point is Z, and the parts MZ, zm, equal and similar; a body by falling through MZ will acquire such a velocity as will carry it to *m*: and, since the velocities at all equal altitudes in the descent and ascent are equal, the whole time of the ascent will be equal to that of the descent.

COR. 3. *The circumstances of the last corollary will obtain in like manner, if the body be retained in the curve by a string which is in every point perpendicular to it:* for the string in that case will sustain the part of the weight which in the former is sustained by the curve.

264. PROP. *The velocities acquired in descending through different arcs of the same circle are as the chords of the arcs, the tangent to the lowest point of each arc being horizontal.*

Let a body fall through the arc AD (fig. 11. pl. XII.), it will (art. 263.) have the same velocity at the point D as if it had fallen through the vertical FD, AF being horizontal: and a body falling through the arc BD will, for the same reason, have a velocity at D equal to that which would have been acquired by falling freely through ED. Now if bodies fall successively from rest at F and E, the velocities acquired on arriving at D will be (art. 243.) as $\sqrt{DF}$ and $\sqrt{DE}$. But, by a well-known property of the circle, $\sqrt{DF} : \sqrt{DE} : : AD : BD$; and consequently the velocities acquired by falling through any arcs ABD, BD, of the circle terminating in D the lower extremity of the vertical diameter, are as the corresponding chords AD, BD.

COR. 1. If we would give to a body on arriving at D a velocity double, triple, &c. the velocity it has at D, after falling through the arc BD, we have only to set off from D the chord DA double, triple, &c. the chord BD, and ABD will be the arc, after falling through which the body will have the required velocity.

COR. 2. If we wish the body on arriving at D to have any proposed velocity, five feet per second, for example: then find by the proper theorem (art. 243.) the height *s* due to that velocity; and having taken on the vertical CD a line DF equal to this height, from a point C above F attach a thread of the length DC to the body, and draw it aside from the vertical till it cut the horizontal line AF in A; from this place let the body fall, and it will have at D the proposed velocity.

265. PROP. *The accelerating force of a body, gravitating in a circular arc, is to its absolute weight, as the sine of its angular distance from the lowest point of the circle, is to the radius.*

Let M (fig. 12. pl. XII.) be the body descending along the arc MA, whose lowest point is A and centre C: let ME be the sine of the arc MA, and let MT be a tangent to the circle at M, meeting the vertical *a*A produced in T. Then will the accelerating force of the body at M be the same as if it were placed on the tangent MT, considered as an inclined plane; and will, therefore, be to its absolute weight as ET to MT (art. 258.); or, because of the similar triangles MET, CEM, as EM to CM, or, as the sine of the arc AM (the measure of the angular distance MCA) is to radius.

COR. 1. In the same circle the accelerating force is every where as the sine of the arc MA; but because the sine and arc approximate to the ratio of equality while the arc decreases, the ratio of the sines to each other will approximate to, and be ultimately equal to, the ratio of their corresponding arcs to each other. Therefore, ultimately, the accelerating force is as the arc MA, the distance of the body from the lowest point A, measured along the circle in which the body moves.

COR. 2. Hence the times of descent down unequal arcs MA, OA, approximate to equality, while those arcs decrease *in infinitum,* and ultimately those times will be equal. For, when the accelerating force varies as the distances (art. 237.) the times of descent will be equal from whatever place the body begins to descend.

266. PROP. *Things being as in the last proposition, draw* MA *the chord of the arc* MOA; *then the accelerating force of the body upon the arc at* M *will be to the accelerating force of the body placed any where upon the chord* MA *ultimately as* 2 *to* 1.

For, draw the diameter *a*CA, join *a*M, CM, let A*t* be a tangent to the circle at A, and MT a tangent to the circle at M, those tangents meeting at *t*. Then, considering the chord MA and tangent M*t* as inclined planes whose altitudes are equal, the accelerating force on the plane M*t* (equal to that on the arc at M) is to the accelerating force on the plane MA, as $\frac{1}{Mt}$ to $\frac{1}{MA}$, or as MA to M*t*; or, by reason of the similar triangles MA*t*, M*a*C, as M*a* to MC, that is, ultimately, as 2 is to 1.

COR. *The time of descent through the arc* MOA *is to the time of descent through the chord* MA *ultimately as* 3·141593 *to* 4, *or as the circumference of a circle is to four diameters.* For, ultimately, the length of the arc and chord are equal, and ultimately the accelerating force of the body on the arc is every where as its distance from the lowest point A (art. 265. cor. 1.); but the accelerating force of the body on the chord is uniform, and equal to half the first force on the arc at M by the prop.; therefore, (art. 237. cor.) the time of the descent through the arc is to the time of descent through the chord as the circumference is to four diameters.

COR. 2. Supposing the periphery of a circle to coincide with the perimeter of a polygon whose sides are increased in number, and diminished in magnitude *in infinitum,* and that the least possible arc of a circle coincides accurately with its chord (which is the language of *indivisibles*), it would follow that the time of a body's descent in such an arc would be *equal* to the time of descent down the chord: and such is the conclusion of Dr. *Keill* in his xvth Lecture, and before him of M. *Parent*. But it appears, from the preceding corollary, that in fact these times are so far from being equal, as Keill and Parent maintain, that the time of descent down the arc is less than that down the chord nearly in the ratio of 3 to 4. And from the method of prime and ultimate ratios applied to this case, it is evident, that while the arc and chord approximate to equality, the times of descending along them do not approximate: for, by the doctrine of limits, no part of a curve, how small soever, can ever be safely taken for its chord; but even when they so far approach each other that their lengths may be considered as equal, the curve still remains a curve; its inclination is different from that of the chord; the accelerating force along the curve perpetually varies, and that in proportion to the distance of the body from the lowest point, while the accelerating force along the chord remains constant; and therefore the times of describing these spaces are unequal, even supposing the lengths could be the same.

Principle of M. D'Alembert.

267. Before we entirely quit the subject of motions along inclined planes, we shall briefly notice a general principle which M. D'Alembert first presented in his *Dynamique*, pa. 73, and which is mentioned here because we mean to shew its application to a part of the subject now under discussion; not meaning to follow it universally. His proposition is this:

Prop. " *In whatever manner several bodies change their actual motions, if we conceive that the motion which each body would have in the succeeding instant, if it were quite free, is decomposed into two others, of which one is the motion which it really takes in consequence of their mutual actions, the second must be such, that if each body were impelled by this force alone (that is, by the force which would produce this second motion), all the bodies would remain in equilibrio.*"

This is evident: for if these second constituent force are not such as would put the system in equilibrio, the other constituent motions could not be those which the bodies really take in consequence of their mutual action, but would be changed by the first.

The use of this proposition will appear from the following examples.

I. Let there be three bodies B, B′, B″, and let the forces F, F′, F″, act upon them, so as to give them the velocities, v, v', v'', in any directions whatever, producing the quantities of motion $\text{B}v$, $\text{B}'v'$, $\text{B}''v''$, which we may call F, F′, F″, because the momenta are the proper measures of the moving forces. Let us farther suppose that by striking each other, or being any way connected with each other, they cannot take these motions F, F′, F″, but really take the motions f, f', f''. It is obvious that we may consider the motion F impressed upon the body B to be composed of the motion f which it really takes, and of another motion φ. In like manner F′ may be resolved into f' which it actually takes, and another φ'; and again F″ into f'' and φ''. The motions will be the same whether B be acted upon with the force F, or the constituent forces f and φ; whether B′ be acted upon by F′, or by f' and φ'; and B″ by the force F″, or the component forces f'' and φ''. Now, by the supposition, the bodies actually take the motions f, f', f'': therefore the motions $\varphi, \varphi', \varphi''$, must be such as will not derange the motions $f, f' f''$: that is to say, if the bodies had only the motions $\varphi, \varphi', \varphi''$, impressed upon them they would destroy each other, and the system would remain at rest.

II. *Motions upon double inclined planes.* Let AC, CB (fig. 7. pl. VI.), represent two inclined planes of equal altitudes, set back to back at DC, the angles of elevation being DAC $= e$, and DBC $= e'$; and let the two weights W, W′ united by a thread WCW′ passing over the pulley C, act the one upon the other. Then, calling w, w', their masses, or their weights, let us determine the circumstances of their motion.

At the end of the time t, w will have a velocity v, and gravity would impress upon it in the instant $\dot{t}$ following a new velocity $= g \sin e\dot{t}$ (art. 258. cor. 3.), provided the weight w were then entirely free: but by the disposition of the system, $\dot{v}$ will be the velocity which obtains in reality. Then estimating the spaces in the direction CW, as the body W′ moves with an equal velocity but in a contrary sense, it is obvious that the decomposition may be made as follows. At the end of the time $t + \dot{t}$. we have for the velocity impressed upon

$$\text{W} \ldots\ldots v + g \sin e\dot{t}, \text{ where} \begin{cases} v + \dot{v} \ldots \text{ effective velocity from C towards A.} \\ g \sin e\dot{t} - \dot{v} \ldots \text{ velocity destroyed.} \end{cases}$$

$$\text{W}' \ldots - v + g \sin e'\dot{t}, \text{ where} \begin{cases} -v - \dot{v} \ldots \text{ effective velocity from C towards B.} \\ \dot{v} + g \sin e'\dot{t} \ldots \text{ velocity destroyed.} \end{cases}$$

If, therefore, gravity impresses, during the time $\dot{t}$, upon the masses w, w', the respective velocities $g \sin e\dot{t} - \dot{v}$, and $g \sin e'\dot{t} + \dot{v}$, the system will be in equilibrio. The quantities of motion being therefore equal, we have

$$wg\sin e\dot{t}-w\dot{v}=w'g\sin e'\dot{t}+w'\dot{v}.$$

Whence we deduce for the effective accelerating force

$$\text{(I.)}\ldots\varphi=\frac{\dot{v}}{\dot{t}}=\frac{w\sin e-w'\sin e'}{w+w'}g.$$

Thus it appears that the motion is uniformly varied: and we readily find

$$\text{(II.)}\ldots v=\frac{w\sin e-w'\sin e'}{w+w'}gt.$$

$$\text{(III.)}\ldots s=\frac{w\sin e-w'\sin e'}{w+w'}\cdot\tfrac{1}{2}gt^2.$$

Such are the equations from which the conditions of the motion are determined. If the two planes are vertical, then is $e=e'=\frac{1}{2}\pi$; by means of which we may find the values that are applicable in the subsequent example.

III. *Motion on the fixed pulley.* The weights P and W are united by the thread PAABW running over the pulley A (fig. 2. pl. VI.); the force of gravity g which solicits them cannot produce its entire effect, because its action upon the body $P=w$ counteracts its effect upon the other body $W=w'$. At the end of the time $t+\dot{t}$, the velocity of w, in the direction from A to P, will be $v+\dot{v}$, instead of $v+g\dot{t}$, which it would have had if the weight had become free at the end of the time t: thus $g\dot{t}-\dot{v}$ will be the velocity destroyed. In like manner, w' having the same velocity but in a contrary direction, will have the velocity $-(v+\dot{v})$ instead of $-(v+g\dot{t})$, which it would have had, if at the end of the time t the connection of the two bodies ceased; so that $g\dot{t}+\dot{v}$ will be the velocity destroyed. Then, as in the preceding example, making the momenta equal, we find for the effective accelerating force

$$\text{(1.)}\ldots\varphi=\frac{\dot{v}}{\dot{t}}=\frac{w-w'}{w+w'}g.$$

This being an expression for a constant accelerating force, we conclude, as in the former instance, that the motion is uniformly varied, and that the circumstances of motion and of velocity ay be found by the equations

$$\text{(2.)}\ldots v=\frac{w-w'}{w+w'}gt\ldots\quad\text{(3.)}\ldots s=\frac{w-w'}{2w+2w'}gt^2.$$

In this investigation we have neglected the inertia of the string and the pulley.

From formulæ similar to these Mr. Atwood deduced (in his valuable treatise on the Motion of Bodies) an ingenious method of verifying the received theorems relative to the effects of gravity on falling and rising bodies. He contrived a machine, in which two weights as w and w' in our theorems, or P and W in the figure last referred to, are connected by a thread which runs over a fixed pulley A: the last given theorems for φ, s, and v, shew that by varying the weights w w', we may change the value of the fraction $\frac{w-w'}{w+w'}$, and thus reduce the accelerating force, and its consequent velocity and space in a given time, in such a ratio with respect to those arising from gravity, that they shall become easily measurable, and the theory verified. Mr. Atwood's apparatus has its several parts so adapted as to be susceptible of great precision: but as this is not the place to enter into detail, the reader may turn to the article *Atwood's Machine* in the second volume, where a more full account is given of its nature and use.

In finding the equations (2.) (3.) above, we supposed that each body began to move from quiescence at a point taken for the origin of the spaces denoted by s. But suppose this is not the case, and that we impress upon w an initial velocity of V carrying it downward. This velocity must be parted between the two masses w, w', according to the same law as if w struck the body w' at rest with the velocity V: thus, the velocity common to the two weights would be $\frac{wV}{w+w'}$. We have this value for the velocity at the end of the time $t=0$; therefore

$$(4.) \ldots v = \frac{wV+(w-w')gt}{w+w'}.$$

From this we readily obtain s in terms of t: and the time may be determined by the equation $wV=(w-w')gt$.

IV. *Motion on the Axis in Peritrochio.* Let the weight P (fig. 4. pl. V.) be denoted by w, and the weight W by w'; to determine the circumstances of the motion. Let the radius of the wheel $=R$, that of the axle $=r$: the velocity of w at the end of the time t will be $v+\dot{v}$, instead of $v+g\dot{t}$, as in the former cases; that of w' will manifestly be to that of w in the ratio of R to r; it will, therefore, be $-\frac{r}{R}(v+\dot{v})$ instead of being $-\frac{r}{R}(v+g\dot{t})$: the velocities lost are therefore $g\dot{t}-\dot{v}$, and $g\dot{t}-\frac{r}{R}\dot{v}$. Then, making the momenta equal with respect to the axe EF, we find

$$(\text{i.}) \ldots \varphi = \frac{\dot{v}}{\dot{t}} = \frac{R^2w - Rrw'}{R^2w+r^2w'}g.$$

Consequently the motion is uniformly accelerated in this case likewise. So that we readily deduce the equations for the motion of w or P.

$$\text{(ii.)} \ldots v = \frac{\mathrm{R}^2w - \mathrm{R}rw'}{\mathrm{R}^2w + r^2w'}gt. \quad \ldots \text{(iii.)} \ldots s = \frac{\mathrm{R}^2w - \mathrm{R}rw'}{\mathrm{R}^2w + r^2w'} \cdot \tfrac{1}{2}gt^2.$$

And the velocity and space, with respect to w', will be

$$\text{(iv.)} \ldots \mathrm{v} = \frac{\mathrm{R}rw - r^2w'}{\mathrm{R}^2w + r^2w'}gt. \ldots \text{(v.)} \ldots \mathrm{s} = \frac{\mathrm{R}rw - {}^2w'}{\mathrm{R}^2w + r^2w'} \cdot \tfrac{1}{2}gt^2.$$

Thus much is here offered on the application of M. D'Alembert's principle: the further use of some of these theorems will appear, when we treat of the maximum of machines in motion, Chap. VI.*

IV. *On the Simple Pendulum, the Cycloidal Pendulum, and the Curve of Swiftest Descent.*

268. We have already seen (art. 263. cor. 2.) that a heavy body after having descended through an arc MZ (fig. 10. pl. XII.) of any curve CZ*m*, will, abstracting from the effects of friction and the air's resistance, mount up the opposite branch Z*m* till it arrives at the point *m*, whose distance from the horizontal line AX is equal to MP, the distance of the point *m* from the same; and that the time of describing Z*m* will be equal to that of describing MZ. Having at the point *m* lost all its motion, it will begin to descend again through *m*Z, and at Z will have acquired a velocity such as will cause it to rise to the point M where its motion first commenced, where again it will be in a state of quiescence; and from this it will move a second time through MZ*m*, and back again through *m*ZM; and so on continually. Such will be the motions whether the body run over a curve surface, in consequence of the joint effects of gravity and the reaction of the surface; or whether it be made to describe the curve in consequence of being fastened to a string CM (fig. 12.) whose centre C is fixed. If we consider the thread CM as inextensible, and without weight, and the body at M as a point with respect to the length of the thread, then is the system known by the name of the *Simple Pendulum*. The motion of such a pendulum in one direction from a state of rest, till it begins to return in an opposite direction is called a *vibration* or an *oscillation*; and the time employed in such motion the *time of vibration*, or *of an oscillation*. Thus, if the pendulum move from quiescence at M, the time it occupies in passing through the arc MA*m* is the time of vibra-

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* Analogous to this is the principle proposed by Mr. Bazley, in Leybourn's Repository, No. 9. N. S.; both being, indeed, immediate and obvious consequences of Newton's 3d law of motion.

tion; and this, it is manifest, is *double the time employed in passing over* MA. This being admitted it would be easy to apply some of the theorems in arts. 264...266. to the vibrations of pendulums in circular arcs; as they would then assume the form below.

I. If a pendulum vibrate in the arc of a circle, the velocity of the ball, at its lowest point, will be as the chord of the arc which it describes in its descent.

II. The force which accelerates a pendulum is to the force of gravity, as the sine of its angular distance from the lowest point to radius.

III. The times of vibrations in very small circular arcs are very nearly equal.

IV. The time of vibration in an indefinitely small circular arc is to the time in which a body would fall through half the length of the pendulum, as the periphery of a circle to its diameter.

V. The time of vibration is as the square root of the length of the pendulum; the force of gravity remaining the same.

And from these the whole doctrine of pendulums vibrating in circular arcs might readily be deduced. But, as the supposition that vibrations in small arcs of equal circles are all performed in equal times is not quite correct, it may be better to draw the chief properties of pendulums from a distinct investigation, by means of which we may also determine the magnitude of the error which can occur in vibrations through any assigned arc. This may be accomplished thus.

269. PROP. *To determine the time of vibration of a pendulum in any circular arc.*

Let MAm (fig. 12. pl. XII.) be the proposed arc; and let the radius or length of the pendulum CM $= l$, AB $= b$, A$p = x$, $po = y$, the variable arc $= s$, the accelerating force or force of gravity $= \varphi$, and the velocity due to the height Ep, or (art. 263.) the velocity acquired by falling through Mo, put $= v = \sqrt{[2\varphi(b-x)]}$ (art. 243). Now it is known, that $v = \frac{\dot s}{\dot t}$ (art. 232. I.); and comparing these two values of v we find $t = \int \frac{\dot s}{\sqrt{[2\varphi(b-x)]}}$. But the known property of the circle gives for the value of the element of the curve $\dot s = \frac{-l\dot x}{\sqrt{(2lx-xx)}}$, which is taken negatively, because the arc s diminishes as the time augments: this value of $\dot s$ introduced into the preceding equation transforms it to $\dot t = \frac{-l\dot x}{\sqrt{[(2lx-xx).2\varphi(b-x)]}} = \frac{-\dot x}{\sqrt{(bx-xx)}} \times \frac{l}{\sqrt{[2\varphi(2l-x)]}}$

$= \frac{\sqrt{l}}{2\sqrt{\varphi}} \times (1 - \frac{x}{2l})^{-\frac{1}{2}} \times \frac{-\dot{x}}{\sqrt{(bx-xx)}} = \frac{1}{2}\sqrt{\frac{l}{\varphi}} \times \frac{-x}{\sqrt{(bx-xx)}} \times (1 + \frac{1}{2} \cdot \frac{x}{2l} + \frac{1\cdot3}{2\cdot4} \cdot \frac{x^2}{4l^2} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6} \cdot \frac{x^3}{8a^3} + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8} \cdot \frac{x^4}{16a^4} + \text{\&c.})$, the factor $(1 - \frac{x}{2l})^{-\frac{1}{2}}$ being expanded by the binomial theorem.

To obtain the time of descent from M to A, we must find the fluent of each term in the series, in such manner that they may vanish when $x = b$. Now we see at once, that the variable factor of all the terms of this series will be of the form $\frac{-x^n\dot{x}}{\sqrt{(bx-xx)}}$; that is to say, these factors will be $\frac{-\dot{x}}{\sqrt{(bx-xx)}}$, $\frac{-x\dot{x}}{\sqrt{(bx-xx)}}$, $\frac{-x^2\dot{x}}{\sqrt{(bx-xx)}}$, &c. Whence it appears that, taking π to 1, the ratio of the circumference of a circle to its diameter, the fluents of each of the terms, taken between $x = b$, and $x = 0$, are π, $\pi \cdot \frac{b}{2}$, $\pi \cdot \frac{1\cdot3\cdot b^2}{2\cdot4}$, $\pi \cdot \frac{1\cdot3\cdot5\cdot b^3}{2\cdot4\cdot6}$, &c. Substituting these values in that of t, it will become

$$t = \frac{1}{2}\pi \sqrt{\frac{l}{\varphi}} (1 + \frac{1^2}{2^2} \cdot \frac{b}{2l} + \frac{1^2.3^2}{2^2.4^2} \cdot \frac{b^2}{4l^2} + \frac{1^2.3^2.5^2}{2^2.4^2.6^2} \cdot \frac{b^3}{8l^3} + \text{\&c.})$$

This, therefore, is the time employed by the body in descending from M to A: but with the velocity acquired at A the body would proceed along the equal branch A*m* of the curve, and would have all its velocity extinguished at *m*, after a time from A equal to the time of descent from M to A: consequently, the time of a complete oscillation will be double the former; that is,

$$t = \pi \sqrt{\frac{l}{\varphi}} \times \left(1 + \frac{1^2}{2^2} \cdot \frac{b}{2l} + \frac{1^2.3^2}{2^2.4^2} \cdot \frac{b^2}{4l^2} + \frac{1^2.3^2.5^2}{2^2.4^2.6^2} \cdot \frac{b^3}{8l^3} + \text{\&c.}\right)$$

The relation $\frac{b}{l}$ or $\frac{\text{AE}}{\text{AC}}$ expresses the versed sine of an arc of the same angular value as AM, having unity for its radius. And as the versed sines of small angles are exceedingly minute, the series $1 + \frac{1^2}{2^2} \cdot \frac{b}{2l} + \frac{1^2.3^2}{2^2.4^2} \cdot \frac{b^2}{4l^2} + \text{\&c.}$ will converge very rapidly. So that in almost every case the two first terms will suffice, or t nearly $= \pi \sqrt{\frac{l}{\varphi}} \times \left(1 + \frac{b}{8l}\right)$; therefore the times of vibrations in different arcs are as $8l + b$, or as 8 times the radius + the versed sine of the arc. Or, when the arc is very small, all the terms but the first may be neglected, and we shall have for the duration of the oscillation,

$$t = \pi \sqrt{\frac{l}{\varphi}}.$$

The quantity b does not enter into this value of t: consequently when the vibrations are made through very small arcs, their duration may be considered as not depending on the magnitude of the arc; and the oscillations may be regarded as isochronal.

270. To determine the numerical error which may be committed, by assuming the isochronism of vibrations in small arcs, let us suppose the arc MA $= 5°$; the length of the oscillation of a second pendulum would be expressed by $1 = \pi \sqrt{\frac{l}{\varphi}}$ or $l = \frac{\varphi}{\pi^2}$ on the supposition of the arc being extremely small. The versed sine of 5° is ·0038053 $= \frac{b}{l}$: and $\frac{1^2}{2^2} \cdot \frac{b}{2l} =$ ·0004757. As to the third term, it is less than ·0000001. We have therefore for the time employed by the pendulum in moving through an arc of twice 5°, $1 + (1 \times \cdot 0004757)$; and the time of oscillating through an arc of 10° differs not from an oscillation in an infinitely small arc more than ·0004757. This multiplied into 86400, the seconds in 24 hours, gives nearly $41\frac{1}{10}$ seconds. So that a pendulum of the same length as that which vibrates in seconds, through infinitely small arcs, would lose about 41″ per day, if it described arcs of 5° on each side the vertical CA. If the arcs described on each side of the vertical were only 1°, of which the versed sine is ·0001523, it would be found by a similar process that the daily retardation would be about $1\frac{2}{3}$″. And for a half degree on each side it would be about $\frac{2}{3}$ of a second.

Or, if D denote the degrees the pendulum describes on each side of the vertical, the time it would be retarded in a second would be nearly expressed by $\frac{D^2}{51150}$, and consequently the time lost in 24 hours would be $24 \times 60 \times 60 \times \frac{D^2}{51150} = \frac{5}{3} D^2$ nearly. In like manner the time lost in 24 hours by describing δ degrees on each side of the vertical would be $\frac{5}{3}\,\delta^2$ nearly. Consequently, if a second pendulum keeps true time in one of these arcs, the seconds lost or gained per day by vibrating in the other will be $\frac{5}{3}\,(D^2 - \delta^2)$. Thus, for example, if a second pendulum measure true time in an arc of 3° on each side of the vertical, it will lose $11\frac{2}{3}$ seconds per day by vibrating through 4° on each side, and nearly 45 seconds per day by describing 6° on each side.

Thus then it appears that the vibrations in very small circular arcs may be regarded as sensibly isochronal: and whenever great accuracy is required, the necessary correction may be easily applied, as above. Reasoning by analogy, we see also that the oscillations in very small arcs of any curve whatever are always sensibly isochronal; because an arc of any curve will, for a small space, coincide with its circle of curvature. The equation $t = \pi\sqrt{\frac{l}{\varphi}}$ will furnish some other important consequences, which we shall now proceed to deduce.

271. I. *The durations of the vibrations of pendulums are respectively as the square roots of their length.* For, let l, l', be the lengths of the pendulums, t, t', the times of vibration; then, we have $t = \pi\sqrt{\frac{l}{\varphi}}$, and $t' = \pi\sqrt{\frac{l'}{\varphi}}$; consequently,

$$t : t' :: \pi\sqrt{\frac{l'}{\varphi}} : \pi\sqrt{\frac{l'}{\varphi}} :: \sqrt{l} : \sqrt{l'}.$$

II. *The time of an oscillation is to the time in which a heavy body would fall through half the length of the pendulum, as the periphery of a circle to its diameter.* For in the former case the time is denoted by $t = \pi\sqrt{\frac{l}{\varphi}}$; and in the latter the theorem $t = \sqrt{\frac{2s}{g}}$ (art. 243.) when accommodated to the present purpose will be $t' = \sqrt{\frac{l}{\varphi}}$. Consequently $t : t' :: \pi : 1$.

III. *If two pendulums of different lengths are solicited by different gravitating forces, the times of oscillations are as the square roots of the lengths of the pendulums directly, and the square roots of the quantities expressing the gravitating forces inversely.* For $t = \pi\sqrt{\frac{l}{\varphi}}$, and $t' = \sqrt{\frac{l'}{g}}$, give $t : t' :: \sqrt{\frac{l}{\varphi}} : \sqrt{\frac{l'}{g}}$.

IV. *The lengths of two pendulums are respectively in the inverse ratio of the squares of the number of oscillations made in the same time.* In the time T let the pendulum make n oscillations, then is $\frac{T}{n}$ = the time of one oscillation: consequently $t = \frac{T}{n} = \pi\sqrt{\frac{l}{\varphi}}$, and $l = \frac{\varphi T^2}{\pi^2 n^2}$. In like manner, if l' be the length of another pendulum, n' its number of oscillations in the same time T, the force of gravity φ being supposed the same (as is always assumed unless otherwise expressed); we shall have $l' = \frac{\varphi T^2}{\pi^2 n'^2}$. Whence $l : l' :: \frac{1}{n^2} : \frac{1}{n'^2} :: n'^2 : n^2$.

By means of this theorem, the length of the seconds pendulum may be readily ascertained experimentally: for, if we take a pendulum of any determinate length l', and count the number of vibrations n' which it makes in a given time, half an hour, for example, or 1800 seconds, then will the length l of the seconds pendulum be found by this analogy, $(1800)^2 : n'^2 :: l' : l$. By this method, among others, has the length of the seconds pendulum in the latitude of London been found $= 39\frac{1}{8}$ inches. It must be recollected though, that the length is not the same at all parts of the earth's surface; the difference arising from the inequality of distance from the centre, and the consequent variations in the centrifugal force (art. 286).

Another good method of ascertaining the length of a pendulum to vibrate in a certain time is that which was first proposed by Mr. *Hatton*, and afterwards executed by Mr. *Whitehurst*. It consists in the application of a moveable point of suspension to the same pendulum; which thus gives the absolute effects of two pendulums, the difference of whose lengths is known, being the interval between the points of suspension in the two cases: and the ratio of their lengths is also known from observing the number of vibrations performed in a given time. Whence, there being two equations and two unknown quantities, the actual lengths of the pendulums themselves are easily deduced. Thus, we shall have $l : l' :: n'^2 : n^2$, and $l - l' = d$, the measured distance between the points of suspension: consequently $l = \frac{dn'^2}{n'^2 - n^2}$ and $l' = \frac{dn^2}{n'^2 - n^2}$.

V. *If a clock keep true time very nearly, the variation in the length of the pendulum necessary to correct the error will be equal to twice the product of the length of the pendulum and the error in time divided by the time of observation in which that error is accumulated.*

Let the time of observation be $= \mathrm{T}$, the time gained or lost by the pendulum $= t'$, the length of the pendulum $= l$, and the space which the pendulum must be lengthened or shortened $= \lambda$: then we have $l : l \pm \lambda :: (\mathrm{T} \pm t')^2 : \mathrm{T}^2$; and from this is found $\lambda = \frac{lt'(\pm 2\mathrm{T} - t')}{(\mathrm{T} \pm t')^2}$. Neglecting t', which is very small with respect to $2\mathrm{T}$, the equation becomes $\lambda = \frac{\pm 2lt'}{\mathrm{T}}$; which is the same as the rule.

COR. If the pendulum be one that should beat seconds, and t' the daily variation be given in minutes, and n be the number of threads in an inch of the screw which raises and depresses the bob of the pendulum, then $\lambda = \frac{\pm 2 \times 39\frac{1}{8} \times nt'}{24 \times 60} = \cdot 05434nt' = \frac{2}{37}nt'$ nearly, for the number of threads which the bob must be raised or lowered, to make the pendulum vibrate truly.

VI. *The length of a seconds pendulum being known in any* *place, the space through which a heavy body would fall freely in the same time and place may be readily found.* Thus, for London, the equation $t = \pi\sqrt{\frac{l}{\phi}}$, becomes $1 = \pi\sqrt{\frac{39\frac{1}{8}}{\phi}}$, whence we have $\phi = \pi^2 l = \pi^2 \times 39\frac{1}{8} = 386{\cdot}14$ inches $= 32_6$ feet nearly, the appropriate measure of the force of gravity; corresponding with the assumed value of g in art. 242. And half this, or $16\frac{1}{12}$ feet, is the space descended by a heavy body in the first second from quiescence.

Or the same conclusion may be obtained rather differently from No. II. of the present article: for $\pi : 1 :: 1' : \frac{1}{\pi}$ the time in which a heavy body would fall through a vertical line of $19\frac{9}{16}$ inches, and by the laws of falling bodies the spaces described are as the squares of the times; therefore $\frac{1}{\pi^2} : 1' :: 19\frac{9}{16} : 19\frac{9}{16} \cdot \pi^2 = 193{\cdot}07$ inches $= 16\frac{1}{12}$ feet nearly, the same as before.

It is obvious also, that by the reverse of this method we may find the length of a seconds pendulum, having given the space fallen through by a heavy body in the first second of time from quiescence. And thus the theory and experiments mutually assist each other, in determining data of such importance as these, in many other branches of science.

272. Defs. If the circumference of a circle be rolled along a right line, until any point O of that circumference which was in contact with the line come in contact with it again, that point O will describe a curve called a *cycloid*. The line on which the circle rolls is the *base* of the cycloid; and the diameter of that circle which is perpendicular to the base when the circle touches its middle point, is the *axis* of the cycloid.

The circle which rolls along the right line is called the *generating circle.*

Cor. The base of the cycloid is equal to the circumference of the generating circle.

273. Prop. *If a line* OFB *be drawn from any point* O *in the cycloid, parallel to its base, and meet the generating circle described upon the axis in* F, *the circular arc* FEV *is equal to the right line* OF.

Let the generating circle touch the base in D (fig. 1. pl. XIII.) when the generating point is at O. DE perpendicular to AC and equal to CV is the diameter of the circle DOE. Draw the chords OE, FV. Then, since DE = CV, and DG = CB, the remainder GE = BV: consequently OG = $\sqrt{(\text{DG}\cdot\text{GE})}$ = FB =

$\sqrt{(CB \cdot BV)}$. Add FG to each of these equals, then will $OF = GB$. Also $OE = \sqrt{(ED \cdot EG)} = VF = \sqrt{(VC \cdot VB)}$; therefore arc $EiO =$ arc VeF: and since every point in OnD has been successively in contact with AD, while the point O moved from A to its present position, $OnD = AD$, and $EOD = AC$: hence $EiO = DC$; and consequently $VeF = EiO = DC = GB = OF$.

COR. Because OF is always equal to the arc VeF or EiO, their cotemporary increments or decrements are equal; that is, the initial motions of the point O which traces out the cycloidal arc, the one parallel to the base AC, the other in the direction of the circle or its tangent at O, are equal to each other.

274. PROP. *If an ordinate* OB *be drawn* (fig. 1. pl. XIII.) *from any point* O *of the cycloidal arc, intersecting the circle* CFV *in* F, *then will the tangent* OE *of the cycloid be parallel to the chord* FV *of the generating circle.*

Draw the tangent Tt to the circle DOE in O, and produce BO, DO, to b and d; then are the initial motions of the point O in the directions Ot, OG, and equal to each other, by corol. to the foregoing proposition. And if the parallelogram of motions be constructed upon the equal sides coinciding with Ot, OG, its diagonal will, by the composition of motion, be in the direction of the tangent: therefore, the contiguous sides of the parallelogram being equal, it will be either a rhombus or a square, and its diagonal will bisect the angle tOG; that is, OE the tangent in O will bisect the angle tOG, or make $tOE = EOG$. But $GOD = ODT$, because OG and TD are parallel; and $ODT = TOD$, because the tangents TO, TD are equal; also $TOD = tOd$, being vertical or opposite angles; therefore $GOD = tOd$. Then adding equals to equals, $DOG + GOE = dOt + tOE$, or $dOE = DOE$; consequently OE is at right angles to Dd, and meets the diameter DE in its extremity E. Again, CD is equal and parallel to FO, therefore DO is equal and parallel to CF; and hence, since $DOE = CFV =$ a right angle, OE is parallel to FV.

COR. 1. Since DO is perpendicular to OE, the tangent to the cycloid at O, it coincides in direction with the radius of curvature at O.

COR. 2. A tangent to the cycloid at the vertex V is perpendicular to the axis VC, and parallel to the base AC.

The method of determining the tangent to the cycloid by the composition of motion was first given by Roberval, in his "*Observations on the Composition of Motion, and on the Tangents of Curve lines,*" published in the Memoirs of the Academy of Sciences, 1666. The same method is again applied to the case of cycloids and epicycloids, by Mr. West, in his *Mathematics*, published in 1762. Afterwards by Dr. Parkinson, in his *Mechanics*, published in 1785. And again by Mr. Ludlam, in his *Essays*, 1787. Neither of these gentlemen seem aware that their method is the same in principle as Roberval's.

275. Prop. *The involute of a semicycloid* AOU (fig. 13. pl. XII.) *is an equal semicycloid* UPV *in an opposite direction, the extremity of the base of the latter being in contact with the vertex of the former.*

From any point O draw OB parallel to AC, cutting the generating circle in F, and join FU. Draw OP a tangent to the cycloid in O, and at E, the point where it cuts the line UW drawn from U parallel to CA, let fall ED perpendicular to UW, and equal to CU. On ED as a diameter describe a circle intersecting the tangent OP in some point P. Then (art. 274.) OE is equal and parallel to FU; and (art. 273.) OF is equal to the arc F*e*U. The circles CFU, DPE are equal, as are likewise the angles FUE and UEP; the chords FU, EP, therefore, are equal, and cut off equal arches. Because OFUE is a parallelogram, UE is equal to FO, or equal to F*e*U, or equal to E*n*P. But if the circle EPD had been placed on the line UW at U, and had rolled from U to E, the arch disengaged would have been equal to UE, and the point which was in contact with U would now be in P, in a periphery of a semicycloid UPV, equal to AOU, having the line UN equal and parallel to AC for its base, and NV equal and parallel to CU for its axis: and since the same may be shewn to obtain with respect to any other point in AOU, the cycloid UPV is the involute of AOU, as in the proposition.

Cor. 1. The arch OU of the cycloid is equal to twice the corresponding chord FU of the generating circle: for this arch is equal to the evolved line OEP: and it has been shewn that $OE = EP$; consequently $OP = OU = 2OE = 2FU$.

Cor. 2. The arch of a semicycloid is equal to twice the diameter of its generating circle; and the whole cycloidal arch equal to four times the diameter of the generating circle.

Cor. 3. The description of the cycloid UPV by the evolution of the cycloid AOU furnishes a simple method of determining the area of any cycloid. For since OP is always parallel to FU, the former OP will sweep over the whole space AOUPV, while UF sweeps over the whole surface of the semicircle UFC; and since OP is always double the simultaneous UF, the space AOUPV will be quadruple the semicircle UFC. Also, the space described by OE in any instant is manifestly one-fourth of the space described in the same interval by the double line OP; therefore, the space AOUN is equal to the semicircle UFC: each of them is one-third of the area NUPV, or its equal AOUC;

and the space ACFUOA is two-thirds of the semicycloid AOUC.

COR. 4. From this proposition it will be easy to make a pendulum oscillate in a given cycloid, as UVW, whose base is parallel to the horizon. Thus, produce the axis VN till VA = VN; through A draw a line AC parallel to UW, and = $\frac{1}{2}$UW; on AC as a semi-base with axis CU describe a semicycloid AOU; and in like manner describe another semicycloid AW turned the contrary way: then a string whose length is AOU or AV, being fixed by one end at A, and in its motion coming gradually into contact with one or other of the cycloidal cheeks AOU, AW, will describe the cycloid UVW by its other extremity.

276. PROP. *To find the time of a pendulum's oscillation, in the arc of a cycloid.*

Let UVW (fig. 13. pl. XII.) be the cycloid in which the body is to oscillate: then by the preceding prop. and corols. VN = NA, and AOU = AW = AV = $\frac{1}{2}$UVW. The accelerating force in any point T of the curve is the same as if the body were placed upon the tangent T*t*: and by the nature of the inclined plane, the force of gravity: force in direction T*t* : : T*t* : TG : : QV : VH (by sin. triangles) : : VN : VQ. In like manner, taking any other point S in the curve, the force of gravity: force in the curve : : VR : VI : : VN : VR. Consequently the accelerating force of a body placed on different points T, S, of the curve, varies as the corresponding chords VQ, VR, of the generating circle, or as the portions VT, VS, of the curve, measuring from the vertex: these portions VT, VS, being the doubles of the corresponding chords VQ, VR, by cor. 1. of the last article. This problem, therefore, is analogous to that discussed in arts. 237. 238. respecting a body which is attracted by a force which varies directly as the distance from the lowest point V: consequently, in this case as well as that, the times of descent from any points on the arc where the body begins to move from quiescence to the lowest point V are equal; and, as equal forces acting in opposite directions will destroy equal quantities of motion in equal times, the times of ascent from V along the other branch of the curve will all be equal likewise: hence the times of oscillation in a cycloid, whether through greater or less arches, are equal.

The expression $\frac{1}{2}\pi\sqrt{\frac{d}{mf}}$ in art. 237. when adapted to the present case becomes $\frac{1}{2}\pi\sqrt{\frac{l}{32\frac{1}{6}}}$, l being equal to WTV or AV; this denotes the time of descent through half the cycloid: hence, in the case of vibration, $t=\pi\sqrt{\frac{l}{32\frac{1}{4}}}$ for the time of oscillation in a pendulum whose radius of curvature at its vertex is equal to l. This expression agrees with that given in art. 269. for pendulums vibrating in very small circular arcs; consequently, the chief inferences made from that equation will equally apply to the cycloidal pendulum.

Cor. *The time of descent through* WV *is to the time of descent through* SV *(after having passed over* WS*) as the semicircumference* NRV *to the arc* RQV. For, on the radius AV describe the quadrant AWZ, make VK = 2VR, and draw KL parallel to UW. Then is $\frac{1}{2}$VR : $\frac{1}{2}$VN : : 2VR : 2VN : : VK : VA. But VK is the sine of the arc LZ to the radius VA, and $\frac{1}{2}$VR is the sine of half the arc VQR to the radius $\frac{1}{2}$VN; consequently LZ and $\frac{1}{2}$VQR are similar portions of their respective circumferences; and therefore since the radii of the two circles are as 4 to 1, the arc LZ equal 2RQV. Now, AV being equal to WSV, and VK=2VR= VTS, and the times of descent through AV, KV, being as the arcs ALZ, LZ (art. 237.) we shall have, time in WSV : time in SV : : ALZ : LZ : : 2NRV : 2RQV : : NRV : RQV.

Cor. 2. *The time of describing any arc* VTS *after having fallen through* UPV, *is to the time of describing* UPV *as the arc* VQR *to the semicircular arc* VRN. This is evident, because the time of describing VS after passing over UPV is manifestly the same as the time of describing SV, after having run through WS.

The isochronism of vibrations in cycloidal arcs is demonstrated upon the supposition that the whole mass of the pendulum is concentrated in a point: a supposition which cannot actually take place in any vibrating body; and when the pendulum is of finite magnitude there is no point given in position which determines the length of the pendulum; for, that which is called the centre of oscillation will not occupy the same place in the given body when describing different parts of the track it moves through, but will be continually moved in respect of the pendulum itself during its vibration. This circumstance has prevented any general determination of the time of vibration in a cycloidal arc except in the imaginary case here considered: the property of isochronism, however, obtaining here, has occasioned the name of *Tautochrones* to be applied to cycloids.

Many other reasons have induced the artists to abandon the use of the cycloidal pendulum, although it was commonly adopted for some time after its first invention by Huygens. The principal are, the difficulty with which the metallic cheeks are bent into the true cycloidal form; the improbability of their long retaining it, supposing it once given; and the changes of which the pendulum is susceptible in consequence of the expansion and contraction by heat and cold. These sources of error are such as the peculiar property of the cycloid cannot obviate: and, as the variations from isochronism in very small circular arcs are very trifling and readily computed, the cycloidal pendulum is now wholly disused in practice.

Curve of Swiftest Descent.

277. At first view it might be imagined that as a right line is the shortest path from one point to another, so it should be the line of quickest descent from one point to another not situated in the same horizontal line: but it has been already seen (art. 266. cor.) that the times of descent through arcs of circles in certain positions are less than the times of running down the chords. And there does not appear any reason why other curves may not be found through which bodies shall pass from one to the other of two given points in less time than they would pass in circular arcs. The general problem was first proposed in 1697, by John Bernoulli, under the title of the *Brachystochronon*, or " that curve along the concave side of which if a heavy body descend it will pass in the least time possible from one point to another, the two points not being in the same vertical line." The problem was truly answered the same year by Leibnitz, Newton, L'Hospital, and James Bernoulli. The problem has been reconsidered more recently by Venturi, who, besides arriving at the same conclusions as the original investigators, has determined that there is a minimum of time of descent even in a circular arc: for an arc of a circle which does not exceed 60 degrees is a curve of speedier descent than any other curve which can be drawn within the same arc; and the arc of 90 degrees is a curve of speedier descent than any other curve which can be drawn without the same arc. And many other theorems might be found, particularly if we were to assume different hypotheses of gravity: but all we shall attempt here is to give the solution of the problem upon the common supposition of gravity being a constant force acting in parallel lines *.

PROP. *To determine the curve along which a body, solicited by gravity, will pass from one given point* A *to another point* B, *not in the same vertical line, in the shortest time possible.*

Let AC (fig. 2. pl. XIII.) be parallel and BE perpendicular to the horizon, intersecting each other in E, and let PM be any ordinate to the curve parallel to EB. Let $AP = x$, $PM = y$, $AM = z$; then the velocity at M will (arts. 243. 263.) be expressed by $y^{\frac{1}{2}}$, and consequently the fluxion of the time of descent through AM will be truly defined by $\frac{\dot{z}}{y^{\frac{1}{2}}}$ or its equal $y^{-\frac{1}{2}} \times (\dot{x}\dot{x} + \dot{y}\dot{y})^{\frac{1}{2}}$. Here, therefore, the fluent of $y^{-\frac{1}{2}} \times (\dot{x}\dot{x} + \dot{y}\dot{y})^{\frac{1}{2}}$ is to be a *minimum* when that of $\dot{x}$ obtains a given value AE. Whence we must have $y^{-\frac{1}{2}}\dot{x} \times (\dot{x}\dot{x} + \dot{y}\dot{y})^{-\frac{1}{2}} =$ a constant quantity; which, in order that the terms may be homologous, may be denoted by $a^{-\frac{1}{2}}$, or $\frac{1}{\sqrt{a}}$. Then $a^{\frac{1}{2}}\dot{x} = y^{\frac{1}{2}} \times (\dot{x}\dot{x} + \dot{y}\dot{y})^{\frac{1}{2}}$; therefore $\dot{x} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{(a-y)}} = \frac{y\dot{y}}{\sqrt{(ay-y^2)}}$; and $\dot{z} = \sqrt{(\dot{x}^2 + \dot{y}^2)} = \frac{a^{\frac{1}{2}}\dot{y}}{\sqrt{(a-y)}}$; consequently $z = 2a - 2a^{\frac{1}{2}} \cdot \sqrt{a-y}$. Hence when $y = a$, $z = 2a$; that is, if these values of y and z be represented by CV and the arch VMA, the latter will be double the former; which, as we have shewn (art. 275. cor. 2.), is a property of the cycloid whose vertex is V, and diameter of its generating circle CV.

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* For more concerning these and kindred enquiries, see Woodhouse's Treatise on Isoperimetrical Problems.

When the point B falls on the other side the vertex with respect to A, as it must do when EB is less with respect to AE than VC to AC, or than the diameter of a circle to half its circumference, the process will be the same, and it will terminate in a similar conclusion. The cycloid therefore is the curve required.

Cor. 1. If the celerity be supposed as any function F of the quantity y, the problem may be resolved in the same manner; in that case, the equation of the curve will be $\frac{\dot{x} \times (\dot{x}^2 + \dot{y}^2)^{-\frac{1}{2}}}{F} = \frac{1}{a}$.

Cor. 2. From the above investigation a very simple construction may be deduced. Thus, A and P (fig. 3. pl. XIII.) being the given points: through A draw the horizontal line AB, on part of which as a base describe any inverted cycloid A*pb*; join A, P, by the line AP cutting the curve A*pb* in *p*; join *pb*, and parallel to it draw PB meeting AB in B: so shall AB be the base of the inverted cycloid AVB, through which the body will pass from A to P in the shortest time possible. This construction is founded upon the property that cycloids are similar curves, having only one constant quantity entering their equation, namely, the diameter of the generating circle. We see, also, that for any two given points A, P, there can be only one cycloid which will answer the conditions of the problem of the brachystochronon.

It should be remarked, that this simple construction was first given by Newton in the Philosophical Transactions, No. 224.

278. PROP. *If* AVP *be the line of swiftest passage from* A *to* P (fig. 4. pl. XIII.), *and* AP *the right line joining the points* A *and* P; *then, if* PS *be drawn perpendicular to the curve in* P, *and* AS *let fall perpendicularly from the point* A *upon* PS, *the time in which a body drawn by gravity describes the right line* AP, *is to the time in which it would pass from* A *to* P *along the curve* AVP, *as* AP *to* AS.

Through P draw PN parallel to the axis of the cycloid, and PT parallel to the base BA, meeting the axis in H, and a circle described on the diameter CV in F and Q, and lastly, meeting the cycloid in T. Draw the chord CF, which (art. 274.) will be parallel to PS. Whence PM = FC, and MC = PF = (art. 273.) FV: and consequently AM = CQVF. By art. 276. the time in arc ATV : time in CV :: semicircumference : diameter; and by art. 276. cor. 2. the time of describing VP (after having passed over AV) : time in ATV :: VF : VQC. Therefore time in AVP : time in CV :: arc CQVF : diam. CV. But, time in CV : time in NP or CH :: CV : CF; therefore, by equality, time in AVP : time in NP :: arc CQVF : chord CF :: AM : MP. Again, time of fall through NP : time in right line AP :: NP : AP; therefore, the ratio of the time along AVP to the time in AP is composed of the ratios of AM to MP, and of NP to AP; that is, time along AVP : time in AP :: AM · NP : MP · PA. But AM · NP = MP · AS, each being the double of the triangle AMP. Consequently, the time in which a body falling from rest runs through the curve of the cycloid AVP, is to the time in which it would pass over the right line AP, as MP · AS to MP · PA, or as AS to AP. And in the same manner (mutat. mutan.) the demonstration proceeds when the point P is between A and V.

COR. 1. Since AP the hypothenuse of the right-angled triangle ASP is always greater than AS, the time of descent through the right line AP is always greater than the time of passing through the arc AP.

COR. 2. When P coincides with V, PS and AS coincide with VC and AC respectively; and then, time in arc AV to time in chord AV, as $\frac{1}{2}\pi$ to $\sqrt{(1+\frac{1}{4}\pi^2)}$, or as 1·5708 to 1·8621 nearly.

CHAPTER III.

On Central Forces.

279. SINCE a body, when once put into motion, will (art. 21.), unless prevented by obstacles, persevere in that state with the same velocity and the same direction, it follows that a body cannot describe a curve line unless it is subjected to the action of a constant force, or meets with obstacles after obstacles which change at every instant the direction of its motion. If the force which acts on a moving body according to any direction different from that in which it is moving, act at finite intervals of time, and communicate at each interval a determinate velocity, the body will describe a polygon. But if the body have received at first a finite velocity, and the force which deflects it from its path act continually or without interruption, the body will then describe a curve line: such is the effect of the constant solicitation of gravity; and such also that of the resistance of fluids.

A body which is moving in a curvilinear track may be considered, at any instant whatever, as if it were moving along the tangent of that point of the curve at which it is found; and if the force which deflects the body at any instant ceases to act, the body will persevere in its motion according to the direction of that tangent.

DEFS. 1. The *centre of attraction or of force* is the point towards which any body is solicited or impelled.

2. *Deflecting force* is the force which tends to bend the course of a body at every instant.

3. *Centripetal force* is the force which tends constantly to solicit or to impel a body towards a certain fixed point or centre.

4. *Centrifugal force* is that by which it would recede from such centre, were it not prevented by the centripetal force.

5. These two forces are called jointly CENTRAL FORCES.

COR. The centrifugal and centripetal forces, being correlatives in circular motions, may be represented by the same line.

6. *Projectile force* is that with which the body would run out in a tangent to its path, if there were no centripetal force to prevent it.

COR. The centripetal and projectile forces are heterogeneous, and cannot be compared. For the action of the one is incessant, but that of the other is impulsive.

7. The path described by a body acted upon by a centripetal force is often called its *trajectory*, or its *orbit.*

8. *Radius vector* is a line drawn from the centre to which the force is referred, or in which it is supposed to act, to any point in the trajectory wherein the body is found.

9. *Angular velocity* is the velocity with which the angle is described which is contained between any two positions of the radius vector; it is measured by an arc of a circle whose radius is a unit of distance, and comprehended between two positions of the radius vector at the interval of a unit of time.

10. When a body moves round a centre in an orbit or trajectory which returns into itself, the time employed by the body after passing a certain point before it returns to that point again is called the *periodical time.*

280. PROP. *If from two points* A *and* D, *equally remote from the centre of attraction* C (fig. 5. pl. XIII.) *two bodies move with equal velocities, the one along the right line* AC, *the other in a curve line* DBQ, *their velocities at all equal distances from the centre will be equal.*

Let DK in the tangent of the curve at the point D be the space which would be described in an evanescent portion of time with the velocity at D, FG the arc of a circle whose centre is C, and GK its tangent; and while AF would be described with the velocity at A, let FH be added to it by the attractive force. Draw the arc HI, and IL a tangent to it at I, meeting DK produced in L : draw also KB parallel to DC, and LB perpendicular to DL. Then, DG : DK : : GI : KL : : KL : KB, by similar triangles; therefore GI : KB : : DG^2 : DK^2; and consequently KB will be the space described by the attractive force, while DK would be described with the velocity at D; for the force may be considered as uniform during the description of the evanescent increments, and the spaces described by the action of a uniform force are as the squares of the times (art. 227.) Hence, the resultant will be DB, which is ultimately equal to DL; and the whole velocity will be increased in the ratio of DL to DK, or of DI to DG, or of AH to AF: consequently, since H, I, and L, are ultimately equi-distant from C, the velocities in EA and ED are always equally increased at equal distances, and will, therefore, always remain equal at equal distances.

COR. The same thing may in similar circumstances be shewn to obtain with respect to the velocities in EC, and any other curve E′D′Q′; and consequently it will hold with regard to the two curves: therefore, generally, *two or more bodies being attracted toward a given centre with equal forces at equal distances, if their velocities be once equal at equal distances, they will remain always equal at equal distances, whatever be their directions.*

281. Prop. *A body revolving in an orbit, by the joint effects of a projectile and a centripetal force, describes by its radius vector equal areas in equal times, and in unequal times areas proportional to the times.*

Let ABCD, &c. (fig. 7. pl. XIII.) be part of an orbit described by a body, which is solicited by a centripetal force towards the point φ. If the projectile force alone acted, the body would move uniformly in a right line as AV: but if the centripetal force be supposed to act by separate impulses or solicitations, as at B, C, D, &c. when the body receives the impulse at B it will be drawn out of its course towards φ, and (art. 217.) will describe the diagonal BC in the same time as the projectile force alone would have made it describe Bc. After equal intervals similar effects will take place at C, D, &c. Since AB, BC, &c. are the lines described in equal times by the body, the areas described in equal times by the radius vector will be AφB, BφC, &c. Now AB, Bc, expressing spaces passed over by a uniform motion, are equal bases of the triangles AφB, Bφc, which being terminated by the same point φ, have likewise equal altitudes, and are therefore equal. And because the body at B is by the joint action of the projectile and centripetal forces carried forwards in the diagonal BC of the parallelogram Gc, the opposite sides GB, Cc, are parallel, and Cc is parallel to Bφ. But Bφ is the common base of the two triangles BφC, Bφc; and these triangles are between the same parallels; therefore they are equal. Consequently AφB, which has been proved equal to Bφc, is likewise equal to BφC; that is, the areas described in equal times are eqaul. And, by composition, any sums of these areas are to each other as the times in which they are described; that is, the areas are universally as the times.

Let the number of these triangles be augmented, and their breadths diminished indefinitely, the centripetal force being now supposed to act continually: then will the ultimate perimeter ABCD, &c. be a curve line, which is always concave towards φ the centre of force; and, the above reasoning being still applicable to those triangles whose breadths are indefinitely diminished, the areas will be as the times.

It does not necessarily follow, that the centripetal force should cause the body always to *approach* the centre of force; it may continue to recede from it, notwithstanding its being drawn by a force residing there: but this property must always belong to its motion, that the trajectory which it describes be concave towards the centre of force.

COR. 1. *The plane in which the trajectory lies passes through the centre of force.* For the diagonal BC of the parallelogram *c*G is in the same plane as its sides; therefore BC, Bφ, are in the same plane: in the same manner B*d*, CD, Cφ, are in one plane, being the same as the former; and so on.

COR. 2. *The projectile velocity of a body revolving in a curve about an immoveable centre of force, is reciprocally as the perpendicular let fall from that centre upon the tangent to that point of the orbit where the body is when its velocity is estimated.* For the area of any of the triangles AφB, BφC, being constant, the base which represents the velocity is reciprocally as the perpendicular demitted upon it from φ.

COR. 3. *The angular velocity at the centre of force is reciprocally as the square of the body's distance from that centre.* For, if the small triangles φCD, φBA (fig. 8.pl. XIII.) are equal, they are described in equal times: and 2 area CφD = φC · CQ; also 2 area φBA = φB·BP; therefore φC·CQ = φB·BP. But, angle CφD : angle AφB : : CQ : *cq* : : φC·CQ : φC · *cq* : : φB · BP : φC · *cq* : : area φBA : : area φ*cq* : : $\varphi B^2 : \varphi c^2$ *or* φC^2.

COR. 4. If AB, BC, and DE, EF (fig. 7.), the arcs described in equal times, be completed into the parallelograms AC, DF, the centripetal forces at B and E will be to each other in the ultimate ratio of the diagonals BG, EZ, when those arcs are indefinitely diminished. For the motions of the body BC, EF, are compounded of the motions B*c*, BG, and E*f*, EZ; but BG and EZ are equal to C*c* and E*f*, which, as appears from this proposition, are generated by the impulses of the centripetal force in B and E, and are therefore proportional to those impulses.

COR. 5. *The forces with which bodies are drawn into curvilinear orbits are to each other as double the versed sines* ½GB, ½ZE, *of the indefinitely small arcs* ABC, DEF, *described in equal times; and these versed sines converge to the centre* φ, *and bisect the chords when those arcs are diminished indefinitely:* for such versed sines are half the diagonals of parallelograms; BG, EZ, being bisected by the diagonals AC, FD.

COR. 6. Conversely, *if the radius vector of a body describe areas proportional to the times about any point* φ, *it is urged towards that point by the centripetal force.* For if φ be not the centre of force, let it be some other point, as S: then, by the prop. the body will so move that its radius vector will describe also areas proportional to the times, about the centre S; which is impossible; for it is manifest a body cannot universally describe areas proportional to the times about two different centres in the same plane.

282. Prop. *To determine the ratio of forces by which bodies tending to the centres of given circles are made to revolve in their peripheries.*

Let AMa (fig. 12. pl. XII.) be the circle in which one of the bodies moves round the centre of force C, and let the indefinitely little arch AO be the distance it moves over in a given or constant particle of time; then, cor. 5. of the preceding prop. the centripetal force at O will be measured by twice Ap. And by the nature of the circle, the chord and arc AO will be ultimately equal in length; whence $AO^2 = Aa \cdot Ap = AC \cdot 2Ap$; consequently, $2Ap = \frac{AO^2}{AC}$. And the same may be shewn with respect to the motion in any other circle. So that, if R, and r, denote the radii of two circles, F, and f, the respective central forces, V and v, the velocities with which the bodies move in their peripheries, we shall have $F : f :: \frac{V^2}{R} : \frac{v^2}{r}$; therefore, *the forces are as the squares of the velocities directly, and as the radii inversely.*

Cor. 1. *In a circle the velocity is uniform, if the centre of force coincide with the centre of the circle.* For the radius which is the perpendicular to the tangent (cor. 2. art. 281.) is a constant quantity.

Cor. 2. Because $F : f :: \frac{V^2}{R} : \frac{v^2}{r}$, it follows that

$$V : v :: \sqrt{RF} : \sqrt{rf}, \text{ and}$$

$$R : r :: \frac{V^2}{F} : \frac{v^2}{f}.$$

Cor. 3. Comparing the analogy $V : v :: \sqrt{RF} : \sqrt{rf}$, with the expression $s \propto \frac{v^2}{2f}$ in uniformly accelerated motions, it follows that *the velocity is every where equal to that which a body would acquire in falling by the same uniform force through half the radius.*

Cor. 4. If the ratio of the periodic times be denoted by that of P to p, then the ratio of the velocities being as $\frac{R}{P}$ to $\frac{r}{p}$, we shall have by equality $\sqrt{RF} : \sqrt{rf} :: \frac{R}{P} : \frac{r}{p}$; whence also

$$F : f :: \frac{R}{P^2} : \frac{r}{p^2}, \text{ and}$$

$$R : r :: FP^2 : fp^2.$$

COR. 5. A few other properties of circular motion under different hypotheses may be thrown together; thus,

If $P : p :: 1 : 1$, then $F : f :: R : r$.

If $V^2 : v^2 :: r : R$, then $F : f :: r^2 : R^2 :: V^4 : v^4$,

also $P : p :: R^3 : r^3$.

If $F : f :: 1 : 1$, then $P : p :: \sqrt{R} : \sqrt{r}$.

If $V : v :: r : R$, then $F : f :: r^3 : R^3 :: V^3 : v^3$.

COR. 6. If the measure of the force, or the velocity which would be uniformly generated in a unit of time, be expounded by any power r^n of the radius AC (fig. 12. pl. XII.), then the distance through which a body would freely descend in the same time by the constant operation of that force will (art. 243.) be expressed by $\frac{1}{2}r$. Hence, since the distances descended by means of the same force uniformly continued are as the squares of the times, it is evident if the time of moving through AO be denoted by t, that the distance Ap descended in that time will be denoted by $\frac{t^2}{1^2} \times \frac{1}{2}r^n$: so that we shall have $AO = \sqrt{(2Ap \cdot AC)}$ $= \frac{t}{1} \times r^{\frac{n+1}{2}}$; which being the distance described by the revolving body in the time t, it follows that the space passed over in the given time 1, will be equal to $r^{\frac{n+1}{2}}$.

COR. 7. Hence, to find the periodic time we have $r^{\frac{n+1}{2}} : \pi \times 2r$ (the whole periphery) $:: 1 : 2\pi r \div r^{\frac{n+1}{2}} = 2\pi^{\frac{1-n}{2}}$, the true measure of the periodic time.

COR. 8. Hence also it follows that if n be expounded by 1, 0, -1, -2, -3 successively, the velocity corresponding will be as r, $r^{\frac{1}{2}}$, 1, $r^{-\frac{1}{2}}$, and r^{-1}; and the time of revolution as 1, $r^{\frac{1}{2}}$, r, $r^{\frac{3}{2}}$, and r^2, respectively.

SCHOLIUM.

283. From the preceding proposition and its corollaries the velocity and periodic time of a body revolving in a circle, at any given distance from the earth's centre, by means of its own gravity, may be deduced. Thus, let the radius of the earth ($=21000000$ feet, nearly) be denoted by r, and the space through which a heavy body falls at the surface ($=16\frac{1}{12}$ feet) by $\frac{1}{2}g$, the force of gravity at the surface being denoted by g;

then will the velocity per second in a circle at the surface (cor. 2.) be $= \sqrt{gr} = 26000$ feet nearly; and the time of revolution $= \frac{2\pi r}{\sqrt{gr}} = \pi \sqrt{\frac{4r}{g}} = 5075$ seconds. Let R be put for the radius of any other circle described by a projectile about the earth's centre: then, because the force of gravitation above the surface varies inversely as the square of the distance, we have, by cor. 8. $r^{-\frac{1}{2}} : R^{-\frac{1}{2}} :: 26000$ feet (velocity per second at the surface): $26000 \sqrt{\frac{r}{R}}$, the velocity in the circle whose radius is R. And $r^{\frac{3}{2}} : R^{\frac{3}{2}} :: 5075''$ (the periodic time at the surface) : $5075 \sqrt{\frac{R^3}{r^3}}$, the periodic time in the circle whose radius is R.

For example, if R be assumed equal to $60r$, the distance of the moon from the earth, the expression for the velocity will become $3356\frac{1}{2}$ feet per second; and that for the periodic time will become $2360035''$ or $27\frac{3}{10}$ days, nearly.

284. Thus also the ratio of the forces of gravitation of the moon towards the sun and the earth may be estimated. For $365\frac{1}{4}$ days being the periodic time of the earth and moon about the sun, and 27·3 days the periodic time of the moon about the earth; also 60 being the distance of the moon from the earth in terms of the earth's radius, and 23920 her mean distance from the sun in the same measure, we have, by cor. 4. $\frac{23920}{365{\cdot}25^2} :: \frac{60}{27{\cdot}3^2} :: F : f :: 2\frac{2}{9} : 1$ nearly; that is, the moon's gravitation towards the sun is to her gravitation towards the earth as $2\frac{2}{9}$ to 1 nearly.

285. Again, from the same principles the centrifugal force of a body at the equator arising from the rotation of the earth is derived. For the proposition and corollaries apply to centrifugal forces as well as centripetal ones; the terms, as before observed, being correlatives (when those two alone keep the body in its orbit). And we have just found (art. 283.) that the time of revolution is $5075''$ when the centrifugal force becomes equal to the gravity; also (cor. 4.) it appears that the forces in circles having the same radii are reciprocally as the squares of the periodic times: hence, therefore, since the earth's rotation is performed in $23^h.\ 56$. or $86160''$., we have $86160^2 : 5075^2 ::$ the force of gravity : the centrifugal force of a body at the equator arising from the earth's rotation $:: 1 : \frac{1}{289}$ nearly.

286. Since the time of revolution of a body under the equator EQ (fig. 6. pl. XIII.), and in any parallel of latitude BG,

is equal, the centrifugal forces (cor. 5.) are as the distances CE, AB, from the axis of motion, or, as radius CB, to the cosine AB of the latitude. But in any latitude, as at B, the centrifugal force is not (as under the equator) opposite to the whole gravity, but only a part of it, which also is to the whole as the cosine of the latitude to radius. For, produce AB the direction of the centrifugal force to D, and CB the direction of gravity, till it meet a perpendicular let fall upon it from D at F; then BD representing the whole centrifugal force at B, BF will represent that part of it which is directly opposed to gravity; but BD : BF : : BC : AB : : rad : cos BE. Therefore, combining these two ratios, it follows, that *the diminution of gravity at the surface of the earth arising from the centrifugal force varies as the square of the cosine of the latitude.*

The law just stated for the diminution of gravity is on the supposition of the earth's sphericity; but as the polar axis of the earth is rather shorter than the equatorial, the former being to the latter nearly as 319 to 320, or what is technically denominated the *compression* being about $\frac{1}{320}$; and as, moreover, the density of the earth is different at different distances from its centre; the preceding theorem is not exact. It would, however, take us too far from the immediate subject of this chapter, were we to trace all the minutiæ here. Let it suffice if there be added a theorem or two for the relations of gravity at different latitudes. In order to which, let γ the gravity under the equator, G that at either pole, g that under any latitude λ; then $g=(1+\cdot 0052848 \sin{}^2\lambda)\gamma$; and therefore $G=1\cdot 0052848\gamma$. Or, if g' denote the gravity at 45°, then for any latitude λ, we shall have $g=g'\ (1-\cdot 002837 \cos 2\lambda)$. And the absolute length of the centesimal second's pendulum expressed in metres

$$=\cdot 739575+0\cdot 0040942 \sin{}^2\lambda.*$$

287. But now to determine more universally the ratio of the force of a body revolving in any given circle to its gravity; we have already given $\pi\sqrt{\frac{4r}{g}}$, for the periodic time at the surface of the earth, when the gravity and centrifugal force are equal: if, therefore, the time of revolution in any circle whose radius is ϱ feet be denoted by t seconds, it will follow, from cor. 4. of the prop. that $\frac{r}{\frac{\pi^2\cdot 4r}{g}} : \frac{\varrho}{t^2}$: : gravity of body : its centrif. force in that circle; which is as unity to $\frac{4\pi^2\varrho}{gt^2}$, or as 1 to 1·2274

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* See Connaissance des Tems, an 1816. p. 332.

$\times \frac{\varrho}{t^2}$ very nearly. Thus, if the length of a sling by which a stone is whirled about be 2 feet, and the time of revolution half a second, the force by which the stone endeavours to fly off will be to its weight as $1{\cdot}2274 \times \frac{2}{{\cdot}25}$ to 1, or as 9·8192 to unity.

The circumference of the circle whose radius is ϱ being $2\pi\varrho$, and t the periodic time, the velocity in the circle will be $\frac{2\pi\varrho}{t}$; and if this be put for v in the general expression $v = \sqrt{2gs}$ (art. 243.) we shall have $\frac{2\pi\varrho}{t} = \sqrt{(2gs)}$, whence $s = \frac{2\pi^2\varrho^2}{gt^2}$ the space a heavy body must descend freely to acquire the velocity in the circle; but it is manifest that $\varrho : \frac{4\pi^2\varrho^2}{gt^2} :: 1 : \frac{4\pi^2\varrho}{gt^2}$, the preceding expression for the centrifugal force; so that our theorem agrees with the comprehensive one first given by the Marquis de l'Hopital, namely, *find from what height the body must have fallen to acquire the velocity in the circle; then, as the radius of the circle to double that height, so is the weight of the body to its centrifugal force.*

288. From the general proportion in the preceding article, the centrifugal force and periodic time of a pendulum describing a conical surface may readily be found. Thus, let CA (fig. 9. pl. XIII.) the length of the pendulum, be denoted by l; CS, the altitude of the cone, by a; the semidiameter AS of the base by ϱ, and the periodic time by t: then, because the body is retained in the circle by three different forces, viz. the centrifugal force $\frac{4\pi^2\varrho}{gt^2}$, in the direction SA, the force of gravity, or the weight, 1, in a direction parallel to CS, and the force of the thread AC, compounded of the two former; it follows that CS : CA or as $a : b ::$ weight of body at A : force upon the thread at A; also, as $1 : \frac{4\pi^2\varrho}{gt^2} ::$ CS : SA $:: a : \varrho$. Whence $gt^2 = 4a\pi^2$, and $t = 2\pi\sqrt{\frac{a}{g}} = 1{\cdot}10784\sqrt{a}$. Consequently, *the periodic time varies as the square root of the altitude of the conic pendulum, let the radius of its base be what it may.* Or, comparing this theorem with that in art. 271. I. it appears that the *semiperiodic time in the cone is equal to the time of oscillation of a simple pendulum whose length is the altitude of the cone.*

Because $\frac{1}{2}gt^2$ or its equal $2a\pi^2$, represents the space a heavy body will descend by its own gravity in t seconds (art. 243.); and because $1^2 : \pi^2 :: 2a : 2a\pi^2 = \frac{1}{2}gt^2$; it hence appears, that *as the square of the diameter of any circle is to the square of its periphery, so is twice the altitude of the cone to the distance a heavy body will freely descend in the time of a complete gyration of the conical pendulum.*

Hence also, if CS be to CA as a square inscribed in a circle to the square of its circumference, or *if the angle* CAS *be nearly* $2° 53' \frac{5}{6}$, *the periodic time of the pendulum will be equal to the time of free descent through* CS *.

289. PROP. *To determine the law of centripetal force tending to a given point* C, *by which a body may describe a given curve* APQ.

Draw PT a tangent to the curve at any point P (fig. 10. pl. XIII.), and from C, the proposed centre of force, demit the perpendicular CT: let PO, the radius of curvature at the point P, be denoted by R, the distance or radius vector CP by ϱ, and the perpendicular CT upon the tangent by p, the velocity in the curve being denoted by v. Then, because the centripetal forces in circles are as the squares of the velocities directly and the radii inversely (art. 282.), it follows that the force tending to the point O, by which the body might be retained in its orbit at P, which must manifestly be the same as the force in the circle whose radius is PO, will be defined by $\frac{v^2}{R}$, or by $\frac{1}{p^2 R}$, since v is inversely as p, by cor. 2. art. 281. Wherefore, by the resolution of forces, we shall have CT $(=p)$: CP $(=\rho)$: : $\frac{1}{p^2 R}$ (force in direction PO) : $\frac{\rho}{p^3 R}$, force in the direction PC.—Now, the general expression for the radius of curvature is $R = \frac{\rho\dot{\rho}}{\dot{p}}$ which value of R substituted for it in the preceding expression for F, the centripetal force towards C, will transform it to this; $F = \frac{\dot{p}}{p^3 \dot{\rho}}$, an equation expressing the law required.

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* When a body is made to describe a circle by being fixed to one extremity of a string (or of an inflexible bar), while the other extremity is attached to an immoveable point, or by moving along the concave superficies of a polished sphere or cylinder; in both cases, whatever be the proportion in which the centrifugal force is increased by increasing the velocity of the projection, the reaction of the string or of the surface will always be increased in the same proportion, so that the body will describe the same circle with different degrees of velocity. But when the centripetal force and the distance from the centre are given, the velocity is given (art. 282. cor. 3.), being that which would be acquired by falling down half the distance. If, therefore, the velocity be increased whilst the centripetal force continues the same, the centrifugal force being increased in the duplicate ratio of the velocity, it will be greater than the centripetal; therefore in the time that the body would have described any distance MT in the tangent (fig. 12. pl. XII.) it will be drawn to a greater distance than A from the centre, and will have described a curve exterior to the circle MA*m*. For a like reason, if the velocity be diminished, the centrifugal force becoming less than the centripetal, the body will describe a curve interior to the circle: but if the centripetal force be at the same time increased or diminished in the same proportion, the body will still be retained at the same distance from the centre, and describe the circle MA*ma*.

Hence it is manifest that when a body describes any orbit exterior or interior to that of the circle, the tangent being perpendicular to the radius vector, the centrifugal force of the body in its orbit is equal to the centripetal force with which the body would describe a circle at the same distance, and with the same velocity in the direction perpendicular to the radius vector. The same will be true if the direction be not perpendicular to the radius vector: for in this case if the motion be resolved into two, one in the direction of the radius vector, and the other perpendicular to it, the latter is the only part which will increase or diminish the centrifugal force.

In this last case the body is retained in its orbit by three forces; the centripetal and centrifugal forces, and that part of the motion in the tangent which is in the direction of the radius vector.—NEWTON *on Ultimate Ratios*.

Another expression, which will be sometimes useful, may be found by taking the value of the radius of curvature in terms of the arc and its rectangular co-ordinates, that is, $R = \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$; for this introduced into the expression $F = \frac{\rho}{p^3 R}$ will convert it to $F = \frac{-\rho\dot{x}\ddot{y}}{p^3 \dot{z}^3}$.

Cor. 1. If the point C be so remote that all right lines drawn from thence to the curve may be considered as parallel to each other, then making Pr perpendicular to Cp (Pp being an evanescent portion of the curve), the force will be as $\frac{-\rho\dot{x}\ddot{y}}{(CP \times Pr)^3}$, or barely as $\frac{-\dot{x}\ddot{y}}{(Pr)^3}$, since $\rho = CP$ may in this case be rejected.

This expression being general in all cases where the force acts in parallel directions, it hence follows that the force which always acting in the direction of the ordinate PM would retain the body in its orbit is every-where as $\frac{-\ddot{y}}{\dot{x}^2}$; because in this case PC coincides with PM, and Pr becomes $= \dot{x}$.

Cor. 2. Since the force tending to the point C is universally as $\frac{CP}{CT^3 \times PO}$ or $\frac{\rho}{p^3 R}$, the force to any other point c will of consequence be as $\frac{CP}{ct^3 \times PO}$. Therefore, the forces to different centres C and *c*, about which equal areas are described in the same time, are to each other in the inverse ratio of $\frac{\text{CT}^3}{\text{CP}}$ to $\frac{ct^3}{c\text{P}}$.

COR. 3. Hence also the ratio of the velocity at P to that by which a body might revolve in a circle about the centre C at the distance CP, is easily obtained. For, since the velocity at P is that by which the body would revolve in a circle about the centre O, and the forces tending to the centre O and C are to each other as $p = \text{CT}$ and $\varrho = \text{CQ}$; it therefore follows, if the ratio sought to be assumed as v to u, that $\frac{v^2}{\text{PO}} : \frac{u^2}{\text{PC}} :: p : \varrho$ (art. 282.) Whence also $v^2 : u^2 :: p \times \text{PO} \ (= p\text{R}) : \varrho \times \text{PC} \ (= \varrho^2)$; and consequently $v : u :: \sqrt{\frac{p\text{R}}{\varrho\varrho}} : 1 :: \sqrt{\frac{p\dot{\varrho}}{\varrho\dot{p}}} : 1 :: \sqrt{\frac{\dot{\varrho}}{\varrho}} : \sqrt{\frac{\dot{p}}{p}}$ because $\text{R} = \frac{\varrho\dot{\varrho}}{\dot{p}}$.

COR. 4. Finally, the law of centripetal force being given, the nature of the trajectory may hence be found: for, since the force F is universally defined by $\frac{\dot{p}}{p^3\dot{\varrho}}$, it is manifest that the fluent of $\text{F}\dot{\varrho} = \frac{-1}{2p^2}$, which, when F is given in terms of ϱ, will become known; and then, the relation between p and ϱ being given, the curve itself is known.

We now shew the application of this proposition by an example or two.

290. Ex. I. *Let it be required to find the force tending to the centre of an ellipse when a body revolves in its periphery.*

Let the semitransverse C′A (fig. 11. pl. XIII.), be denoted by a, the semiconjugate axe C′E by b, the radius vector C′P′ by ϱ, and its semiconjugate C′R by n: then, by the nature of the ellipse, (Emerson's Con. I. 34.) $\varrho\varrho + nn = aa + bb$, whence $n = \surd(a^2 + b^2 - \varrho^2)$: again (ibid. I. 37.) $n\ [= \surd(a^2 + b^2 - \varrho^2)] : b :: a : p\ (= c't) = \frac{ab}{\surd(a^2 + b^2 - \varrho^2)}$; therefore $\dot{p} = \frac{ab\varrho\dot{\varrho}}{(a^2 + b^2 - \varrho^2)^{\frac{3}{2}}}$. Consequently, $\frac{\dot{p}}{p^3\dot{\varrho}} = \frac{ab\varrho\dot{\varrho}}{(a^3 + b^2 - \varrho^2)^{\frac{1}{2}}} \times \frac{(a^2 + b^2 - \varrho^2)^{\frac{3}{2}}}{a^3 b^3 \dot{\varrho}} = \frac{\varrho}{a^2 b^2}$. So that the force tending to the centre of the ellipse is directly as the radius vector.

Ex. II. *To find the law of the centripetal force, by which a body tending to the focus* C *is made to revolve in the periphery of an ellipse.*

From the other focus F (fig. 11.) draw FS parallel to CT, meeting the tangent TP at right angles in S; join FP: then put AB $=2a$, C′D $=b$, as before, and the latus-rectum or parameter $\left(\frac{2b^2}{a}\right)=\lambda$. Denoting CP, and CT, as before, we have FP $=$ AB $-$ CP $=2a-\varrho$; whence, by reason of the similar triangles CPT, FPS, it will be $\varrho : p :: 2a-\varrho :$ FS $=\frac{p(2a-\varrho)}{\varrho}$. But, by the nature of the curve, FS × CT $=$ C′D^2: whence, $\frac{p^2(2a-\varrho)}{\varrho}=b^2$; and consequently $\frac{1}{p^2}=\frac{2a}{b^2\varrho}-\frac{1}{b^2}$. The fluxion of the latter expression is $-\frac{2\dot{p}}{p^3}=-\frac{2a\dot{\varrho}}{b^2\varrho^2}$. So that we have (art. 289.) $\frac{\dot{p}}{p^3\dot{\varrho}}=\frac{a}{b^2}\cdot\frac{1}{\varrho^2}=\frac{2}{\lambda\varrho^2}$: and (art. 289. cor. 3.) $\sqrt{\frac{p\varrho}{\varrho\dot{p}}}=\sqrt{\frac{2.(2a-\varrho)}{2a}}=\sqrt{\frac{\text{FP}}{\text{AC}'}}$. Hence it appears, that the centripetal force is in this case reciprocally as the square of the distance ϱ or CP; and that the velocity at P is to that by which the body might describe a circle at the distance CP, every-where in the ratio of $\sqrt{\text{FP}}$ to $\sqrt{\text{AC}'}$.

Otherwise thus: Let another body descending in a right-line begin to fall with the same velocity, then among the general equations for variable motions we have (art. 232. III.) $\varphi=\frac{v\dot{v}}{\dot{s}}$, where s is equivalent to ϱ in the present case; and $\dot{\varrho}$ being considered as negative, the equation becomes $\varphi=-\frac{v\dot{v}}{\dot{\varrho}}$; the same being likewise true in the curve, by art. 280. Now $v=\frac{1}{p}$ (art. 211. cor. 2.) and its fluxion $\dot{v}=\frac{-\dot{p}}{p^2}$, therefore $\varphi=\frac{1}{p}\cdot\frac{\dot{p}}{p^2\dot{\varrho}}=\frac{\dot{p}}{p^3\dot{\varrho}}$: which is the same expression as at art. 289. but deduced from art. 280. The rest may then be determined from the properties of the ellipse, as above.

Ex. III. *Required the law which would cause the body to move in a hyperbola, the force tending to the focus.*

In this instance, proceeding as before, we shall have $\frac{p^2(2a+\varrho)}{\varrho}=b^2$, instead of $\frac{p^2(2a-\varrho)}{\varrho}=b^2$, as in the ellipse. From this we obtain $\frac{1}{p^2}=\frac{2a}{b^2\varrho}+\frac{1}{b^2}$; whence there will result $\frac{\dot{p}}{p^3\dot{\varrho}}=\frac{2}{\lambda\varrho^2}$, the very same as in the ellipse.

EX. IV. *To find the law which, when the force tends to the focus, would cause the body to move in a parabola.*

Here the equation will be $\frac{2a+0}{\varrho} \cdot p^2 = b^2$, or $\frac{p^2}{\rho} = \frac{b^2}{2a} = \frac{1}{4}\lambda$; and the force again as $\frac{2}{\lambda\varrho^2}$. But the measure of the velocity will in this instance become $\sqrt{\frac{p\varrho}{\varrho p}} = \sqrt{\frac{2\cdot(2a-0)}{2a}} = \sqrt{2}$: consequently the velocity in a parabola is to that by which the body might describe a circle at the same distance from the centre of force, in the constant ratio of $\sqrt{2}$ to unity.

291. PROP. *To determine the ratio of the velocities of bodies revolving in different orbits, about either the same or different centres; the orbits themselves and the forces tending to the centres being given.*

Let APD (fig. 11. pl. XIII.) be any orbit which a body describes about the centre of force C; let the force itself at the principal vertex A be denoted by F; let r denote the semiparameter, or the radius of curvature at A, and let CT be perpendicular to the tangent TP. Then (art. 282. cor. 2.) the velocity at A is always as $\sqrt{\text{F}r}$; and (art. 281. cor. 2.) we have CT : CA : : $\sqrt{\text{F}r}$ (the velocity at A) : $\frac{\text{CA}}{\text{CT}}\sqrt{\text{F}r}$, the velocity at P. This answers, however the values of AC, r, and F, may vary.

COR. 1. If the centripetal force be as the square of the distance inversely, or $\text{F} \propto \frac{1}{\text{AC}^2}$, the velocity at P will become $\frac{\text{AC}}{\text{CT}}\sqrt{\frac{r}{\text{AC}^2}}$ or $\frac{\sqrt{r}}{\text{CT}}$. Consequently *the velocities of bodies revolving in different orbits about a common centre are directly as the square roots of the parameters, and reciprocally as the perpendiculars from the centre of force to the tangents to the curve at the points where the bodies are.*

Cor 2. If the velocity at P be denoted by Pp, and Cp be drawn, then, since $\text{P}p \propto \frac{\sqrt{r}}{\text{CT}}$, it follows that $\sqrt{r} \propto \text{P}p \cdot \text{CT}$, or as the triangle CP$p$. Hence, *the areas described about a common centre of force, by the radius vector, in a given time, are in the subduplicate ratio of the parameters.*

COR. 3. And, since the area of the curve APDB, when an ellipse, is known to be as $\text{AC}' \cdot \text{C}'\text{D}$, or as $\text{AC}' \cdot \sqrt{(r \cdot \text{AC}')}$; whence, if this be applied to $\sqrt{r}$, expressing, by the last cor. the area described in a given part of time, we shall obtain $\text{AC}' \cdot \sqrt{\text{AC}'}$ or $\text{AC}'^{\frac{3}{2}}$ for the relative measure of the time of a complete revolution. Therefore, it appears, *that the periodic times, let the species of the ellipses be what they will, are in the sesquiplicate ratio of the principal axes.*

This, of consequence, obtains when the ellipse becomes a circle: agreeing with what has been previously shewn, under the fourth case of cor. 8. art. 282.

Cor. 4. Hence, therefore, it follows *that the periodic time in an ellipse is the same as in a circle whose diameter is equal to the transverse axis, or radius equal to the mean distance* CD.

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It appears from this proposition and corollaries that the periodic time of a planet under the influence of a force varying inversely as the square of the distance, depends *on its mean distance alone*, and will be the same whether the planet describe a circle or an ellipse having any degree of excentricity whatever. Now suppose the shorter axis DE of the ellipse ADBE (fig. 11. pl. XIII.) to diminish continually, the longer axis AB remaining the same: then, as the extremity of the invariable line DC moves from D towards C′, the extremity C will move towards A, so that when D coincides with C′, C will coincide with A, and the ellipse will be transformed into a straight line AB, the length of which is equal to 2CD. In all the successive ellipses produced by this gradual diminution of C′D the periodic time remains unchanged, if the force acting at C continues unchanged. Just before the perfect coincidence of D with C′ the ellipse may be conceived as undistinguishable from the line AB; and the revolution in such ellipse undistinguishable from the ascent of the body along the right line from A to B, and the subsequent descent in an equal time, from B to A. Consequently a body solicited by such a central force will descend from B to A in half the time of the revolution in the ellipse ADBE; and the time of descending through any distance DC (supposing the projectile force extinguished) is half the periodic time of a body revolving at *half* that distance from the sun. Hence we see that *the squares of the times of falling to the sun, or other centre of force, are as the cubes of the distances from it:* agreeing with the conclusion in art. 236. To find this time of descent in particular instances, multiply half the periodic time by the square root of the cube of $\frac{1}{2}$, or the whole periodic time by $\frac{1}{2}\sqrt{\frac{1}{8}}$, or by $\frac{1}{8}\sqrt{2}$, or by ·1767767. This process give us 4*d*. 20*h*. for the time of descent of the moon to the earth. Mercury to the sun 15*d*. 13*h*. Venus, 39*d*. 17*h*. The earth, 64*d*. 13$\frac{3}{4}$*h*. Mars, 121*d*. 10$\frac{1}{2}$*h*. Jupiter, 765*d*. 21$\frac{2}{3}$*h*. Saturn, 1901*d*. 22$\frac{1}{2}$*h*. Herschell, 5433*d*. 17*h*. And the same rule will apply to the descent of the satellites of Jupiter, Saturn, &c. to their respective primaries. See Thorp's Newton, vol. I. p. 193. Whiston's Mathematical Philosophy, p. 173.

292. PROP. *The centripetal force tending to a given point* C (fig. 11.) *being inversely as the square of the distance, and the direction and the velocity of a body at any point* P *being given; to determine the path in which the body moves, and, if it returns, the periodic time.*

It is manifest from art. 290. that the trajectory is a conic section, of which the point C is one of the foci. Let the other focus be F. Upon the tangent demit the perpendiculars CT, FS, and draw CP, FP. Let $AC'=a$, $CQ=\rho$, as before; the sine of the angle of direction $CPT=m$, to radius 1; and let the given velocity at P be to that by which the body might revolve in a circle about the centre at the distance CP, in any given ratio of n to unity. Then, by art. 290. $n : 1 :: \sqrt{PF} : \sqrt{AC'}$; therefore, $n^2 : 1 :: PF \; (=2a-\rho) : AC'=a$: whence we find $AC'=\frac{\rho}{2-n^2}$. Again, since $CT=m\cdot CP$, and $FS=m\cdot FP$ (the angles CPT, FPS being equal), we have $C'D^2=CT\cdot FS=m^2CP\cdot FP=\frac{m^2n^2\rho^2}{2-n^2}$, so that the semiconjugate axis C′D is likewise given.

Lastly, by cor. 3. of the preceding article, it will be $CN^{\frac{3}{2}} : CP^{\frac{3}{2}} :: T$ (the periodic time in any given circle whose radius is CN) : $T\cdot CP^{\frac{3}{2}} \div CN^{\frac{3}{2}}$, the required time of revolution when the orbit is an ellipse; that is, when n^2 is less than 2. For, when $n^2=2$, the axis of the curve (expressed by $\frac{2\rho}{2-n^2}$) becomes infinite, and the orbit degenerates into a parabola: and if n^2 exceeds 2, the axis becomes negative, and the curve is a hyperbola, whose principal diameters are $\frac{2\rho}{n^2-2}$ and $\frac{2mn\rho}{\sqrt{(n^2-2)}}$ respectively.

COR. 1. Because neither the value of AC′ nor the expression for the periodic time is affected with m, it follows that the principal axis and the periodic time will remain invariable, if the velocity at P be the same, whatever the direction at that point may be.

COR. 2. We may readily apply this proposition to the case of bodies projected from the surface of the earth: for it has already been shewn (art. 283.) that a body projected with a velocity of 26000 feet, or 4·92424 miles, per second, would describe a circle at its surface: and by this prop. $1 : \sqrt{2} ::$

26000 feet : 36769 feet or 6·96393 miles, velocity of projection when a parabola would be the trajectory. Hence appears the truth of what is often remarked in popular treatises of astronomy; that if a body were projected from the earth's surface with a velocity exceeding 7 miles per second, it would (if not resisted by the air) describe a hyperbola; if with a velocity of rather less than 7 miles per second, it would describe a parabola; and if the initial velocity were between 7 and 5 miles per second, the body would describe an ellipsis: if the velocity be less than 5 miles (or 4·92424) per second, the body would not describe a complete ellipse, as its periphery would in part, if not entirely, fall within the circumference of the earth, and of course the motion of the body would be stopped at the first point of intersection of the two curves.

293. Prop. *The attraction of a corpuscle to a sphere is just the same as if all the matter of the sphere were collected into its centre: the force being supposed to vary inversely as the square of the distance.*

In order to prove the truth of this proposition, we must first investigate the force by which a corpuscle would be attracted towards a circular plane. Let O, therefore, be the centre of the circle ABCD (fig. 12. pl. XIII.) to which the corpuscle P is attracted. Suppose *abcd* a smaller circle in the same plane as ABCD, and having the same centre, and by supposition the attraction of P to any particle c will be as $\frac{1}{\mathrm{PC}^2}$. Put $\mathrm{PO}=d$, $\mathrm{PC}=x$, the number 3·14159 being as heretofore represented by π: then $\mathrm{O}c^2=x^2-d^2$, and $\pi(x^2-d^2)=$ area of circle *abcd*. The fluxion of that area is therefore $=2\pi x\dot{x}$; and, by the resolution of forces, $x : d :: \frac{1}{x^2} : dx^{-3}$, the attraction of P to c in the direction PO. Hence the fluxion of the whole force is truly defined by $2\pi x\dot{x}\times dx^{-3}$ or its equal $2\pi x^{-2}\dot{x}$; and the force itself by $\int 2\pi x^{-2}\dot{x}$: which when properly corrected becomes $-\frac{2\pi d}{x}+\frac{2\pi d}{d}=2\pi\left(1-\frac{d}{x}\right)$. So that when $x=\mathrm{PC}$ the attraction to the whole circle becomes $2\pi\left(1-\frac{\mathrm{PO}}{\mathrm{PC}}\right)$, or the attraction varies as $1-\frac{\mathrm{PO}}{\mathrm{PC}}$.

Now, let ABCD (fig. 13.) be a sphere, and P a corpuscle attracted towards it. Draw PAOC through the centre O; and let BID be the diameter of a section aBcD of the sphere perpendicular to PC. Put the radius $\mathrm{AO}=r$, $\mathrm{PO}=d$, $\mathrm{AP}=d-r$

$= c$, PI $= y$ and PB $=$ PD $= c + x$: then will AI $= y - c$, CI $= 2r - y + c$, and consequently AI·CI $=$ BI$^2 = (y-c)\cdot(2r-y+c) =$ PB$^2 -$ PI$^2 = (c+x)^2 - y^2$. From this equation there arises $y = \frac{2rc+2c^2+2cx+x^2}{2r+2c} = \frac{2dc+2cx+x^2}{2d}$, because $d = r + c$. Whence it follows, that $2\pi\left(1 - \frac{\text{PI}}{\text{PB}}\right)$ the value of the attraction towards the circle *a*BCD is equal to $2\pi\left(1 - \frac{2dc+2cx+x^2}{2d\cdot(c+x)}\right) = \frac{2\pi(2rx-x^2)}{2d(c+x)}$: this multiplied by $\frac{c\dot{x}+x\dot{x}}{d} = \dot{y}$ gives $\pi\frac{(2ar\dot{x}-x^2\dot{x})}{d^2}$ for the fluxion of the required force. The fluent of this is $\frac{\pi(rx^2-\frac{1}{3}x^3)}{d^2}$, which expresses the attraction of the segment ABD. And this, when B and D coincide with C and $x = 2r$, becomes $\frac{4\pi r^3}{3d^2}$, for the measure of the attraction of the whole sphere, which therefore varies as $\frac{r^3}{d^2}$. Now if the density δ of the sphere should vary, the attraction must *(cæt. par.)* vary as δ: so that for all spheres the attraction varies as $\frac{\delta r^3}{d^2}$. But the quantity of matter Q varies as δr^3; and consequently the attraction varies as $\frac{Q}{d^2}$. Therefore, if the spheres were evanescent in magnitude, or the same quantity of matter condensed into the centres, the attraction would be the same. Q. E. D.

Cor. 1. When $d = r$, the value of the attraction $\frac{4\pi\delta r^3}{3d^2}$, becomes $\frac{4}{3}\pi\delta r$. So that *at the surface of the sphere the attraction is directly as the radius.*

Cor. 2. *If the moleculæ of two spheres* S, S′, *attract each other by a force varying inversely as the square of the distance, the attraction is the same as if the whole quantity of matter in each sphere were collected into its respective centre.*

Cor. 3. Hence what has been proved respecting the attraction of two corpuscles when the force varies inversely as the square of the distance, holds true for two spheres, the particles of which attract each other according to the same laws. Consequently, *if the moleculæ of two spheres act upon each other according to that law, one sphere will describe a conic section about the other in one of its foci.*

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294. Since it is known that the sun and all the planets are nearly spherical, and that they revolve in ellipses in one of the

foci of which the sun is always very nearly; the preceding propositions, therefore, furnish astronomers with just reason for concluding, that *each planet is attracted to the sun by a force which is directly as the quantity of matter in the central body, and inversely as the square of the distance of their centres: and that the constituent particles also attract each other by a force which varies in like manner.* But we need not stop here. It is one of the constant laws of nature (art. 21.) that a body cannot act upon another, without being subject to an equal contrary reaction. Thus, the planets and the comets being drawn towards the sun, likewise attract the sun towards them by the same law: thus also the satellites are attracted towards the planets, and the planets contrarily towards the satellites; and both these again towards the sun. This attractive property is, therefore, common to the sun, planets, their satellites, and to comets; and, of consequence, we may regard the mutual gravitation of the celestial bodies as a property generally obtaining throughout the universe. This consideration will lead to a few more general propositions which will immediately follow: we may just observe previously, that what is already done will enable the student to understand the principles on which the quantity of matter and density of the planets are commonly ascertained. For the effects of attraction at different distances being as stated at the beginning of this scholium, and the quantity of matter being jointly as the magnitude and density of a body; if therefore the effects of the attraction of different bodies are known, and their diameters, we can find their densities, and thence their quantities of matter. Now to find the densities, let δ represent the density of the central body, r its radius, Q its quantity of matter, T the periodic time of the revolving body, d its mean distance from the central body, and s the natural sine of the angle under which r appears at the distance d. Then $Q \propto \delta r^3$, while $T^2 \propto \frac{d^3}{Q}$, and this again, $\propto \frac{d^3}{\delta r^3}$: hence $\delta \propto \frac{d^3}{r^3 T^2}$. But $s = \frac{r}{d}$, or $\frac{1}{s} = \frac{d}{r}$; whence $\frac{1}{s^3} = \frac{d^3}{r^3}$, and consequently $\delta \propto \frac{1}{s^3 T^2}$. From this general expression the comparative densities of most of the planets have been determined. But this brief sketch of the method must suffice: for this is not a place to enter more into detail.

295. **Prop.** *The common centre of gravity of two bodies is not affected by their mutual attractions.*

Since a body B attracting another body B′ (fig. 14. pl. XIII.) exerts its influence equally upon every particle of B, the acceleration of B′ to B is the same whatever the quantity of matter in B′ may be, and will vary as the quantity of matter in B: the magnitudes of the bodies being supposed indefinitely small with respect to their distance. And in like manner the acceleration of B towards B′, from the attraction of the latter, is in proportion to *its* quantity of matter. Let therefore B and B′ attract each other, G being their centre of gravity; then the acceleration of B′ towards B from B's action : acceleration of B towards B′ from B's action :: B : B′ :: GB′ : GB (art. 110). Hence, the spaces B*e*, B′*e*′, which the bodies pass over in indefinitely small portions of time, in consequence of their mutual attractions, will be as GB′, and GB. Consequently the remainders G*e*′, G*e*, must be in the same ratio; and, therefore, G continues to be the common centre of gravity of the two bodies.

COR. 1. *If while the bodies act on each other they be projected from* B′ *and* B *in opposite and parallel directions, with velocities proportional to their distances from the centre of gravity, they will describe similar figures about that centre.*

For let B′*d*′ and B*d* be the spaces which the bodies urged by the projectile force would describe while their attractions would have carried them to *e*′ and *e*; then, completing the parallelograms *e*′*d*′, *ed*, the bodies at the end of that time will be found at *b*′ and *b*. Now the spaces described being as the velocities, B′*d*′ or *e*′*b*′ : B*d* or *eb* :: G*e*′ : G*e*; hence the angle BG*b*′ = BG*b*, and consequently *b*′G*b* is a right line: also G*b*′ : G*b* :: GB′ : GB; so that G is the common centre of gravity of the bodies in their new situation *b*′, *b*; and the same may be shewn after a second and after a third interval of time, and so on.

COR. 2. If we conceive each body to be acted upon, at the same time, by equal accelerative forces in the same direction, the *relative* motions of the two bodies will not be altered, and they will still continue to describe similar figures about G which is then in motion. And by varying the motion of the system, the absolute initial velocities of B and B′ may be varied *ad libitum. Hence, if* B *and* B′ *be projected with any velocities, they will continue to revolve about their centre of gravity, and describe similar figures about it; and the centre of gravity, not being disturbed by their mutual attractions, will continue to move on uniformly in a right line.*

COR. 3. *If the attractions are inversely as the squares of the distances, the bodies will, with regard to their common centre of gravity, describe similar conic sections about that point as a focus.*

296. PROP. *The periodic time of two bodies* B, B′, *attracting each other with any forces revolving about their common centre of gravity* G, *is to the periodical time of one of the bodies* B′

revolving about the other unmoved, and describing a similar figure, in the subduplicate ratio of the other body B *to the sum of the bodies* $B + B'$.

Let BD' be the orbit described about the centre C (fig. 14.), and $B'D''$ that described about B. Draw the common tangent $B'R''$ and taking $B'D'$, $B'D''$, indefinitely small, draw $GD'R'$, and parallel to it $BD''R''$; then will BD', $B'D''$, be similar parts of the similar curves. The times in which the bodies are drawn from the tangent $B'R''$ through the spaces $R'D'$, $R''D''$, with the same force, will be as $\sqrt{D'R'}$ and $\sqrt{D''R''}$ (art. 232.), that is, because of the similar figures $GB'R'D'$, $GB'R''D''$, as $\sqrt{GB'}$ to $\sqrt{BB'}$; or again, since GB' is to BB' as B to $B + B'$ (art. 110.) the times will be as $\sqrt{B}$ to $\sqrt{(B + B')}$. But time in $R'D'$ = time in $B'D'$, and time in $B'D''$ = time in $R''D''$: also the whole periodic times vary as the times of describing similar portions. Therefore period. time in wh. curve $B'D'$: period. time in wh. curve $B'D'' :: \sqrt{B} : \sqrt{(B + B')}$.

Cor. *Bodies revolving about their common centre of gravity describe, by their radii vectores, areas proportional to the times of description.*

297. Prop. *If two bodies* B *and* B', *attracting each other mutually with forces reciprocally proportional to the square of their distances, revolve about their common centre of gravity* G; *then will the principal axis of the ellipse which the body* B *describes about the other* B' *by this motion, be to the principal axis of the ellipse which the same body* B *might describe about the other* B' *quiescent in the same periodic time, as* $\sqrt[3]{(B + B')}$ to $\sqrt[3]{B'}$.

Conceive a body B'' to be placed at G, whose attraction upon B shall be equal to that of B': then, as the attraction varies as the quantity of matter directly and square of the distance inversely, we have $\frac{B''}{BG^2} = \frac{B'}{B'B^2}$, and therefore $B'' = B' \cdot \frac{BG^2}{B'B^2}$. Now the squares of the periodic times of bodies revolving about the focus of an ellipse vary as the cubes of the major axes directly, and the absolute forces inversely; therefore, if the periodic time be given, the major axis must vary as the cube root of the absolute force. Consequently, major axis of ellipse described by B about B'': major axis of ellipse described by B about B' at rest in the same periodic time $:: B'^{\frac{1}{3}} \cdot \frac{BG^{\frac{2}{3}}}{B'B^{\frac{2}{3}}} : B'^{\frac{1}{3}} :: BG^{\frac{2}{3}} : B'B^{\frac{2}{3}}$.

Again, by similar figures, the major axis of B about B' at rest : major axis of B about $B'' :: BB' : BG$. Therefore, *componendo*, major axis of ellipse described by B when both bodies revolve about their centre of gravity : major axis of ellipse described by B about B′ in quiescence in the same time :: $B'B^{\frac{1}{3}} : BG^{\frac{1}{3}} :: \sqrt[3]{(B + B')} : \sqrt[3]{B'}$.

COR. *If two bodies attracting each other, move about their common centre of gravity, their motions will be the same as if they did not attract each other, but were both attracted with the same forces by a third body placed in the centre of gravity.*

298. PROP. *If a body projected in a given direction be constantly drawn towards two fixed points which are not both in the same plane with the initial direction, it will describe equal solids in equal times about the right line joining the said points: and the converse.*

Let the time be divided into any number of equal parts, and in the first moment let the body describe the line AQ (fig. 1. pl. XIV.); in the second, if not prevented, it would proceed to describe the line QT equal to AQ; but at Q it is acted on by centripetal forces tending to the centres C and B: let these forces be expressed by the lines QS, QV, which would be described in consequence of their individual energy, while the body would be carried by the projectile motion from Q to T. Complete the parallelopiped whose sides are the lines QS, QV, QT, the body by the joint action of these forces will describe QD the diagonal of the parallelopiped (arts. 64. 217.). But the solid QCBA = QCBT, because they have the same base QCB, and the same altitude (for the line TA cuts the plane QBA in Q, and TG = QA); also the solid QCBT = QCBD, because they stand on the same base, and are between the same parallel planes QCB, DT. In the same manner it is evident that equal solids will be described in other equal moments of time, round the same points. If, therefore, the number of intervals of time, and of the right lines AQ, QD, be indefinitely increased, the path of the body will ultimately become a curve, and the body impelled by continued forces will describe round the points C, B, equal solids in equal times; and in any times solids proportional to the times.

Conversely, if these solids QCBA, QCBD, described in equal times, be equal, the line TD will be parallel to the plane QCB; and therefore the body will be urged by forces QS, QV, tending to the points C, B.

COR. *The orbit* AQD &c. *is not contained in one plane, except in some particular cases.* For, that the orbit may not deviate from any plane, the forces on both sides of it ought to be equal and similarly situated.

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299. By means of the law just laid down, philosophers explain the motion of the moon and satellites, which are attracted towards two principal centres. But it would be incompatible with the objects of this work to enlarge here upon these and other particulars, connected with the doctrine of attraction and centripetal forces. The student who wishes to obtain a profound acquaintance with such matters should consult Newton's *Principia*, lib. 1.; Clairaut's *Théorie de la Lune;* Euler's *Theoria Motuum Lunæ;* D'Alembert's *Recherches sur differens Points du Système du Monde;* Simpson's *Essays and Tracts;* Dealtry's *Fluxions;* Frisi's *Cosmographia and De Gravitate Universali Corporum;* La Place's *Mécanique Céleste;* Vince's *System of Astronomy*, vol. 2.; *Dawson's* paper on the *Inverse Problem of Central Forces,* in the Manchester Memoirs, vols. 4 and 5, or corrected in Leybourn's Math. Repository, Nos. 4 and 5, N. S.

CHAPTER IV.

On the Rotation of Bodies about fixed Axes, and in free Space; with Theorems relative to the Centres of Oscillation, Gyration, Percussion, Spontaneous Rotation, &c.

300. In demonstrating the chief propositions relative to rectilinear motion, whether uniform or variable, accelerated or retarded, we have either considered the bodies as physical points, or we have imagined that the impelling force has been impressed, and the resistance exerted, in the direction of a right line, passing through the centre of gravity of the body moved; and, therefore, that every particle of such body must partake of the same velocity as that with which the centre of gravity moves. But in numerous instances which occur to the mechanist, a body or system of bodies is so situated that when any force or any number of forces are impressed upon it, it cannot take any other motion than one of rotation about a fixed axis, which may either pass through the body or system, or be at an extremity of it: so that the velocity of the constituent moleculæ of the system shall be greater or less according to the greater or less distance of any individual particle from the axis about which the motion is performed. And in such cases it is necessary to call to our aid other considerations than what were required when discussing the properties of acceleration and retardation,

Again, with respect to the motion of bodies at liberty to move freely by the action of any force impressed, if any such body receive an impulsion in any direction which does not pass through the centre of gravity, the motion which will ensue is a rotatory one: for if at the same moment a body is impelled according to any direction AB (not passing through the centre of gravity), an equal and opposite force is exerted upon the body, in a parallel direction, CG, passing through the centre of gravity, the centre of gravity will manifestly be kept at rest: nevertheless, it is clear that the other parts of the body will not be in a state of quiescence; because the two forces, though equal, are not directly opposite: so that the only motion that the body can have, its centre of gravity being at rest, is evidently a motion of rotation about that centre. Now, the fixed axis round which a body revolves is pressed by the impelling force while it generates rotatory motion; but the axis being (by hyp.) immoveable, reacts equally against that pressure, and when it passes through the centre of gravity would, as above stated, cause each particle to move with the same velocity, and in the direction of the force. If, then, the force which presses against a fixed axis in given circumstances be ascertained, the motion of the body in free space when the axis is removed will be known: for the latter motion will consist of the rotatory motion about the axis passing through the centre of gravity considered as fixed, compounded with the motion of the centre of gravity caused by the force now at liberty to impel that centre, the fixed axis which passed through it being removed.

When a solid body receives an impulse on any one point, or that point is urged in any way by a moving force, it cannot move unless the other points with which it is connected by the force of cohesion move also (except the force of impulsion is sufficient to overcome that of cohesion; a case which is not meant to be considered here). And whatever is the motion of any particle, that particle must be conceived as urged by a force precisely competent to the production of that motion, by acting immediately on the particle itself. The particle immediately impelled by the external force is either pressed towards its neighbouring particles, or is drawn from them; and by this endeavour to change its place the connecting forces are excited, or brought into action. We are but little acquainted with the nature of these connecting forces: but this is not of much consequence in a mechanical point of view; for the fact that the forces by which the moleculæ of bodies act on each other are equal is sufficient for our present purpose.

In the propositions we are about to enter upon, the attention will be chiefly called to two objects, viz. The moving force by which the revolving motion is generated; and the inertia of the different parts of the system moved. And both the effects of the moving force and of the inertia of the particles depend upon their distance from the axis of motion, all other things being the same: if both these be ascertained the absolute acceleration will be determined, and consequently the absolute velocity generated in a given time.

These preliminary observations may be determined by remarking, that, since in rotatory motions all the particles of a body are supposed to turn together without changing their relative positions; therefore, when a body has made a complete rotation, each particle has described the circumference of a circle, and the whole paths of the various particles will be in the ratio of their respective circumferences, or of their radii; and the same is true of any portion of a rotation: so that the velocities of the different particles are proportional to their distances from the axis of rotation, and are not adequate measures of the *rotatory velocity of the body;* its proper measure is the angle described by any radius vector of the body in a unit of time.

301. PROP. *If a body revolve about an axis, the particles of which that body is composed resist, by their inertia, the communication of motion to any given point, with forces which are as the particles themselves, and the squares of their distances from the axis of motion jointly.*

Let a force φ be applied at any point A (fig. 2. pl. XIV.) in order to communicate motion to a system of particles p, p', p'', &c. revolving at determinate distances round the centre of motion C. Let A be such a quantity of matter as will, if concentrated in A, have the same effect in resisting the communication of motion to that point, by its inertia, when any particle p is removed from the situation P, as that particle would have, revolving at the distance PC. Now the effect of the given force φ acting at the point A (in a direction φ A perpendicular to CA), to move a body at that point, is to its effect to move a body at P, inversely as those distances; or as PC to AC, by the nature of the lever: and if these bodies be moved with equal angular velocities, their distances from the axis being then (art. 300.) as the spaces described in a given time, the moving forces are inversely as the spaces described. But taking for a moment the notation of art. 228. we there have $F \propto \frac{BV^2}{S}$, or $FS \propto BV^2$; in which, if $F \propto \frac{1}{S}$, then will $B \propto \frac{1}{V^2}$: consequently, the quantities of matter must be inversely as the squares of the velocities; or, in the present case, inversely as the squares of the distances from the axis; that is, $A : p :: PC^2 : AC^2$; whence we have $A = \frac{p \cdot PC^2}{AC^2}$, which indicates that the resistance of the particle p at the distance PC is equivalent to the resistance of the mass $\frac{p \cdot PC^2}{AC^2}$, at the distance AS. In like manner, taking another particle p', at the distance P′C, and a corresponding quantity of matter A′ concentrated into the same point A, we shall have the resistance of the particle p' at its distance equal to the resistance of the mass $\frac{p' \cdot P'C^2}{AC^2}$ at the distance AC. And the same may be shewn of other particles p'', p''', &c. Consequently, if we use, as hitherto, the character $\int$ to denote the whole fluent, or sum of all the separate resistances, we shall have the resistance of the whole revolving body expressed by $\int \frac{p \cdot \text{PC}^2}{\text{AC}^2}$.

Cor. 1. *The force which accelerates the point* A *of any body revolving on an axis, to which point that force* ϕ *is applied, is equal to the product of the force into the square of the distance* AC, *divided by the sums of the products of all the moleculæ into the squares of their respective distances from* C, *the centre of motion.*

For the mass moved is $\int \frac{p \cdot \text{CP}^2}{\text{AC}^2}$, and the moving force is ϕ: but the accelerating force is equivalent to the quotient of the moving force by the mass, and is therefore represented by $\int \frac{\phi \cdot \text{AC}^2}{p \cdot \text{PC}^2}$.

Cor. 2. *The angular velocity of a system, generated in a given time, by any force* ϕ *at* A, *perpendicular to* AC, *is proportional to the rectangle of the force into the distance at which it acts, divided by the sums of the products of all the moleculæ into the squares of their respective distances.*

For the absolute velocity of the point A is as the accelerating force, and the angular velocity is as the absolute velocity directly and the distance reciprocally; therefore the angular velocity is as $\int \frac{\phi \cdot \text{AC}^2}{p \cdot \text{PC}^2} \times \frac{1}{\text{AC}}$, or as $\phi \cdot \text{AC} \div \int p \cdot \text{PC}^2$.

Cor. 3. *The angular motion of any system, generated by a uniform force, will be a motion uniformly accelerated.*

This is evident, because the accelerating force is in a constant ratio to the uniform force ϕ.

Cor. 4. What has been here shewn with respect to moleculæ situated on a right line passing through a centre of motion will hold equally with regard to a body or system moving upon an axis: for all the particles of such body may be conceived to be transferred to the plane in which the axis of suspension CP performs its motion, by an orthographical projection, the lines of transference being all parallel to the axis of motion; this supposition will, it is obvious, neither affect the place of the centre of gravity (with regard to the axis of motion) nor the angular motion of the body.

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302. The preceding investigation has been conducted upon the supposition that the forces were each applied in a direction

perpendicular to the line drawn from each point of application to its respective centre of motion: but the same conclusion would result if the moving forces were applied in any direction, provided that, instead of these forces, we had taken their equivalents when reduced to the perpendicular direction. As this proposition and its corollaries are, however, of great importance in all that depends upon rotatory motion; and as the same conclusions, when drawn from different methods, strike the mind with greater force; we shall here consider the matter in another way.

Let an axis of rotation pass through C (fig. 4. pl. XIV.) perpendicular to the plane of the figure, and let a body fixed firmly to this axis be acted upon by several accelerating forces; we are to enquire into the circumstances of the motion produced. Suppose, at first, that a particle p, situated at P in the figure, is urged about the fixed axis by a force φ, applied in the direction PD: that force tends to impress upon the particle p a certain velocity in the direction PD, yet, in consequence of the mutual cohesion between the different moleculæ of the body, and the connection of the whole with the fixed axis, the velocity can only be produced actually in the initial direction Pd perpendicular to CP. Drawing, therefore, Pd perpendicular to CP, the force φ must be decomposed into two others, in the directions Pd, PC, of which the one Pe will be extinguished by the resistance of the axis, and the other Pd has place; where, of consequence, we have P$d=\varphi$ cos DPd (art. 59.). If, therefore, we denote the distance CP of p from the axis of rotation by r, and the perpendicular distance DC of the force from the axis by δ, we have, in the triangle CPD, sin DPC $=$ cos DP$d=\frac{\delta}{r}$; and, consequently, P$d=\varphi\,\frac{\delta}{r}$. This expression would represent the effective accelerating force of the particle p, if that particle were alone; but the connection of this particle with the others, and the operation of the forces acting on the latter, change this effect: if, then, the arc a be run over at the end of the time t by a particle at a unit of distance from the fixed axis, ar will be the arc described in the same time by the particle p; so that the velocity, and the effective accelerating force, (art. 232. I. VI.) of this latter will be $r\,\frac{\dot{a}}{\dot{t}}$, and $r\,\frac{\ddot{a}}{\dot{t}^2}$ respectively.

Now comparing this force with the former, and observing that they are both directed in the same right line, we may have recourse to the principle of D'Alembert, stated in art. 267. The accelerating force $\phi\,\frac{\delta}{r}$ which is impressed in the direction of the initial motion, must be decomposed into

$r\frac{\ddot{a}}{\dot{t}^2}$ . . . effective accelerating force.

$\varphi\frac{\delta}{r} - r\frac{\ddot{a}}{\dot{t}^2}$ . . . $\left\{\begin{array}{l}\text{accel. force destroyed by the action of} \\ \text{the other powers in the system.}\end{array}\right.$

In like manner we may proceed to investigate the effects of other forces φ', φ'', &c. acting upon the moleculæ p', p'', &c. (whose distances are r', r'', &c.) in the directions P′D′, P″D″, &c., and at the distances δ', δ'', &c. the expressions being the same with the letters accented similarly: the whole, therefore, will be in equilibrio when impressed by the moving forces

$$\frac{\delta}{r}\varphi p - \frac{\ddot{a}}{\dot{t}^2}rp, \frac{\delta'}{r'}\varphi' p' - \frac{\ddot{a}}{\dot{t}^2}r'p', \frac{\delta''}{r''}\varphi'' p'' - \frac{\ddot{a}}{\dot{t}^2}r''p'', \text{\&c.}$$

Now as the system is attached to a fixed axis, only one equation (art. 98.) is necessary to express the state of equilibrium. And, if we suppose that the forces φ, φ', φ'', &c. are parallel to the plane of the figure, or perpendicular to the plane of rotation (and they may all be resolved to such planes by an obvious process), it will be merely requisite to make the sum of the moments of the powers (art. 60.) with respect to the fixed axis equal to nothing. Here, that of the first-force will be $\delta\varphi p' - \frac{\ddot{a}}{\dot{t}^2}r^2 p$, and the moments of the other forces will be expressed in the same manner, adding the accents. Thus, then, we shall have

$$\delta\varphi p + \delta'\varphi' p' + \text{\&c.} = (r^2 p + r'^2 p' + \text{\&c.})\ \frac{\ddot{a}}{\dot{t}^2}.$$

Or taking the character $\int$ as before, and denoting the angular velocity by U, we shall have $\int\varphi\delta p = \frac{\ddot{a}}{\dot{t}^2}\int r^2 p$: whence, $\frac{\ddot{a}}{\dot{t}^2}$ $= \frac{\dot{U}}{\dot{t}} = \int\frac{\varphi\delta p}{r^2 p}$.

Hence, if the quantity $\int r^2 p$, which is the sum of the products of the several moleculæ into the squares of their respective distances from the axis, be called the *momentum of inertia*, and if $\frac{\dot{U}}{\dot{t}}$ be called the *angular accelerating force*, the equation just given may be thus stated in words at length:

The angular accelerating force is the quotient of the sum of the momenta of the moving forces, or (art. 60.) *of their resultant, divided by the momentum of inertia.*

303. From the above equation we may readily obtain an expression for the angular velocity. For (art. 232. II.) $\varphi = \frac{\dot{v}}{\dot{t}}$, &c. give $\int \varphi \delta p = \frac{\dot{v}}{\dot{t}} \delta p = \frac{\dot{v}}{\dot{t}} \delta p + \frac{\dot{v}'}{\dot{t}} \delta' p' +$ &c. whence we have $\dot{\text{U}} = \frac{\dot{v}\delta p + \dot{v}'\delta' p' + \&c.}{r^2 p + r'^2 p' + \&c.}$; and, taking the fluents,

$$\text{U} = \frac{v\delta p + v'\delta' p' + \&c.}{r^2 p + r'^2 p' + \&c.} = \int \frac{v\delta p}{r^2 p},$$

From which it appears that cor. 2. art. 301. is not confined to a single force; but may be extended to as many forces as we please, and applied to the body in any directions whatever.

304. As to the part $\text{P}e$ of the force φ, which operates as a pressure upon the fixed axis, and is entirely destroyed by its reaction, it may be easily determined. For $\text{P}e = \varphi \cos \text{DPC}$, and $\text{DP} = \sqrt{(r^2 - \delta^2)}$; therefore, $\cos \text{DPC} = \frac{\text{DP}}{\text{PC}} = \sqrt{\left[1 - \left(\frac{\delta}{r}\right)^2\right]}$

$$\text{consequently, } \text{P}e = \varphi \sqrt{\left[1 - \left(\frac{\delta}{r}\right)^2\right]}.$$

And the same may be shewn with regard to the effect of any other forces φ', φ'', &c. upon the axis of motion.

305. DEF. The *centre of oscillation* is that point in the axis of suspension of a vibrating body in which, if all the matter of the system were collected, any force applied there would generate the same angular velocity in a given time as the same force at the centre of gravity, the parts of the system revolving in their respective places.

Or, since the force of gravity upon the whole body may be considered as a single force (equivalent to the weight of the body) applied at its centre of gravity, the *centre of oscillation* is that point in a vibrating body into which, if the whole were concentrated and attached to the same axis of motion, it would then vibrate in the same time the body does in its natural state.

COR. From the first definition it follows that the centre of oscillation is situated in a right line passing through the centre of gravity, and perpendicular to the axis of motion*.

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* Since, according to the former of these definitions, the centre of oscillation is a definite point, and according to the latter it is variable; it might, perhaps, be better to say, that a body has a *centre* of oscillation when it is attached to a centre of motion, and has a *line* of oscillation when it moves upon an axis; the *line* of oscillation in the latter case being parallel to the axis of motion. But this is merely hinted by the way, and not introduced into the text.

306. **PROP.** *If a body vibrate about an axis by the force of gravity, the distance of the centre of oscillation from that axis is equal to the quotient arising from dividing the sum of the products of each particle into the square of its distance from the axis, by the product of the mass or body into the distance of its centre of gravity from the axis.*

Let AB (fig. 3. pl. XIV.) be a plane passing through the centre of gravity G of the body perpendicular to the axis of vibration on which the body is orthographically projected; C the centre of motion of that plane, or of the whole body reduced to that plane; O the centre of oscillation in the line CG produced, and p, p', &c. the constituent moleculæ of the body thus projected. Through C draw CX parallel to the horizon, and upon that line demit the verticals or perpendiculars pd, $p'd'$, Gg, Oo, &c. Now since the angular velocity of the several particles is not changed by this projection, we have $p \cdot p\mathrm{C}^2 + p' \cdot p'\mathrm{C}^2 + \&\mathrm{c}. = \int p \cdot p\mathrm{C}^2$, the momentum of inertia of the whole body B, while $\mathrm{B} \cdot \mathrm{OC}^2$ will express the momentum of inertia of an equal body concentrated into the point O. And since the force of gravity φ acts in the parallel directions pd, $p'd'$, &c. we shall have $\varphi \cdot p \cdot d\mathrm{C} + \varphi \cdot p' \cdot d'\mathrm{C} + \&\mathrm{c}.$ for the effect of gravity to turn the body about C, by the nature of parallel forces, and this is equal to $\varphi \cdot \mathrm{B} \cdot \mathrm{C}g$, by the nature of the centre of gravity (art. 108.). And if B were concentrated in the point O, then would $\varphi \cdot \mathrm{B} \cdot \mathrm{C}o$ be the accelerating force of gravity to turn the body about C. But, that the same angular velocity may be generated in both cases, the quotients of the accelerating forces with respect to C, by the momenta of inertia, must be equal (art. 301. cor. 2. 303.), that is, $\frac{\varphi \cdot \mathrm{B} \cdot \mathrm{C}g}{\int p \cdot p\mathrm{C}^2} = \frac{\varphi \cdot \mathrm{B} \cdot \mathrm{C}o}{\mathrm{B} \cdot \mathrm{OC}^2}$, whence $\mathrm{OC}^2 = \frac{\varphi \cdot \mathrm{B} \cdot \mathrm{C}o \cdot \int p \cdot p\mathrm{C}^2}{\varphi \cdot \mathrm{B} \cdot \mathrm{B} \cdot \mathrm{C}g}$. Now, the triangles CG$g$, CO$o$ being similar, we have $\frac{\mathrm{CO}}{\mathrm{CG}} = \frac{\mathrm{C}o}{\mathrm{C}g}$: hence, substituting the former fraction for the latter in the preceding value of OC^2, it becomes $\mathrm{OC}^2 = \frac{\mathrm{CO}}{\mathrm{CG}} \cdot \frac{\varphi \cdot \mathrm{B} \cdot \int p \cdot p\mathrm{C}^2}{\varphi \cdot \mathrm{B} \cdot \mathrm{B}.} = \frac{\mathrm{CO}}{\mathrm{CG}} \cdot \frac{\int p \cdot p\mathrm{C}^2}{\mathrm{B}}$: consequently

$$\mathrm{CO} = \frac{\int p \cdot p\mathrm{C}^2}{\mathrm{B} \cdot \mathrm{CG}} = \frac{\int pr^2}{\mathrm{B} \cdot \mathrm{CG}}.$$

This expression, being independent of the line CX, will continue the same for all inclinations of the line CO; so that the centre of oscillation O, thus determined, is a *fixed point*, agreeably to the definition.

COR. 1. *Hence any body* AB, *suspended at* C, *and vibrating, may, so far as regards the time of vibration, be considered as a simple pendulum whose length is* CO.

COR. 2. *That* CO *may be the equivalent pendulum, and* O *the centre of oscillation,* O *must be in the line* CO, otherwise it would not rest in the same position with the body, when no force was keeping it out of its vertical position.

COR. 3. *If a body be turned about its centre of gravity in a direction perpendicular to the axis of motion, the place of the centre of oscillation will remain unaltered.* For the quantity $\int pr^2$ will be not at all affected by such a motion of the body.

COR. 4. The distance of the centre of gravity from that of oscillation (if the plane of the body's motion remain unaltered) will be reciprocally as the distance of the former from the point of suspension. *If, therefore, that distance be found when the point of suspension is in the vertex, or so situated that the operation may become the most simple, the value thereof in any other proposed position of that point will likewise be given by one single proportion.*

COR. 5. The product of the distances of the centre of gravity, and that of oscillation, from the axis of motion, is manifestly a constant quantity for the same plane of vibration. *If, therefore, the centre of oscillation be made the point of suspension, the point of suspension will become the centre of oscillation.*

COR. 6. *Hence also, if upon the plane of vibration passing through the centre of gravity of any body, two concentric circles be described, having the common centre* G, *and radii* GC, GO, *the body suspended from any point in the periphery of either circle will perform an oscillation in the same time.*

Some other corollaries depending upon the relation subsisting between the centres of oscillation and of gyration will be given after we have treated of that centre: when also we shall apply the propositions to the determination of these centres in various bodies. Previous to this, however, we may here shew how

307. *To find the distance of the centre of oscillation from the point of suspension experimentally.*

Suspend the body proposed very freely by the given point, and make it vibrate in small arcs; then count the number n of vibrations it makes in a minute, by a good stop-watch, applying when necessary the correction indicated in arts. 269, 270; so shall the distance CO of the centre of oscillation and point of suspension be denoted by $\frac{140850}{n^2}$. For the length of the simple second pendulum being 39⅛ inches; and the lengths of pendulums being reciprocally as the square of the number of vibrations made in a given time; therefore $n^2 : 60^2 :: 39\frac{1}{8} :$

$\frac{60^2 \times 39\frac{1}{8}}{n^2} = \frac{140850}{n^2}$, the length of the simple pendulum which oscillates n times in a minute, or the distance CO in the compound pendulum.

Or, two-thirds of the length of a thin cylindrical rod, suspended at one end, and vibrating in the same time as the body, will give CO; the reason of which will soon appear.

308. Prop. *In a compound pendulum, consisting of several bodies revolving about a common axis, the centre of oscillation may be determined by this process: add together the several products of each mass into the distances of the respective centres of oscillation and gravity from the axis of motion, and divide the sum by the product of the whole system into the distance of the common centre of gravity from the axis of motion; the distance of the centre of oscillation from the same axis will be represented by that quotient.*

For it appears from art. 306. that with respect to any body B in the compound mass, $\int pr^2 = \mathrm{B}\cdot\mathrm{CO}\cdot\mathrm{CG}$; and again, with regard to another body B′ in the same system $\int p'r'^2 = \mathrm{B}'\cdot\mathrm{CO}'\cdot\mathrm{CG}'$, and so on; consequently $\mathrm{S}\cdot\int\cdot pr^2 = \int \mathrm{B}\cdot\mathrm{CO}\cdot\mathrm{CG}$; that is, the sum of all the pr^2 in the whole pendulum is equal to the sum of all the products in each part of the pendulum, which arise from multiplying each part into the product of the distances of its centres of gravity and of oscillation from the common axis of motion. If, therefore, this latter sum be divided by the product of the compound mass into the distance of the common centre of gravity from the axis of motion, the quotient (art. 306) will be the distance of the centre of oscillation from the same axis: for $\frac{\int\cdot\int pr^2}{\int \mathrm{B}\cdot\mathrm{CO}\cdot\mathrm{CG}} \propto \frac{\int pr^2}{\mathrm{B}\cdot\mathrm{CO}\cdot\mathrm{CG}}$: whence $\mathrm{CO} = \frac{\int pr^2}{\mathrm{B}\cdot\mathrm{CG}}$, as well with respect to the compound mass as to any of its constituent parts.

309. Def. The *centre of gyration* is that point in which, if all the matter contained in a revolving system were collected, the same angular velocity will be generated in the same time by a given force acting at any place as would be generated by the same force acting similarly in the body or system itself.

When the axis of motion passes through the centre of gravity, then is this centre called the *principal* centre of gyration.

310. Prop. *If the sum of the products formed by multiplying each particle of a system into the square of its distance*

from the axis of motion, that is, if the momentum of inertia be divided by the whole mass, the square root of the quotient will be the distance of the centre of gyration from the axis of motion.

For, if CR be the distance from the axis of motion to the centre of gyration, the expression for the angular motion $\frac{\varphi\cdot\mathrm{AC}}{\int p\cdot\mathrm{RC}^2}$ or $\frac{\varphi\cdot\mathrm{AC}}{\int pr^2}$ (art. 301. cor. 2.) will be transformed to $\frac{\varphi\cdot\mathrm{AC}}{\mathrm{B}\cdot\mathrm{CR}^2}$; and these by the def. must be equal. Consequently, $\mathrm{B}\cdot\mathrm{CR}^2 = \int pr^2$, and $\mathrm{CR} = \sqrt{\frac{\int pr^2}{\mathrm{B}}}$.

COR. 1. *The distance of the centre of gyration from the axis of motion is a mean proportional between the distances of the centres of oscillation and gravity, from the same axis.*

For $\mathrm{CR}^2 = \frac{\int pr^2}{\mathrm{B}}$, while $\mathrm{CO} = \frac{\int pr^2}{\mathrm{B}\cdot\mathrm{CG}}$; therefore $\mathrm{CO}\cdot\mathrm{CG} = \frac{\int pr^2}{\mathrm{B}} = \mathrm{CR}^2$, and $\mathrm{CO} : \mathrm{CR} :: \mathrm{CR} : \mathrm{CG}$. (Fig. 2. pl. XIV.)

COR. 2. *The distance between the centre of gravity and principal centre of gyration is a mean proportional between the distances of the centres of motion and oscillation, from the centre of gravity.*

For $\mathrm{PC}^2 = \mathrm{PG}^2 + \mathrm{GC}^2 \pm 2\mathrm{PG}\cdot\mathrm{GC}$; therefore, by art. 307, we have $\mathrm{OC} = \int p \cdot \frac{\mathrm{PG}^2 + \mathrm{GC}^2 \pm 2\mathrm{PG}\cdot\mathrm{GC}}{\mathrm{B}\cdot\mathrm{CG}}$. But from the nature of the centre of gravity, $\int p\cdot\mathrm{PG}$ on each side of G are equal, and, because $2\mathrm{CG}$ is constant, $\int \pm 2\mathrm{PG}\cdot\mathrm{GC} = 0$. Also, because $\int p\cdot\mathrm{GC}^2 = \mathrm{B}\cdot\mathrm{GC}^2$, we have $\mathrm{OC} = \mathrm{GC} + \int \frac{p\cdot\mathrm{PG}^2}{\mathrm{B}\cdot\mathrm{GC}}$, and $\mathrm{OG} = \int \frac{p\cdot\mathrm{PG}^2}{\mathrm{B}\cdot\mathrm{GC}}$; whence this corollary is manifest.

COR. 3. *The time in which a body vibrates will be the least possible when the axis of motion passes through the principal centre of gyration.*

For if D and d represent the distances from the centre of gravity to the point of suspension, and to the centre of gyration, it will be $\mathrm{D} : \mathrm{D}+d :: \mathrm{D}+d : \frac{(\mathrm{D}+d)^2}{\mathrm{D}}$, the distance from the point of suspension to the centre of oscillation (cor. 1.), and this latter expression will obviously be a minimum when $\mathrm{D} = d$.

COR. 4. If we wish to know what quantity of matter M must be placed at any other distance AC from C, so that the inertia may remain the same, we must have $\mathrm{AC}^2\cdot\mathrm{M} = \mathrm{CR}^2\cdot\mathrm{B}$, whence $\mathrm{M} = \frac{\mathrm{B}\cdot\mathrm{CR}^2}{\mathrm{AC}^2}$.

COR. 5. And, for a like reason, if any number of bodies B, B′, B″, be put in motion about a common axis passing through C, by a force acting at A, the system will have the same angular velocity, if, instead of those bodies placed at the distances CB, CB′ CB″, there be substituted bodies equal to $\frac{CB^2}{CA^2}B$, $\frac{CB'^2}{CA^2}B'$, $\frac{CB''^2}{CA^2}B''$, all concentrated into the point A.

SCHOLIUM.

311. Having now demonstrated the chief properties of the centres of oscillation and gyration, we may apply them to a few examples of finding the position of those points in bodies of given shape. And, first, let us speak of

The Centre of Oscillation.

I. Let CP (fig. 2, pl. XIV.) be a right line, or very slender cylinder, moving freely about its extreme point C as a centre of suspension.

Then, if CA, considered as variable, be denoted by x, the force of $\dot{x}$ particles at A will be defined by $x^2\dot{x}$: its fluent, therefore, that is, $\frac{1}{3}x^3$, denotes the momentum of inertia of all the particles in CA; and this, when CA becomes equal to CP, will be $\frac{1}{3}CP^3$. In this case, too, the body $B = CP$, and $CG = \frac{1}{2}CP$: consequently (art. 306.) $\frac{\int pr^2}{B \cdot CG} = \frac{\frac{1}{3}CP^3}{\frac{1}{2}CP \cdot CP} = \frac{\frac{1}{3}CP^3}{\frac{1}{2}CP^2} = \frac{2}{3}CP = CO$, the distance of the centre of oscillation from the point of suspension.

COR. In a line CP of uniform thickness, but the density of whose particles *increases* as their distance from C the point of suspension increases, we have $CO = \frac{\int x^3\dot{x}}{\int x^2\dot{x}} = \frac{\frac{1}{4}x^4}{\frac{1}{3}x^3} =$ (when $x = CP$) $\frac{3}{4}CP$.

II. Let the point of suspension of the slender cylinder CP be S, to find the centre of oscillation.

Put $CP = v$, $CS = y$, $PS = v - y$, and $SO = x$, then $(v-y)^2 \times \dot{v}$ will express the fluxion of all the pr^2 in CP, while $(v-y) \cdot \dot{v}$ represents the fluxion of the product of $B \cdot CG$. Hence $x = \int \frac{(v-y)^2\dot{v}}{(v-y)\dot{v}} = \frac{2v^2 - 6vy + 6y^2}{3v - 6y}$, where y and v may be taken in any ratio to each other. If $y = 0$, or C and S coincide, then $x = \frac{2}{3}v = \frac{2}{3}CP$, as in the preceding example. If $y = \frac{1}{3}v$, then also $x = \frac{2}{3}v = \frac{2}{3}CP$; in which case O and P will coincide. So that, *whether a very slender cylinder be suspended at one end, or at* $\frac{1}{3}$ *of its length, its vibrations will be performed in the* *same time.* (See art. 306. cor. 5.) When $y=\frac{1}{2}v$, then $x=\pm\infty$, or the centre of oscillation is at an infinite distance : *the tendency to oscillate will then cease, and if any extraneous force produce motion it will be completely rotatory.*

COR. When the vibrations are to be the quickest possible $x=\frac{2v^2-6vy+6y^2}{3v-6y}$, must be a minimum; or its fluxion making y variable $=0$. Hence $(12y\dot{y}-6v\dot{y})\cdot(3v-6\dot{y})-6\dot{y}(2v^2-6vy+6y^2)=0$, and $y=\frac{1}{2}v\pm\frac{1}{2}v\sqrt{\frac{1}{3}}$. When s is taken between A and G the negative sign manifestly obtains, and $y=\frac{1}{2}v\ (1-\sqrt{\frac{1}{3}})=\cdot211325\,v$.

Hence it appears that the time of vibration of a slender cylinder (however short) with a moveable point of suspension, may be made to increase from the time assigned by the minimum limit, up to infinity: and the like would obviously obtain with respect to a wire with a sphere at each end; though the absolute time of vibration would not be the same with as without the balls. (See the general results at the end of this article.)

III. Let ACB (fig. 5. pl. XIV.) be an angular pendulum, composed of two thin cylindric wires AC, CB, vibrating upon the point of suspension C, in the plane of the figure.

Bisect AC, and CB, in g, γ, and bisect $g\gamma$ in G, so shall g, γ, G, be the centres of gravity of AC, CB, and of AC+CB. In this instance we may have recourse to art. 308. Denote AC or CB by a, and the natural secant of angle ACG $=\frac{1}{2}$ACB, by s; then O and O′ being the centres of oscillation of the separate masses AC, CB, we have (ex. I.) CO $=$ CO′ $=\frac{2}{3}a$: and by trigonometry, $s:1::\frac{1}{2}\text{AC}:\text{CG}=\frac{a}{2s}$. Hence we have $\frac{\text{AC}\cdot\frac{1}{2}\text{AC}\cdot\frac{2}{3}\text{AC}+\text{CB}\cdot\frac{1}{2}\text{CB}\cdot\frac{2}{3}\text{CB}}{(\text{AC}+\text{CB})\frac{a}{2s}}$ $=\frac{2(a\cdot\frac{1}{2}a\cdot\frac{2}{3}a)}{2a\frac{a}{2s}}=\frac{2}{3}as=\text{CO}$.

COR. 1. Since CO varies with the secant of $\frac{1}{2}$ACB, it follows that the time of vibration of the angular pendulum may be increased *sine limite*, while its length remains constant; the time of vibration depending upon the magnitude of the angle ACO.

COR. 2. When the angle ACO vanishes, or AC and CB coincide throughout, the time of vibration will be equal to that of a simple pendulum whose length is $\frac{2}{3}$AC : when ACO is a right-angle, the time of vibration is infinite, or the pendulum will rest upon its centre of gravity in all positions.

COR. 3. Hence if AC $=$ CB be given, and the time of vibration, the angle ACB may readily be found : for the above expression for CO gives $s=\frac{3\text{CO}}{2a}$.

Thus, if AC$=15$ inches, and the time were 1 second, CO would be $39\frac{1}{8}$ inches, and $s = \frac{117\frac{3}{8}}{30} =$ sec of 75° $11\frac{1}{2}'$ nearly: hence ACB$=150°\ 23'$.

If, while AC$=15$ inches, the time of vibration should be 5 seconds, then CO$=25 \times 39\frac{1}{8}$, and $s = \frac{75 \times 39\frac{1}{8}}{30} = 97{\cdot}8125 =$ sec of 89° $24\frac{3}{4}'$; consequently ACB$=178°\ 49\frac{1}{2}'$ nearly.

IV. Let the solid formed by the rotation of the curve VmM about its absciss VA vibrate about an axis passing through C; to find the centre of oscillation.

Put CV$=d$, the variable absciss V$a=x$, its corresponding ordinate $am=y$, VA$=a$, AM$=r$, $3{\cdot}1416=\pi$; then, as may easily be shewn (though that part of the investigation is omitted, to save room,) the sum of the products of each particle in the circle mn into the square of its distance from the axis$=$ $(\text{C}a^2 + \frac{1}{4}am^2)\pi y^2 = \pi\left((d+x)^2 y^2 + {}^1 y^4\right)$. Hence $\pi\dot{x} \times \left((d+x)^2 y^2 + \frac{1}{4}y^4\right)$ is the fluxion of the sum of all such products for the whole body. Consequently we shall have

$$\text{CO} = \frac{\int \pi\dot{x} \times \left((d+x)^2 y^2 + \frac{1}{4}y^4\right).}{\text{B}\cdot\text{CG}}.$$

V. Let the general theorem in the preceding example be applied to the case of the cone whose vertex is V, and radius of its base AM.

In this case VmM will become a right line, so that $a : r :: x : y = \frac{r}{a}x = nx$, putting $n = \frac{r}{a}$. Hence $\int \pi\dot{x} \times (\overline{d + x}^2\ y^2 + \frac{1}{4}y^4) = \int \pi\dot{x} \times (d+x)^2\ n^2x^2 + \frac{1}{4}n^4x^4) = \frac{1}{3}\pi d^2n^2x^2 + \frac{1}{2}\pi d\ n^2x^4 + \frac{1}{5}\pi n^2x^5 + \frac{1}{20}\pi n^4x^5$. But, by the rules of mensuration, the body B$= \frac{1}{3}\pi n^2\ x^3$; and (art. 124.) VG$= \frac{3}{4}x$, whence CG$= d + \frac{3}{4}x$. Therefore B$\cdot$CG$= \frac{1}{3}\pi dn^2x^3 + \frac{1}{4}\pi n^2\ x^4$, by which dividing the preceding fluent, there results CO$= \frac{20d^2 + 30d\dot{x} + 12x^2 + 3n^2x^2}{20a + 15x}$; which for the whole cone when $x=a$, and $nx=y=r$, is transformed to this:

$$\text{CO} = \frac{20d^2 + 30ad + 12a^2 + 3r^2}{20d + 15a}.$$

Cor. 1. When C coincides with V, or the cone is suspended by its vertex, d vanishes, and the expression becomes

$$\text{CO} = \text{VO} = \frac{4a^2 + r^2}{5a} = \frac{4}{5}a + \frac{r^2}{5a}.$$

COR. 2. If, when d vanishes, r be $= a$, then will CO $= a$. Consequently, *if a right-angled cone be suspended at its vertex, its centre of oscillation will coincide with the centre of its base, and the cone will vibrate in the same time as a simple pendulum whose length is equal to the altitude of the cone.*

REMARK. The conclusion in the last corollary flows also very naturally from the geometrical method of finding the centre of oscillation of a cone suspended at its vertex; which is this: *Bisect* AN, *the radius of the base* (fig. 7. pl. XIV.), *in* F; *join* VF, *and make* IF $= \frac{1}{5}$ VF; *then* IO *drawn perpendicular to* VF *will meet* VA *or* VA *produced, in* O, *the centre of oscillation.* For VF $= \sqrt{VA^2 + AF^2} = \sqrt{a^2 + \frac{1}{4}r^2}$, and VI $= \frac{4}{5}\sqrt{a^2 + \frac{1}{4}r^2}$. But the triangles VAF, VIO, are similar, therefore VA : VF :: VI : VO, that is, $a : \sqrt{a^2 + \frac{1}{4}r^2} :: \frac{4}{5}\sqrt{a^2 + \frac{1}{4}r^2} : \frac{\frac{4}{5}a^2 + \frac{1}{5}r^2}{a} = \frac{4}{5}a + \frac{r^2}{5a} =$ VO, which is the same expression as in cor. 1.

When O coincides with A, we have (fig. 8.) VI : IO :: IO : IF :: VA : AF. So that IO $= \sqrt{VI \cdot IF} = \sqrt{4IF \cdot IF} = 2IF$; therefore, VA $= 2$AF, and consequently VA $=$ AN, by construction: which agrees with cor. 2.

COR. 3. *When* VA *is less than* AN, O *falls below* A; *and when* VA *is greater than* AN, O *falls above* A.

COR. 4. From the above it likewise appears, that if cones be made to vibrate as pendulums, about their vertices as points of suspension, the time of vibration may be increased *sine limite*, while VN, the slant side of the cone, remains constant. In practice, however, this cannot be carried so far in conical pendulums as the similar thing in angular pendulums, noticed in cor. 1. exam. III.

COR. 5. Whether a right-angled cone be suspended at its base or its vertex, it will vibrate in equal times (art. 306. cor. 5).

These examples must suffice as general specimens of the method of finding the centre of oscillation: we shall here add the results of other investigations in some of the most useful cases.

In surfaces when the vibration is flatwise, or perpendicular to the plane of the figure, the distance CO of the centre of oscillation from the point of suspension will be as below:

In an isosceles triangle suspended at its vertex........	$\frac{3}{4}$ the altitude.
In a circle suspended at its circumference,................	$\frac{5}{4}$ the radius.
In the common parabola suspended at its vertex,....	$\frac{5}{7}$ the axis.
Any parabola, ditto..........................	$\frac{2m+1}{3m+1} \times$ axis.

When the surfaces are moved edgewise, or in the plane of the figure; then

In a circle		$\frac{3}{4}$ of the diameter.
In a rectangle suspended by one angle,		$\frac{2}{3}$ of the diagonal.
In a parabola	its vertex,	$\frac{5}{7}$ axis + $\frac{1}{3}$ param.
Ditto	middle of base,	$\frac{4}{7}$ axis + $\frac{1}{2}$ param

In a parabola whose distance CO shall equal its axis, we have base $= \frac{2}{7}$ axis $\sqrt{42} =$ axis $\times 1{\cdot}85164$. Such a parabola will manifestly vibrate in equal times whether suspended at its vertex, or at the middle of its base.

In an isosceles triangle $\frac{3a^2+b^2}{4a}$, where $a =$ alt. and $b = \frac{1}{2}$ base; and if $a = b$ or the $\triangle$ right-angled at the vertex, CO $= a$, as in the right-angled cone.

In a square pyramid whose altitude is a, and side of its base $2b$, CO $= \frac{4}{5}a + \frac{4b^2}{15a}$, the vibrations being performed in vertical planes parallel to that passing through a side of the base.

If a sphere whose radius is r, be suspended at the distance d from its centre, then is δ the distance from the point of suspension to the centre of oscillation $= d + \frac{2r^2}{5d}$. If the point of suspension be at the extremity of a radius, then $\delta = \frac{7}{5}r$. When δ and r are given, we have $d = \frac{1}{2}\delta \pm \sqrt{\frac{1}{4}\delta^2 - \frac{2}{5}r^2}$; whence we learn that a sphere may be suspended at two different distances from the axis of motion, and yet *vibrate in the same time*. When $\frac{1}{4}\delta^2 = \frac{2}{5}r^2$ the radical expression vanishes, and $d = \frac{1}{2}\delta = r\sqrt{\frac{2}{5}} = {\cdot}632495\, r$ is the only value of d; the point of suspension being then the centre of gyration, and δ, which varies with the time of vibration, is a *minimum*.

If the weight of the thread is to be taken into the account, we have the following distance between the centre of the ball and that of oscillation, where B is the weight of the ball, d the distance between the point of suspension and its centre, r the radius of the ball, and w the weight of the thread or wire, GO $= \frac{(\frac{1}{3}w + \frac{2}{5}B)4r^2 - \frac{1}{6}w(2dr + d^2)}{(\frac{1}{2}w + B)d - rw}$; or, if B be expressed in terms of w considered as a unit, then GO $= \frac{\frac{1}{6}d}{B + \frac{1}{2}}$.

If two weights W, W′, be fixed at the extremities of a rod of given length WW′, C being the centre of motion between W and W′; then if $d =$ CW, D $=$ CW′, and m the weight of an unit in length of the rod, we shall have CO $= \frac{mD^3 + 3W'D^2 + md^3 + 3Wd^2}{mD^2 + 2W'D - md^2 - 2Wd}$; the radii of the balls being supposed very small in comparison with the length of the rod.

In the bob of a clock pendulum, supposing it two equal spheric segments joined at their bases, if the radii of those bases be each $=\varrho$, the height of each segment v, and d the distance from the point of suspension to G the centre of the bob, then is GO$=\frac{1}{3}d\cdot\frac{\varrho^4+\frac{1}{2}\varrho^2v^2+\frac{1}{10}v^4}{\varrho^2+\frac{1}{3}v^2}$; which shews the distance of the centre of oscillation below the centre of the bob.

If r the radius of the sphere be known, the latter expression becomes GO$=\frac{\frac{2}{3}r^2v-\frac{1}{2}rv^2+\frac{1}{10}v^3}{d(r-\frac{1}{3}v)}$. In a solid of this form if $d=40$ inches, $r=6$, $v=\frac{1}{2}$, then will GO$=\frac{2703}{56000}$, or the centre of oscillation will fall about $\frac{1}{20}$ of an inch below the centre of gravity.

The Centre of Gyration.

312. Ex. I. Let a slender rod whose length is CP$=l$ (fig. 2.) revolve on the point C; to find its centre of gyration.

Let any variable distance CA$=x$; then will the general expression for (art. 310.) $\sqrt{\frac{\int p\rho^2}{\text{B}}}$, become $\sqrt{\frac{\int x^2\dot{x}}{l}}=\sqrt{\frac{x^3}{3l}}$; and this when $x=l$, gives CR$=\sqrt{\frac{1}{3}l^2}=l\sqrt{\frac{1}{3}}=\cdot57735l$.

The same conclusion may be deduced from cor. 1. art. 310. and ex. 1. art. 311. For CG$=\frac{1}{2}l$, and CO$={}^2l$: consequently CR$=\sqrt{(\frac{1}{2}l\cdot\frac{2}{3}l)}=l\sqrt{\frac{1}{3}}=\frac{1}{3}l\sqrt{3}$.

Ex. II. To determine the centre of gyration in a circle, or in a circular wheel of uniform thickness, so that its weight may be as at its area.

Let $\pi=3\cdot1416$, and r the radius of the circle; then is $\pi r^2=$ the area of the circle; its fluxion is $2\pi r\dot{r}$; and therefore $\int pr^2=\int 2\pi r^3\dot{r}$. Consequently $\frac{\int pr^2}{\text{B}}=\int\frac{2\pi r^3\dot{r}}{2\pi r\dot{r}}=r\sqrt{\frac{1}{2}}=\frac{1}{2}r\sqrt{2}=\cdot707107r=$CR; the distance from C the centre of motion, to R the centre of gyration. The same evidently holds for a cylinder turning on its axis.

Ex. III. Let ABDE, and *abde* (fig. 9. pl. XIV.), be two concentric circles whose respective radii are R, r;—if the plane or solid wheel *abed* be taken away, and the ring ABDE*abde*, revolve about an axis passing through C perpendicular to the plane of the figure, it is then proposed to determine the distance CR.

Here $\pi\text{R}^2-\pi r^2=$ the area of the annulus, the fluxion of which is $2\pi\text{R}\dot{\text{R}}$, R being supposed variable: consequently $2\pi\text{R}^3\dot{\text{R}}$ is the fluxion of all the pr^2. And by the general rule we have CR $= \sqrt{\frac{\int\cdot 2\pi R^3\dot{R}}{\pi R^2-\pi r^2}} = \sqrt{\frac{R^4}{2R^2-2r^2}}$.

Cor. 1. When r vanishes, or the ring becomes a circle, we have CR$=$R$\sqrt{\frac{1}{2}}$, as in the preceding example.

Cor. 2. The sectors cab, CAB, being to each other as the areas of their respective circles, if $\frac{1}{n}\pi$ be put for the corresponding part of the circumference, the same result will be obtained with respect to the sectors as to the whole circles; so that if the part ABba revolved about the centre C, its centre of oscillation R′ would be determined by means of the equation CR′$= \sqrt{\frac{R^4}{2R^2-2r^2}}$.

Ex. IV. To find the centre of gyration of a cone which revolves about its vertex.

Here we may again have recourse to cor. 1. art. 310. For we know that VG, the distance from the vertex to the centre of gravity, is $=\frac{3}{4}a$ (art. 124.); and VO$=\frac{4}{5}a+\frac{r^2}{5a}$ (ex. v. art. 311.). Hence VR $=\sqrt{(\text{VG}\cdot\text{VO})}=\sqrt{\frac{12a^2+3r^2}{20}}=\frac{1}{2}\sqrt{\left(\frac{12}{5}a^2+\frac{3}{5}r^2\right)}$.

Cor. When $a=r$, as in the right-angled cone, we shall have VR$=\frac{1}{2}\sqrt{(\frac{12}{5}a^2+\frac{3}{5}a^2)}=\frac{1}{2}a\sqrt{3}=\cdot86602a$.

The results in a few other useful cases are as follow:

In the periphery of a circle revolving about the diam.. CR$=$rad $\sqrt{\frac{1}{2}}$.
In the plane of a circle.......................ditto.. CR$=\frac{1}{2}$ rad.
In the surface of a sphere...................ditto.. CR$=$rad $\sqrt{\frac{2}{3}}$.
In a solid sphere...............................ditto.. CR$=$rad $\sqrt{\frac{2}{5}}=\frac{7}{11}$ r nearly.
In a cone.......................................the axis.. CR$=r\sqrt{\frac{3}{10}}=\cdot54772\,r$.
In a paraboloid...............................ditto.. CR$=r\sqrt{}=\frac{1}{3}\cdot57735\,r$.

In a straight lever whose arms are B and b, the distance CR$=\sqrt{\frac{B^3+b^3}{3B+3b}}$.

If the matter of any gyrating body were actually to be placed as if in the centre of gyration, it ought either to be disposed in the circumference of a circle whose radius is CR, or into two points R, R′, diametrically opposite, and at distances from the centre each $=$CR.

For a practical method of finding the centre of gyration, see art. 314. cor. 3.

313. Prop. *When a pendulum is at rest, if a body impinge on it in a horizontal direction, the same velocity will be communicated to the point of impact, as if the mass of the pendulum were removed, and instead of it an equivalent mass were concentrated in the point of impact, the quantity of the equivalent mass being to that of the pendulum in a duplicate ratio of the*

distances of the centre of gyration and the point of impact from the axis of motion.

Let CADE (fig. 10. pl. XIV.) represent a pendulum whose axis of motion passes through C, and let any impact be impressed upon the point P, in an horizontal direction, and perpendicular to the vertical plane ABCD, the pendulum being at rest. Let Q be the quantity of matter which being concentrated in P, the same angular velocity will be communicated by the impact, as when B the whole body is concentrated into its centre of gyration R. Then, since the resistances arising from the inertia of the two masses at their respective distances must be equal, in order that the angular velocities may be equal; we must, as in cor. 4. art. 310. make $B \cdot CR^2 = Q \cdot CP^2$; whence $Q = B \frac{CR^2}{CP^2}$.

COR. Hence V, the velocity of the impinging body I, may be determined. For, let v be the velocity communicated to the point of impact: then, by the laws of the collision of non-elastic bodies (chap. v. art. 343.), $I : I+Q :: v : V = v \frac{I+Q}{I}$. To determine v, measure the arc described by O, the centre of oscillation in its ascent after impact; and call its versed sine s, then (arts. 243. 263.) the velocity of the centre of oscillation at its lowest point $= \sqrt{2gs}$: whence, the velocity of the point of impact $= \frac{CP}{CO} \sqrt{2gs}$.

SCHOLIUM.

On the principles developed in this proposition and corollary, the celebrated mathematician Mr. Benjamin Robins founded the use of his ingenious contrivance, the *Ballistic Pendulum*, for ascertaining the velocities of military projectiles. (New Principles of Gunnery, prop. 8.) This machine consists of a large block of wood annexed to the end of a strong iron stem having a cross steel axis at the other end, placed horizontally, about which the whole vibrates together. When the machine is at rest, a ball discharged horizontally from a piece of ordnance strikes and enters the block, and causes the pendulum to vibrate more or less, according to the velocity of the projectile; and by the extent of the vibration the velocity of the ball at the time of the impact becomes known. A more minute and particular description of this instrument may be seen in Dr. Hutton's Tracts. See, also, our 2d volume, art. BALLISTIC *pendulum*.

314. PROP. *If motion be communicated by a hanging weight to a system revolving on a fixed axis passing through the centre of gravity, and the moving force act always at a given di-*

stance from the axis of motion, it will generate in the revolving system the same motion as the weight would acquire in the same time by falling freely by its gravity from a state of quiescence.

Let DGB (fig. 11. pl. XIV.) represent a body or system moveable upon a horizontal axis passing through its centre of gravity G; let R be the centre of gyration, P the weight of the body which gives motion to the system whose weight is W, by means of a cord Pdc wound round the circumference dCE. If GR be denoted by r, and Gd by d, the inertia of the whole system will be equivalent to the weight $W\frac{r^2}{d^2}$ uniformly diffused through the periphery dCE, every point of which manifestly moves with the same velocity as P. The moving body P being supposed either destitute of inertia, or incomparably less than the weight of the system, the accelerating force φ will be $\frac{Pd^2}{Wr^2}$; consequently in the time t, the velocity generated will be $= \varphi gt = \frac{gtPd^2}{Wr^2}$; which multiplied into the quantity of matter $W\frac{r^2}{d^2}$, gives gtP for the whole quantity of motion generated. But if P were to descend by the force of gravity, it would acquire, in the time t, the velocity gt (art. 243.); which multiplied into the quantity of matter P, produces gtP, the same quantity of motion as before.

Cor. 1. Hence it follows, that the permanence of motion, estimated by the product of the quantity of matter and velocity, obtains in bodies which revolve on fixed axes.

Cor. 2. If P be supposed to possess inertia, the quantity Pd^2 must be added to the denominator of the fraction expressing the accelerating force, which will then become $\frac{Pd^2}{Pd^2 + Wr^2}$. This leads to a useful practical purpose, explained in the next corollary.

Cor. 3. Hence, *to find the centre of gyration in any system experimentally.* Suppose s to be the space described by the weight P descending from rest (as in the prop.) in t seconds: then the accelerating force being as in cor. 2. we shall have $s = \frac{1}{2}\varphi t^2 = \frac{gPt^2d^2}{2Pd^2 + 2Wr^2}$; whence we obtain $r = \sqrt{\frac{gPt^2d^2 - 2sPd^2}{2sW}}$; in which equation, if the numerical values of s, d, P, W, as resulting from the experiment, be used, r becomes known.

315. Prop. *Let the body* DB (fig. 11. pl. XIV.), *whose centre of gravity* G *is the centre of the circle* CED, *be attached to a fixed point* A *by means of a cord* ACED, *wound about* CED *as an*

axle; and let O *be the centre of oscillation when* C *is the point of suspension or momentary centre of motion of the body; then, if while the body descends by the force of gravity, it is made to turn round by the unwinding of the cord* ACEd, *the space actually descended by the body* DB, *is to the space described in the same time by a body falling freely, as* CG *to* CO.

Through the point of contact C, and the centre of gravity G, draw the horizontal line CGO, and suppose the motion of the body to commence when these centres are so situated; then (art. 305.) the angular velocity of the body at such commencement of its motion will be the same as if the whole were placed at O: and if a body were placed in O, its initial velocity would be the same as that of a body falling freely. Drawing, therefore, c*go* indefinitely near CGO, and the small arcs O*o*, G*g*, having the common centre C, we shall have, veloc. of O : veloc. of G :: O*o* : G*g* :: OC : GC; but the velocity of the descending body is that of its centre of gravity; therefore, velocity in free descent : velocity of body :: OC : GC. Now, since the points C, G, O, are always in a horizontal line, and the radius CG is given as well as the distance CO; the velocity of a body falling freely, and of the body unwinding from the thread, is always in the ratio of CO to CG; and consequently, when CO to CG is a constant ratio, the whole spaces described in a given time will be as stated in the proposition.

COR. 1. *The weight of the body* DB, *is to the tension of the cord* AC, *as* CO *to* GO.

For, if the body were supported at O as well as C, G being the centre of gravity, or estimate place of the body, we should have (art. 133. cor. 6.) weight of DB : pressure at C :: CO : GO. But, if the point O be set free, the force acting there will generate a motion about C whilst the pressure there, and consequently the tension of the cord, will be neither increased nor diminished.

COR. 2. *This motion of the body* DE *by unwinding the thread is uniformly accelerated, and the tension of the cord is invariable through the whole descent.*

COR. 3. *If any body whose transverse sections are circular run down an inclined plane, while the cord or ribband* CEd *unfolds; or if such round body roll down an inclined plane, being hindered by friction from sliding: the space it describes in any time, is to the space described by a body sliding down freely without friction, as* CG *to* CO.

For the forces which generate the motions are both decreased in the same ratio; that is, as the absolute gravity to the relative gravity upon the plane; therefore, the spaces described will, in either case, remain in the ratio of CG to CO.

SCHOLIUM.

The force by which spheres, cylinders, &c. are caused to revolve as they move down an inclined plane (instead of sliding) is the adhesion of their surfaces occasioned by the pressure against the plane: this pressure is part of the body's weight; for the weight being resolved into its components, one in the direction of the plane, the other perpendicular to it, the latter is the force of the pressure; and, while the same body rolls down the plane, will be expressed by the cosine of the plane's elevation. Hence, since the cosine decreases while the arc or angle increases, after the angle of elevation arrives at a certain magnitude, the adhesion may become less than what is necessary to make the circumference of the body revolve fast enough; in this case the body descends partly by sliding and partly by rolling[*]. And the same may happen in smaller elevations, if the body and plane are very smooth. But at all elevations the body may be made to roll by the uncoiling of a thread or rib-band wound about it.

316. If w denote the weight of a body, *s* the space described by a body falling freely, or sliding freely down an inclined plane, according as the supposition is that in the prop. or in cor. 3. then the spaces described by rotation in the same time by the following bodies, will be in these proportions.

1. A hollow cylinder, or cylindrical surface, s $= \frac{1}{2}s$; tension of the cord in the first case $= \frac{1}{2}$w.

2. In a solid cylinder, s $= \frac{2}{3}s$, tens. $= \frac{1}{3}$w.

3. In a spheric surface, or thin spherical shell, s $= \frac{3}{5}s$, tens. $= \frac{2}{5}$w.

4. In a solid sphere, s $= \frac{5}{7}s$, tens. $= \frac{2}{7}$w.

Something of the same kind as we have stated in the proposition obtains in common pendulous bodies. *A ball hung by a flexible thread not only oscillates, but also makes part of a rotation;* whence its oscillations differ from those of a heavy point hanging by the same thread. The curious reader who wishes to see how oscillations of this nature are to be determined, supposing the arcs described to be very small, is referred to *Bezout's Mechanics*, art. 732. The investigation, being long, is omitted here, in order to make room for more useful matter.

317. Def. The *centre of percussion* is that point in a body

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[*] If a cylinder *roll* down a plane whose elevation is e, and the friction be supposed to vary as the pressures against planes, then on another plane where the spaces passed over by the sliding and rolling motions in the same time shall be as $n : 1$, the elevation E will be known by means of the formula $\sin E = \frac{(2n+3)\sin e}{\sqrt{[9\cos^2 e + (2n+3)^2 \sin^2 e]}}$. Leybourn's Repository, vol. ii. part 2. p. 6.

revolving about an axis, at which, if it struck an immoveable obstacle, all its motion would be destroyed, or it would not incline either way: so that if at the instant of the impact the supports of the axis were annihilated, the body would remain in absolute rest.

During the vibration of a system of bodies round a fixed axis passing through c (fig. 12. pl. XIV.) if such an obstacle be opposed to any point o, as entirely to destroy the motion of that point, every other particle of the system will endeavour, by its inertia, to proceed in the direction of its motion, that is, of the tangent of the circular arc it was describing the instant o was stopped. These forces, therefore, will act on the system to turn it round o; and if the sum of the forces on each side of o should be unequal, the motion of the system will not be destroyed when o is stopped; but since the forces which act upon the pendulous body between o and c have an effect to continue the motion of the system, contrary to those which are impressed on the other side of o, if this point o be so situated that the sum of the forces to turn the system round o, on each side of that point, may be exactly equal, then the instant in which o is stopped, the whole motion of the system will be destroyed.

COR. 1. When a pendulum, vibrating with a given angular velocity, strikes an obstacle, the effect of the impact will be the greatest if it be made at the centre of percussion.

For, in this case the obstacle receives the whole revolving motion of the pendulum; whereas, if the blow be struck in any other point, a part of the motion of the pendulum will be employed in endeavouring to continue the rotation.

COR. 2. If a body revolving on an axis strike an immoveable obstacle at the centre of percussion, the point of suspension will not be affected by the stroke.

We can ascertain this property of the point o when we give a smart blow with a stick. If we give it motion round the joint of the wrist only, and, holding it at one extremity, strike smartly with a point considerably nearer or more remote than $\frac{2}{3}$ of its length, we feel a painful wrench in the hand: but if we strike with that point which is precisely at $\frac{2}{3}$ of the length, no such disagreeable strain will be felt. If we strike the blow with one end of the stick, we must make its centre of motion at $\frac{1}{3}$ of its length from the other end; and then the wrench will be avoided.

318. PROP. *The distance of the centre of percussion from the axis of motion is equal to the distance of the centre of oscillation from the same: supposing that the centre of percussion*

is required in a plane passing through the axis of motion and centre of gravity.

Let CBE (fig. 12.) be a plane passing through the centre of gravity G of the body, and perpendicular to the axis of suspension which passes through C; and conceive the whole body to be projected upon this plane in lines perpendicular to it, or parallel to the axis of motion; for then, as each particle will fall at the same distance from the axis as in the body itself, the effect from the rotatory motion will not be changed, neither will the place of the centre of gravity. Through C and G draw the line CGON, and let P be the place of one of the particles p composing the system. Now, since the angular motion of all the particles is the same, the absolute velocity will be proportional to the distance from the axis of motion; and if at the distance 1 the velocity be expressed by unity, the velocity of p will manifestly be denoted by PS, and its quantity of motion will be $p\cdot\mathrm{PC}$; which will act in the direction PR perpendicular to PC: produce PR till it meets OD, drawn parallel to PC, in D; then the force $p\cdot\mathrm{PC}$ acting in the direction PR will act upon O as though it had the advantage of the lever OD, and consequently, $p\cdot\mathrm{PC}\cdot\mathrm{OD}$ will represent the force of the particle p to move the system round O. But by reason of the similar triangles CPR, ODR, we have $\mathrm{RC}:\mathrm{CP}::\mathrm{RO}:\mathrm{OD}=\frac{\mathrm{CP}\cdot\mathrm{RO}}{\mathrm{CR}}=\mathrm{CP}\cdot\frac{\mathrm{CO}-\mathrm{CR}}{\mathrm{CR}}$; and, if PA be perpendicular to CO, we shall have $\mathrm{RC}:\mathrm{CP}::\mathrm{CP}:\mathrm{CA}=\frac{\mathrm{CP}^2}{\mathrm{CR}}$. Hence the entire force $p\cdot\mathrm{PC}\cdot\mathrm{OD}$ becomes $=p\cdot\mathrm{PC}^2\cdot\frac{\mathrm{CO}-\mathrm{CR}}{\mathrm{CR}}=p\cdot\mathrm{CA}\cdot\mathrm{CO}-p\cdot\mathrm{PC}^2$. But, when O is the centre of percussion the sum of all the $p\cdot\mathrm{CA}\cdot\mathrm{CO}-p\cdot\mathrm{PC}^2$, must be equal to zero, or $\int p\cdot\mathrm{CA}\cdot\mathrm{CO}=\int p\ \mathrm{PC}^2=\int pr^2$. Whence it follows that $\mathrm{CO}=\frac{\int pr^2}{\int p\cdot\mathrm{CA}}=\frac{\int pr^2}{\mathrm{B}\cdot\mathrm{CG}}$; the denominators of the two fractions being equal, by arts. 108. 109. and this value of CO obviously corresponds with that given for the centre of oscillation in art. 306.

Cor. 1. If the body be symmetrical with regard to the plane BGE, or if it be a solid of rotation, the centre of percussion found in the axis of the body will coincide with the centre of oscillation.

Cor. 2. If the centre of percussion be required in a plane which does not pass through G, as Co for instance, we must proceed thus: from G, the centre of gravity, let fall on Co the perpendicular Gg; and, by the same argument as above, Co =

$\frac{\int pr^2}{\int p\cdot \mathrm{C}a} = \frac{\int pr^2}{\mathrm{B}\cdot \mathrm{c}g}$. Now, C$o$: CO :: CG : Cg; hence the angles oOC and GgC are equal, and, consequently, the former is a right angle.

COR. 3. Hence it follows, that *a body has several centres of percussion* according to the plane passing through the axis of motion in which the impact is made, *and the right line* Oo, *at right angles to* CO, *is their locus.*

319. PROP. *If* CB (fig. 13. pl. XIV.) *be the axis of a body's motion,* CAO *a plane perpendicular to* CB *and passing through the centre of gravity* G, pA *a perpendicular let fall from any particle* p *of the body on the plane* CA, *and* P *the centre of percussion, then will* PO = CB = *the sum of all the* $\frac{p\cdot \mathrm{A}p\cdot \mathrm{AG}}{\mathrm{Body}\cdot \mathrm{CG}}$.

For the sum of all the forces with which the body is liable to be turned in one direction round PB as an axis = sum of all the $p\cdot \mathrm{CA}\cdot(\mathrm{CB} - \mathrm{A}p) = \int p\cdot \mathrm{CA}\cdot(\mathrm{CB} - \mathrm{A}p)$, and the sum of all the forces which tend to turn it in the contrary direction = sum of all the $p\cdot \mathrm{CA}\,(\mathrm{A}p - \mathrm{CB}) = \int p\cdot \mathrm{CA}\,(\mathrm{A}p - \mathrm{CB})$. Therefore, in the case of no motion either way, we have $\int p\cdot \mathrm{AC}\cdot \mathrm{CB} = \int p\cdot \mathrm{A}p\cdot \mathrm{AC}$. But, $\int p\cdot \mathrm{CB}\cdot \mathrm{AC} = \int p\cdot \mathrm{CB}\cdot \mathrm{CG} + \int p\cdot \mathrm{CB}\cdot \mathrm{GA}$; and $\int p\cdot \mathrm{CB}\cdot \mathrm{GA} = 0$, from the nature of the centre of gravity. In like manner $\int p\cdot p\mathrm{A}\cdot \mathrm{AC} = \int p\cdot p\mathrm{A}\cdot \mathrm{AG} + \int p\cdot p\mathrm{A}\cdot \mathrm{CG}$, and $\int p\cdot p\mathrm{A}\cdot \mathrm{CG} = 0$: consequently, $\int p\cdot \mathrm{CB}\cdot \mathrm{CG} = \int p\cdot p\mathrm{A}\cdot \mathrm{AG}$; and $\mathrm{CB} = \mathrm{PO} = \int \frac{p\cdot p\mathrm{A}\cdot \mathrm{AG}}{p\mathrm{CG}} = \frac{\int p\cdot p\mathrm{A}\cdot \mathrm{AG}}{\mathrm{B}\cdot \mathrm{CG}}$.

COR. Hence the centre of percussion of a body turning round the axis CB, is determined by these conditions. 1st. It is in a line PO passing through the centre of oscillation and parallel to CB. 2dly. Its distance OP from the centre of oscillation is $\frac{\int p\cdot p\mathrm{A}\cdot \mathrm{AG}}{\mathrm{B}\cdot \mathrm{CG}}$.

SCHOLIUM.

320. As several writers on mechanics have asserted that the centres of percussion and oscillation reside universally in the same point, we shall, in addition to the preceding investigations, give a simple instance here, in which the contrary will be obvious, if we adhere to the first definition in art. 305. To this end let two equal balls B, B′ (fig. 14. pl. XIV.) (of evanescent magnitude with respect to their distance from each other), be connected with an axis of motion CC′, by inflexible lines CB (= 3) and C′B′ (= 7), void of gravity, perpendicular to CC′ (= 6), and parallel to each other; to find the respective di-

stances of the centres of gravity, oscillation, and percussion, of this system from each other. Join BB' and bisect it in G, the centre of gravity, so shall $Gg = 5$. Then (art. 306.) $\frac{\text{B}\cdot\text{BC}^2+\text{B}'\cdot\text{B}'\text{C}'^2}{(\text{B}+\text{B}')\,\text{G}g} = \frac{9+49}{2\times 5} = 5{\cdot}8 = g\text{O}$, the distance of the centre of oscillation from the axis; whence $\text{GO} = {\cdot}8$. Through O draw bb' parallel to CC', then (art. 318.) will the centre of percussion P be somewhere in this line; and, by the last article we find $\text{PO} = \frac{\text{B}\cdot\text{BA}\cdot\text{AG}+\text{B}'\cdot\text{B}'\text{A}'\cdot\text{A}'\text{G}}{(\text{B}+\text{B})g\text{G}} = \frac{6+6}{10} = 1{\cdot}2$. Here the products 6 and 6 are added together, because, while B and B' are on opposite sides of Gg, they are also on contrary sides of a line passing through G parallel to CC'; so that, if BA be +, $\text{B}'\text{A}$ will be −, and AG being +, $\text{A}'\text{G}$ will be −, and consequently $\text{BA}\cdot\text{AG}$, and $\text{B}'\text{A}'\cdot\text{A}'\text{G}$, are both affirmative products. — ... Or, the centre of percussion P must be so situated that $\text{B}\cdot\text{BC}\cdot b\text{P}$ must be $= \text{B}'\cdot\text{B}'\text{C}'\cdot b'\text{P}$; whence, since $\text{B} = \text{B}'$, we have $\text{B}'\text{C}' : \text{BC} :: b\text{P} : \text{P}b' = \frac{3}{7}\,b\text{P} = \frac{3}{10}\,bb' = 1{\cdot}8$. Consequently, $\text{OP} = 3 - 1{\cdot}8 = 1{\cdot}2$; and $\text{GP} = \sqrt{({\cdot}8^2 + 1{\cdot}2^2)} = 1{\cdot}4422$; the triangle GOP being right-angled at O. See also arts. 392, 393.

321. **Prop.** *In the wheel and axle, when a weight* P *acting at the distance* R, *elevates a weight* W *acting at the distance* r, *from the mathematical axis, the pressure sustained by that axis will be* $= \text{PW}\cdot\frac{(\text{R}+r)^2}{\text{PR}^2+\text{W}r^2}$; *the weight of the wheel and axle and the friction of the cord, &c. not being considered*[*].

Suppose P and W to be at their respective extremities of the horizontal line passing through the centre of motion, and in that situation let o and g be the respective distances of their centres of oscillation and of gravity from the centre of motion. Then, since the whole system revolves with the same angular velocity as if P + W were placed at the distance o from the centre of motion (art. 305.); and, since the accelerative forces are as the velocities generated in a given time (art. 225.) or as the distances from the centre of motion; we have $o : g :: \text{P} + \text{W} : \frac{g}{o}(\text{P}+\text{W})$, the force with which the centre of gravity descends. But $o = \frac{\text{PR}^2+\text{W}r^2}{\text{PR}-\text{W}r}$ (art. 306.) and $g = \frac{\text{PR}-\text{W}r}{\text{P}+\text{W}}$ by the nature of the centre of gravity. Therefore $\frac{g}{o} = \frac{(\text{PR}-\text{W}r)^2}{\text{PR}^2+\text{W}r^2}\cdot\frac{1}{\text{P}+\text{W}}$, and $\frac{g}{o}(\text{P}+\text{W}) = \frac{(\text{PR}-\text{W}r)^2}{\text{PR}^2+\text{W}r^2}$. Now the force with which the centre of gravity tends to descend is manifestly that by which the pressure upon the axis $P + W$ arising from the weight is diminished by the motion: consequently the pressure upon the axis is $= P + W - \frac{(PR - Wr)^2}{PR^2 + Wr^2} = PW \cdot \frac{(R+r)^2}{PR^2 + Wr^2}$.

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[*] For an estimate of the pressure including the friction and the weight of the machine, see Phil. Trans. vol. xlix. or New Abridgment, vol. x. p. 559.

COR. 1. If $R = r$, as in the case of the single fixed pulley, then the pressure $= \frac{4PW}{P+W}$.

COR. 2. The same reasoning will evidently apply to the pressure upon the fulcrum of a straight lever.

322. DEF. The centre of *spontaneous rotation*, or spontaneous gyration, is that point which remains at rest the instant any body is struck, or the point about which the body begins to revolve.

When a body B (fig. 15. pl. XIV.) of any shape whatever receives an impulse, the direction of which does not pass through the centre of gravity, and takes, in consequence, the two motions of which we have before spoken (art. 300.), it is evident that, for an instant of time, we may consider it as having only one motion, namely, a motion of rotation about a point or fixed axis C, which may be either within the body or out of it, according to the shape of the body, and the distance GS between the centre of gravity and the direction of the impact. If while the line GS is transported parallel to itself from GS to G′S′, we imagine that it turns about the moveable point G, as the particles of the body have greater or less velocities as they are more or less distant from G, it is manifest that there is upon SG a certain point C, which will be found to describe from C′ towards C an arc equal to GG′, which during an evanescent instant may be regarded as a right line; in that case the point C will have been carried as far backward by its motion of rotation as it will have been advanced parallel to GG′ by the velocity common to all the parts of the body; the point C has, therefore, during this instant, been actually at rest in C′; and may, consequently, be considered as a fixed point about which the body during such instant has a rotatory motion.

323. PROP. *The centre of spontaneous rotation is the same with the centre of suspension corresponding to the centre of percussion, the centre of percussion being that point where the body is struck.*

The arcs CC′, SI, which the points C and S describe during an instant, being regarded as right lines perpendicular to GS or parallel to GG′, the similar triangles CGG′, G′S′I, give GS : GC :: SI : GG′: now the velocity GG′ will be expressed by $\frac{\phi}{B}$, and the velocity S′I by $\frac{\phi \cdot GS^2}{\int pr^2}$ (art. 301.); therefore GS : GC :: $\frac{\phi \cdot GS^2}{\int pr^2}$: $\frac{\phi}{B}$; and consequently GC $= \frac{\int pr^2}{B \cdot GS}$, the r, r', &c. being reckoned from G. Hence CS $=$ GS $+ \frac{\int pr^2}{B \cdot GS} = \frac{B \cdot GS^2 + \int pr^2}{B \cdot GS}$, which value of CS corresponds with the value of CO in art. 318; because when we have the momentum of inertia with regard to an axis passing through the centre of gravity G, we obtain that momentum with regard to any other axis SI parallel to the former, by adding to it the product of the mass into the square of the distance GS between the two axes.

The same conclusion may be otherwise deduced, thus:—The action of the body against an immoveable obstacle in the centre of percussion must have the same effect upon the body as if the body had been at rest and the obstacle had struck it; in which latter case the centre of suspension would not be affected, and therefore it becomes the centre of spontaneous rotation.

Cor. 1. Since ϕ the force does not enter the above value of CS, we see that *the position of the centre of spontaneous rotation is independent of the magnitude of the impact.*

Cor. 2. *The distance* CG *is always greater as the force* ϕ, *or the result of all the forces, acts nearer the centre of gravity; and vice versa.* So that when ϕs coincides with GG, GC will be infinite; or there will be no motion of rotation.

Cor. 3. *If an impact be made on any point of the axis of a symmetrical body, or a solid of rotation, and that point be considered as the point of suspension, the corresponding centre of oscillation will be the centre of spontaneous rotation.*

This will be evident on comparing this proposition with art. 306. cor. 5. and 318. cor. 1.

Cor. 4. *To whatever point of a right line drawn through the centre of gravity of a body the impact be applied, the velocity of the centre of gravity will be the same.*

For the expression $\frac{\phi}{B}$ denoting the velocity, contains no distance. The same thing will readily appear otherwise, thus: To whatever point of QP (fig. 15.) the force ϕ is applied, the incipient motion $Q \cdot Qq + P \cdot Pp$ will be the same by the second law of motion, the particles Q, P, being connected by the inflexible line QP; and, consequently, the velocity of the centre of gravity, or GG′, is always the same as if both particles were placed at G, and impelled by the same force ϕ; for GG′ $= \frac{P \cdot Pp + Q \cdot Qq}{P + Q}$, by the nature of the centre of gravity. And the same thing may be shewn of as many other points as we please.

COR. 5. *Hence, in rotations about a centre of spontaneous gyration, the permanency of the quantity of motion obtains.*

324. PROP. *When* φ S, *the direction of the impact, passes through the centre of the impelling body, the centre of gravity of the body struck will move with a velocity equal to the product of the quantity of motion of the impelling body into the distance between the centres of gravity and spontaneous rotation, divided by the sums of the products of the impelling body into the distance of the point of impact from the centre of spontaneous rotation, and of the impelled body into the distance between the centres of spontaneous rotation and of gravity.*

Let the quantity of matter of the impinging body be b, its velocity v, or $bv = \varphi$; and when the body B is struck in the direction φS (in which the centre of the body b is always found,) let the velocity of its centre of gravity be V, the centre of spontaneous conversion being at C. Then $\mathrm{CG} : \mathrm{CS} :: \mathrm{V}$: the velocity of the point S, which is therefore $= \mathrm{V}\frac{\mathrm{CS}}{\mathrm{CG}}$; consequently, $v - \mathrm{V}\frac{\mathrm{CS}}{\mathrm{CG}}$ = the velocity lost by b in the direction φS: whence, by the third law of motion, $b \cdot \frac{v \cdot \mathrm{CG} - \mathrm{V} \cdot \mathrm{CS}}{\mathrm{CG}} = \mathrm{B} \cdot \mathrm{V}$, and, by reduction, $\mathrm{V} = \frac{b \cdot v \cdot \mathrm{CG}}{\mathrm{B} \cdot \mathrm{CG} + b \cdot \mathrm{CS}}$.

COR. If the inertia of the striking body be evanescent, the velocity V will become $\frac{bv}{\mathrm{B}}$ or $\frac{\varphi}{\mathrm{B}}$; being *the same as would be generated in the centre of gravity if the body* b *impinged directly on it with the velocity* V.

325. PROP. *The conditions in the last corollary being retained, the angular velocity of the centre of the system about the centre of gravity, is equal to the momentum of the impelling body, divided by twice the product of the mass of the impelled body and the distance* GC, *into the periphery of a circle whose diameter is unity.*

If a fixed axis passed through C, the centre of gravity would describe a circle whose radius is CG with the velocity $\frac{bv}{\mathrm{B}}$. But in the present case the motion of the system will (art. 322.) be compounded of the uniform rectilinear motion of the centre of gravity in the direction GG$'$, perpendicular to CS, and the angular motion $x\mathrm{G}'p = \mathrm{GCG}'$, generated round the centre of gravity. And, since the periphery of a circle whose radius is CG is $2\pi \cdot \mathrm{CG}$, we have this analogy, $\frac{bv}{\mathrm{B}} : 2\pi \cdot \mathrm{CG} :: 1' : \frac{2\pi \cdot \mathrm{B} \cdot \mathrm{CG}}{bv}$, the time of one revolution in seconds: whence it follows, that the number of revolutions, or parts of a revolution, in a second, or the angular volocity U, will be $1 \div \frac{2\pi\cdot B\cdot CG}{bv} = \frac{bv}{2\pi\cdot B\cdot CG}$.

Cor. 1. Since C is the centre of percussion to S as a centre of motion, if Q be the centre of gyration with respect to G as a centre of motion, (that is, if Q be the principal centre of gyration,) we have (art. 310. cor. 2.) $GC\cdot GS = GQ^2$, or $CG = \frac{GQ^2}{GS}$; this value of CG being substituted for it in the preceding expression for the angular velocity, it becomes $U = \frac{b\cdot v\cdot GS}{2\pi\cdot B\cdot GQ^2}$.

Cor. 2. *The centre of spontaneous rotation, during the motion of the system, describes the common cycloid.*

For the motion of any point in the system is compounded of the uniform rectilineal motion of the centre of gravity, and of the angular motion generated round that centre: but (art. 322.) the velocity with which the centre of spontaneous rotation would move round the centre of gravity, if there only existed a rotatory motion in the system, would be equal to that with which the centre of gravity would move round it, if the centre C were fixed: consequently, since the centre C has both a rotatory and a progressive motion, each of which is equal to that of the centre of gravity, it will describe a cycloid.

326. Prop. *In the body or system* B (fig. 15.) *to which, when quiescent, motion has been communicated by the impulse of a force* ϕ *without inertia, that is, rectilinear motion to the centre of gravity measured by the space* V *which that centre would describe uniformly in any given time, and angular motion measured by the revolutions* U, *or parts of a revolution, which it would describe uniformly round* G *in the same time; then, if the notation in the preceding propositions be retained, and* Q *be the principal centre of gyration when the system revolves about its centre of gravity, the perpendicular distance from the centre of gravity at which the impelling force must act so as to have generated these progressive and rotatory motions will be* $GS = \frac{2\pi\cdot U\cdot GQ^2}{V}$.

Let ϕS be the direction of the impulse, and let ϕ be equal to the momentum of an evanescent body b moving with the velocity v, B being the weight of the system; then (art. 324. cor.) the velocity communicated to the centre of gravity will be $= \frac{bv}{B}$; and the angular velocity generated in the system (art. 325. cor. 1.) $= \frac{b\cdot v\cdot \text{GS}}{2\pi\cdot \text{B}\cdot \text{GQ}^2}$. But, by the prop. the velocity communicated to the centre of gravity is V, and the angular motion, that is, the number of revolutions, or parts of a revolution, described while the centre of gravity passes over the space V, is U: so that, from the conditions there arises this equation, $\text{U} = \frac{b\cdot v\cdot \text{GS}}{2\pi\cdot \text{B}\cdot \text{GQ}^2} = \frac{\text{V}\cdot \text{GS}}{2\pi\cdot \text{GQ}^2}$, by putting V for its equal $\frac{bv}{\text{B}}$. Hence $\text{GS} = \frac{2\pi \text{U}\cdot \text{GQ}^2}{\text{V}}$.

COR. If the body B be a sphere whose radius is r, then, art. 312. $\text{GQ}^2 = \frac{2}{5}r$, and if u be the absolute velocity of rotation of an equator of the sphere, we have $\text{U} = \frac{u}{2\pi r}$; whence the preceding value of GS is transformed to this, $\text{GS} = \frac{2}{5}r\,\frac{u}{\text{V}}$.

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327. This proposition may be applied to the double motion of the planets. The earth revolves about an axis passing through its centre of gravity, while, by a motion of translation, that centre is carried on in free space in an orbit nearly circular. And a similar kind of double motion has been discovered in several of the planets; and analogy leads us to believe it obtains in the others. Now, supposing the bodies of the planets to be spherical, as they are nearly, the use of this proposition at once appears. Having given, for instance, the magnitude of an impulse with respect to the mass of the earth, and the direction ϕS in which it was applied, at any given distance SG from the centre of gravity, the angular motion round G would be inferred: and, conversely, if the actual rotatory velocity of the earth's equator and the velocity of its orbit be ascertained, the distance GS from the centre at which it may have received a single impulse ϕS, adequate to produce the double motion, may readily be found.

Thus, any point in the earth's equator passes over 25020 miles by rotation in a sidereal day; and, if the mean distance of the earth from the sun be 95 millions of miles, the earth will pass over nearly 596904000 miles by its orbital motion, in a year, or about 366 sidereal days : hence 596904000 ÷ 366 = 16308852, will be $=$ V in our theorem, while $25020 = u$. Consequently $\text{GS} = \frac{2}{5}r\,\frac{u}{\text{V}} = \frac{r}{163\cdot 2}$. So that if an impulse be impressed on a quiescent sphere, and the direction of the force should be at a distance GS from its centre of gravity of about $\frac{1}{163}$ part of the radius, the angular motion of that sphere, and the absolute motion of its centre, will have the same relation to each other, as those which actually obtain in the earth.

The time of the rotations of Mercury, Venus, Herschel, and the other planets, is unknown; but for the following planets it is ascertained; so that by the same theorem we obtain these values of GS. Mars $\frac{r}{195}$; Jupiter $\frac{r}{2 \cdot 1258}$; Saturn $\frac{r}{2 \cdot 588}$; the moon $\frac{r}{555}$. We have not sufficient data to determine this for the sun. "But (as Dr. Robison remarks) the very circumstance of his having a rotation in 27d. 7h. 47m. makes it very probable, that he, with all his attending planets, is also moving forward in the celestial spaces, perhaps round some centre of still more general and extensive gravitation; for, the perfect opposition and equality of two forces, necessary for giving a rotation without a progressive motion, has the odds against it of infinity to unity. This corroborates the conjectures of philosophers, and the observations of Herschel and other astronomers, who think that the solar system is approaching to that quarter of the heavens in which the constellation Aquila is situated."

328. Prop. *If a body revolve about an axis passing through its centre of gravity with the angular velocity* U, *while this axis is carried round another axis, also passing through its centre of gravity, with the angular velocity* U, *these two motions compose a motion of every particle of the body round a third axis lying in the plane of the other two, and inclined to each of the former axes in angles whose sines are inversely as the angular velocities round them; and the angular velocity* V *about this new axis is to that about one of the primitive axes as the sine of the inclination of the latter axis, to the sine of the inclination of the new axis to the other primitive axis.*

Thus, if a body turn round an axis AG*a* (fig. 1. pl. XV.) passing through its centre of gravity G with the angular velocity U, while this axis is carried round another axis BG*b* with the angular velocity *u*, and if GD be taken to GE as U to *u*, (the points B and E being taken on that side of the centre where they are moving towards the same side of the plane of the figure,) and the line DE be drawn, the whole and every particle of the body will be in a state of rotation about a third axis CG*c*, parallel to DE, lying in the plane of the other two; and the angular velocity *v* about the axis CG*c*, will be to U, and to *u*, as DE is to GD and to GE.—For, let P be any particle of the body, and suppose a spherical surface whose centre is G to pass through P. Draw PR perpendicular to the plane of the figure: then is PR the common section of the circle of rotation IP*i* round the axis A*a*, and the circle KP*k* of rotation round the axis B*b*. Let F and O be the centres of these circles of rotation, and I*i*, K*k*, their diameters. Draw the radii PF, PO, and the tangents PM, PN: these tangents are in a plane MPN which touches the sphere in P, and cuts the plane of the axis in a line MN, to which a line drawn from G through R would be perpendicular. Suppose PN to represent the velocity of rotation of the point P about the axis B*b*, while P*f* represents its velocity of rotation about A*a*; and complete the parallelogram PN*tf*: then is P*t* the direction and velocity of the resultant of the composition of PN and P*f*, and it is manifestly in the same plane as the constituent lines. Let perpendiculars *f*F, *t*T, be drawn to the plane of the axis, and the parallelogram PN*tf* will be orthographically projected on that plane, its projection being also a parallelogram RNTF, (art. 68.)*. Draw the diagonal RT. Then, since PR is perpendicular to the plane of the primitive axis, PR*t*T is so likewise: and, consequently, the compound motion P*t* is in the plane of a circle of revolution about some axis situated in the plane of the other two. Produce TR, and draw GC intersecting it perpendicularly in H; and let LP*l* be the circle of rotation, its diameter being L*l* = 2LH: then is P*t* a tangent, and perpendicular to PH; and it will meet RT in some point Q of the line MN. The particle P is in a state of rotation about the axis CG*c*, and its velocity is to the velocities round A*a*, or B*b*, as P*t* to PF or to PN. Now PN the tangent is perp. to OP, and PR is perp. to ON; therefore, OP : PN : : PR : RN, and $\mathrm{RN} = \frac{\mathrm{PR}\cdot\mathrm{PN}}{\mathrm{OP}}$. But the velocity of P about the axis B*b* is $u\cdot\mathrm{OP}$: whence $\mathrm{RN} = \frac{u\cdot\mathrm{OP}\cdot\mathrm{PR}}{\mathrm{OP}} = u\cdot\mathrm{PR}$. In like manner $\mathrm{RF} = \mathrm{U}\cdot\mathrm{PR}$. Consequently RF : RN : : U : *u* : : GD : GE. But NT : RN : : sin NRT : sin NTR, and GD : GE : : sin GED : sin GDE; hence, sin NRT : sin NTR : : sin GED : sin GDE. Now since NR is perpendicular to EG, and NT (parallel to IF) perpendicular to DG, we have RNT = EGD. Hence TR is perpendicular, and C*c* parallel to ED, and the rotation of the particle P is about an axis parallel to ED. Also, since RN, RF, RT, are as the velocities *u*, U, *v*, about these different axes, and vary respectively as EG, DG, DE, we have *v* : U : *u* : : ED : GD : GF, which was to be demonstrated.

COR. 1. Hence, if every particle of a body, whether solid or fluid, receive at the same instant two separate impulses, the one competent to the production of a motion of the particle round an axis with a certain angular velocity, and the other competent to the production of a rotatory motion about another axis with a certain velocity, the combined effect of these impulsions will be a motion of the whole system about a third axis given in position, with an angular velocity which is also given; and this motion will obtain *without any separation or disunion of parts*, except such as may be occasioned by the action of the centrifugal forces resulting from the rotation.

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* The coincidence of F with the centre of the circle IP*i* is purely accidental.

Cor. 2. Hence, also, if a body be turning round any axis, and every particle in one instant get precisely such an impulse as is competent to produce a given angular velocity round another axis, the body will turn round a third axis given in position with a given angular velocity.

Cor. 3. Lastly, when a rigid body acquires a rotation about an axis by an impulse on one part of it, and either at the same time or afterwards receives an impulse on any part, which, alone, would have produced a certain rotation about another axis, the joint effect of these impulses will be a rotation about a third axis, in conformity with this proposition. For when a rigid body acquires a motion about an axis, not by the simultaneous impulse of the precisely competent force on each particle, but by an impulse on one part, there has been propagated to every particle (by means of the cohesive forces) an impulse precisely competent to the production of that motion which the particle actually acquires; and when a rigid body already turning round an axis A*a* receives an impulse which makes it actually turn about another axis C*c*, there has been propagated to each particle a force precisely adequate to the production, not of the motion, but of the *change* of motion which takes place in that particle, that is, a force which, when compounded with the inherent force of its primitive motion, produces the new motion; that is, (by the prop.) a force which alone would have caused it to turn about a third axis B*b*, with a rotation making the other component of the actual rotation about C*c*.

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This elegant theorem, the enunciation of which is due to P. Frisi, is very important, and gives a great extension to the doctrine of the composition of motion. It is of great use in many curious problems, and particularly that of the precession of the equinoxes. Such as wish for further information on various points connected with the subjects of this chapter, may consult *Simpson's Tracts*, *Frisi's Cosmographia*, *Mr. Vince's Paper in the Phil. Trans.* 1780, *Landen's Memoirs*, *Atwood on the Rotatory Motion of Bodies*, the article Rotation in the *Encyclopædia Britannica*, *Euler's Treatise de*

Motu Corporum rigidorum, &c. There may here be added, however, Mr. *Thomas Knight's* Analytical View of the Composition of Rotatory Motions.

Let a body have forces impressed on it, that would separately give it a movement of rotation, about an axis cx, with the angular velocity v; and about another axis cy with the angular velocity v'. It is plain that a particle in the line cy will not be affected in its motion about cx, by the other rotation about cy. The same observation (mutatis mutandis) may be applied to a particle in the line cx: but in any intermediate line in the plane ycx, as cm, the motion round the one axis will impede that round the other; and if we can find a line in which all the particles are at rest, *that* will be the momentary axis of rotation composed of and equivalent to the two others.

Put c$m=r$, angle xc$m=\theta$, angle yc$m=\theta'$.
A perpendicular from m on cx will be $r\sin\theta$.
A perpendicular from m on cy will be $r\sin\theta'$, and it is plain that m will be at rest if

$$v' : v :: r\sin\theta : r\sin\theta' :: \sin\theta : \sin\theta' \quad \ldots\ldots \quad (\alpha),$$

which determines the situation of the momentary axis.

To find the velocity round this, let us suppose ymx to be a circle drawn through m, the centre being c. Let ay be the velocity of the point y; we have $ay=vr\sin xcy$. Now the velocity about cm at the distance r is plainly $ay\dfrac{r}{r\sin\theta'}=\dfrac{ay}{\sin\theta'}$, consequently the velocity at distance unity, or the angular

$$\text{vel.} = \frac{ay}{r\sin\theta'}=v\cdot\frac{\sin xcy}{\sin\theta'} \quad \ldots\ldots\ldots\ldots\ldots \quad (\beta).$$

The theorems in equations (α) and (β) were first given by *Frisi*. If, besides the two rotations already mentioned, there had been given, at the same time, another round a third axis cz, with an angular velocity $=v''$; it is plain that the axis of rotation, arising from these three, would be in the plane zcm, as at cm', suppose it to decline from cm by the angle θ'', and from cz by θ'''; we find, as we did (α) and (β),

$$v'' : v\frac{\sin xcy}{\sin\theta'} :: \sin\theta'' : \sin\theta''' \quad \ldots\ldots\ldots\ldots \quad (\gamma)$$

$$\text{and angular velocity} = v\cdot\frac{\sin xcy}{\sin\theta'}\cdot\frac{\sin zcm}{\sin\theta'''} \quad \ldots\ldots\ldots \quad (\delta).$$

If, moreover, a rotation had been given round a fourth axis cu, with an angular velocity $=v'''$, we should have

$$v''' : v\frac{\sin xcy\cdot\sin zcm}{\sin\theta'\cdot\sin\theta'''} :: \sin\theta'''' : \sin\theta''''' \quad \ldots\ldots\ldots \quad (\varepsilon),$$

θ'''' and θ''''' being angles that the momentary axis declines from cm' and cu; and the angular velocity $=v\dfrac{\sin xcy\cdot\sin zcm\cdot\sin ucm'}{\sin\theta'\cdot\sin\theta''\cdot\sin\theta''''}$, and thus we may proceed to any number of axes.

We will now consider the case in which the angles *xcy*, *zcm*, *ucm'* &c. are right, and their sines unity, We have $\sin\theta = \cos\theta'$, $\sin\theta'' = \cos\theta'''$ &c. here (α) becomes $v' : v :: \cos\theta' : \sin\theta' = \surd(1-\sin^2\theta') : \sin\theta'$; whence $\sin\theta' = \frac{v}{\surd(v^2+v'^2)}$. We get in like manner from (γ), $\sin\theta''' = \frac{\surd(v^2+v'^2)}{\surd(v^2+v'^2+v''^2)}$, and from ($\varepsilon$) $\sin\theta''''' = \frac{\surd(v^2+v'^2+v''^2)}{\surd(v^2+v'^2+v''^2+v'''^2)}$, and so on: and if the number of axes be n, and the velocities round those axes $v, v', v'', \ldots \ldots v'''^{\cdot\cdot n}$, the angular velocity round the axis arising from the composition of them all will be $\surd\left\{v^2+v'^2+v''^2+ \ldots \ldots +(v'''^{\cdot\cdot n})^2\right\}$. Of this very general theorem the case considered by La Grange of three axes *perpendicular to each other*, is a particular case; in which ang. vel. $= \surd(v^2+v'^2+v''^2)$, and $\sin\theta' = \frac{v}{\surd(v^2+v^{2\prime})}$, $\sin\theta''' = \frac{\surd(v^2+v'^2)}{\surd(v^2+v'^2+v'^2)}$, or, the angle *zmy* being a right one, we easily find that c*m'* makes angles with c*x*, c*y*, c*z*, whose cosines are

$\frac{v}{\surd(v^2+v'^2+v''^2)}$, $\frac{v'}{\surd(v^2+v'^2+v''^2)}$, $\frac{v''}{\surd(v^2+v'^2+v''^2)}$. In the general case treated of above, the angles *zmy* *um'm*, &c. are any whatsoever, and if their magnitude be given, the situation of the composed axis is also given, by means of the values of $\sin\theta'$, $\sin\theta'''$, $\sin\theta'''''$, &c.

CHAPTER V.

Physico-Mathematical Theory of Percussion.

329. Defs. In the ordinary theory of percussion, or collision, bodies are regarded as either *hard*, *soft*, or *elastic*. A *hard* body is that whose parts do not yield to any stroke or percussion, but retains its figure unaltered. A *soft* body is that whose parts yield to any stroke or impression, without restoring themselves again, the shape of the body remaining altered. An *elastic* body is that whose parts yield to any stroke, but presently restore themselves again, so that the body regains the same figure as before the stroke. When bodies which have been subjected to a stroke or a pressure return only in part to their original form, the *elasticity* is then *imperfect*: but if they restore themselves entirely to their primitive shape, and employ just as much time in the restoration as was occupied in the compression, then is the elasticity *perfect*.

It has been customary to treat only of the collision of bodies perfectly hard or perfectly elastic: but as there do not exist in nature any bodies (which we know) of either the one or the other of these kinds, the usual theories are of no service in practical mechanics. In fact, if the effects which are experienced in the mutual impact of such bodies as nature furnishes were the same as those which would result from bodies perfectly hard, it would appear impossible to oppose to the moving bodies any pressure capable of destroying their effects. For the quantity of motion, or the momentum of a moving body, being valued by the product of its mass and velocity, while the effect produced by a body which merely presses is estimated by the product of its mass into a velocity denoted by 0; one of the factors of the latter product will be nothing, while both factors of the former will be finite, and of consequence the magnitudes are such as cannot be compared.

Yet we may easily be convinced by daily experience, that the advantage which bodies moving swiftly have over those which oppose merely a resistance of pressure, though very great, is by no means infinite. Numerous circumstances which will be suggested to every mind, prove that, physically speaking, we may balance a percussive force by one of mere pressure, and even that the latter may overcome the former and produce a greater effect*. It argues nothing to ascribe any part of this to *friction;* for friction itself is a species of pressive force, or is, certainly, more analogous to pressure than to percussion.

The causes to which we should attribute the difference which subsists between the effects of impact and of simple pressure may be readily traced. When a body, supposed perfectly hard, in the course of its motion strikes another body, perfectly hard also, the variation of motion ought to be produced in an indivisible instant, in such sort, that between the initial velocity and the velocity after the shock there shall not be any intermediate velocity. But if the motion of the body were modified by a pressure, or by a constant force, as gravity, then it would change by sensible degrees, and have undergone a determinate variation at the end of a certain time.

It is therefore the law of continuity so usually though not constantly manifested in nature, which distinguishes the effects of pression and those of impact, when the *hardness* is infinite: but as such hardness no where exists, since matter always possesses a certain degree of elasticity, and a limited cohesion of particles which may be surmounted, we may enquire if percussion, considered physically, conforms to the law of continuity. Now when a body strikes another, two effects have place in each. 1st. The parts in contact yield to the action of the stroke, and become compressed, so that the figure of the bodies is altered by a flattening or impression, which obtains in the parts in contact, and in the neighbourhood of those parts. 2d. When the flattening or impression has arrived at the greatest degree of which the bodies are susceptible, then their inherent elasticity tends to destroy the impression, and effaces it either wholly or in part; this produces a mutual action and reaction of the bodies, which is continued till they are no longer in contact*. Hence it appears, that the law of continuity is followed actually in the impact of bodies; and that no alteration in their motion takes place, without the previous taking place of all the intermediate alterations. It is true this alteration is always produced during an extremely short interval of time, and this occasions the great disproportion which is observed between the effects of impact and those of pressure: but it is notwithstanding inconceivable, as Mr. Atwood remarks, "that any really existing body should pass from quiescence into finite motion, or from one degree of finite motion to another, without having possessed all the intermediate degrees of velocity:" and hence it follows that the phenomena of collision may be considered of a kindred class to those which are occasioned by accelerating or retarding forces, and act by insensible degrees in order to produce a finite effect.

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* Thus, in driving the piles for the foundation of the bridge of some mills, where the soil upon which the bridge was to be erected was a pure sand of *uniform* density to the depth of 47 feet, no piles could be driven lower than 15 feet, although the rammers in the engines employed were enormously great. So that percussion is brought into a state of equilibrium with mere pressure, and cannot therefore be infinitely greater. Indeed it is not a little inconsistent that many authors, while they admit that impact and pressure are comparable in the theory of Hydrodynamics, and Pneumatics (as when, for example, they assume *the weight of a column of water equal to the force of impact* of that fluid on a wheel), should deny the same thing when writing on Dynamics.

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* A very distinct view of this process has been given by Dr. Robison in the article IMPULSION, *Supp. Ency. Britan.* from which the following is extracted:

"As soon as the bodies come into sensible contact, compression must begin; for we may suppose the bodies to be two balls, which will therefore touch only in one point. The mutual pressure, which is necessary in order to produce the retardation of A, and the acceleration of B, is exerted only on the foremost particle of A, and the hindmost particle of B; but no atom of matter can be put in motion, or have its motion any way changed, unless it be acted on by an adequate force. The force urging any individual particle must be precisely competent to the production of the very change of motion which obtains in that particle. Except the two particles which come into contact in the collision, all the other particles are immediately actuated by the forces which connect them with each other; and the force acting on any one is generally compounded of many forces which connect that particle with those adjoining. Therefore, when A overtakes B, the foremost particle of A is immediately retarded: the particles behind it would move forward, if their mutual connection were dissolved in that instant; but, this remaining, they only approach nearer to the foremost striking particles, and thus make a compression, which gives occasion for the inherent elasticity to exert itself, and, by its reaction, retard the following particles. Thus each stratum (so to conceive it), continuing in motion, makes a compression, which occasions the elasticity to react, and, by reacting, to retard the stratum immediately behind it. This happens in succession: the compression and elastic reaction begin in the anterior stratum, and take place in succession backward, and the whole body gets into a state of compression. Things happen in the same manner in B, but in the contrary direction, the foremost strata being the last which are compressed. All this is done in an instant (as we commonly, but inaccurately, speak), that is, in a very small and insensible moment of time; but in this moment there is the same gradual compression, increase of mutual action, greatest compression, common velocity, subsequent restitution, and final separation, as in the case of bodies with a slender spring interposed, or even in the case of mutually repelling magnets. In all the cases, the changes of motion are produced by the elasticity or the repulsion, and not by the transfusion of the force of motion. The changing force is indeed inherent in the bodies, but not because they are in motion; the use of the motion is to give occasion, by continued compression, for the continued operation of the inherent elasticity."

After these preliminary observations, we shall now present the reader with a physical theory of percussion, which will comprehend all the circumstances of the motion, both during and after the stroke. From the same source the laws of the collision of hard, soft, and perfectly or imperfectly elastic bodies, will be deduced. All that we shall give which is different from the usual theory, we wish to be considered as originally due to *Don George Juan*, a Spanish author, and published in 1771, in a work entitled *Examen Maritimo.* The same theory has been lately adopted by *M. Prony*, and some other French authors. As the mode of investigation is analytical throughout, it shall be presented nearly in its original state, without announcing the various particulars as they arise, in separate propositions.

330. Def. By the *Depth of the impression* we mean its greatest lineal measure in the direction of the motion; as if the whole impression were a spheric segment the *depth* would be its versed sine. The *Amplitude of the impression* is the greatest section which can be taken perpendicularly to the direction of the motion: in the case of a spherical body it would be the circular base of a segment.

Through the whole of this investigation the various quantities will be thus represented:

m and μ, the two bodies which strike each other;
b and β, the amplitudes of impression;
r and ϱ, the comparative hardness of the bodies;
x and z, the depths of the impressions;
f and φ, the quantities of motion which the bodies acquire in a unit of time, in virtue of the action of the constant powers;
w and ϖ, the velocities with which the shock commences;
u and v, the velocities in any instant of the shock;
s and σ, the spaces described in the same time;
t, that time.

Any other characters that may be adopted will be explained as they first occur.

331. Let it be supposed that the body m follows and strikes the body μ, and, of consequence, that the velocity w is greater than ϖ; without this the impact can never take place, unless the velocity ϖ be negative, or contrary to w: but for the greater facility, we shall suppose all along (except it be otherwise expressed) that the powers f and φ, as well as the velocities w and ϖ, are positive; for it will be easy to render either of the quantities negative in the result of the computation.

Let it be supposed, moreover, that the two bodies are moving in the same direction, and that the impact is so effected that it does not produce any gyratory motion; otherwise we should render the investigation very intricate and perplexing.

Finally, let the bodies m and μ be considered as so large that the impressions made upon them by the shock shall not produce any sensible change in the place of the centre of gravity, or that the motions of these centres shall not be sensibly affected by the relative change of situation in the various moleculæ of the bodies.

322. All the quantities mentioned in art. 330. may be readily introduced into the calculus, since each of them has a certain relation to determinate units; the quantities r and ρ only, which represent the degrees of hardness, appear to present some little difficulty. Now to form an idea of hardness in that point of view under which we must consider it,-that is to say, as a *measurable* object, we may observe, that since this property is one of the most intimate and hidden properties of bodies, and since to measure is to compare, we cannot (properly speaking) measure hardness, because in an unknown object we can make no comparison. But as hardness is only interesting to us by its effects, it is the measure of these effects which it is important for us to know; and this measure for the respective bodies we denote by r and ρ.

If a perfectly hard body strike another which is not perfectly hard, and if the surface of contact to which the direction of the motion is supposed perpendicular is plane and constantly equal to a given surface, as a foot square (the foot being the unit), the resistance of the body struck will take from the impinging body, during an evanescent instant of time $\dot{t}$, a quantity of motion infinitely small, which let be called ω. If ω be multiplied by the fraction expressing the ratio of the unit of time to the element of time, the product $\omega \frac{1}{\dot{t}}$ will be precisely equal to the quantity ρ, which denotes the hardness of the body struck during the instant which corresponds to the value of ω, or rather which denotes the effect that this hardness would produce by continuing its action for a unit of time. We have, therefore, $\frac{\omega}{\dot{t}} = \rho$, or $\omega = \rho\dot{t}$.

Thus it appears that ω is independent of the masses and of the velocities, whether absolute or relative (12. 15.); it represents a quantity of motion which may result from an infinite number of combinations of masses and velocities; ω, therefore, depends solely on the nature of the body struck: and although the masses and the velocities of the two bodies enter into the value of ϱ, they do so merely as causes producing the impression, and we determine ϱ from experiments by the relation which they bear to this impression.

A similar reasoning will apply to the quantity r, by supposing that the body struck is perfectly hard, while the impinging body is not; and hence will arise the analogous equation $\omega' = r\dot{t}$.

If the surface of contact be not plane, we must substitute for it the amplitude of the impression: but this substitution will change nothing; for all the points struck in the direction of the motion being contained in the amplitude of impression, it may represent the sum of all those points.

333. *To find the quantities of motion lost*, viz. the values of ω and ω', which answer to any amplitudes of impression whatever b and β, we must make these proportions; $1 : \beta :: \rho\dot{t} : \beta\varrho\dot{t} = \omega$, and $1 : b :: r\dot{t} : br\dot{t} = \omega'$; then will the equations $\omega = \beta\rho\dot{t}$, and $\omega' = br\dot{t}$ obtain, whatever are the amplitudes of impression.

In consequence of the equality of action and reaction, the values $\omega = \beta\rho\dot{t}$, and $\omega' = br\dot{t}$, of motion lost by the impinging bodies, are likewise those of the *motion gained by the bodies struck*.

334. *To find the value of the motion lost by the body, or of the percussive force*, when neither of the two bodies is perfectly hard. In this case both are compressible, and if F be the fluxion of the quantity of motion, we shall have to determine $\frac{1}{\dot{t}}\,\text{F}$ or $\frac{\text{F}}{\dot{t}}$. Now it is manifest that F will be always the greater, as $\beta\rho$ and br are greater; and thence (supposing $\dot{t}$ constant) $\frac{\text{F}}{\dot{t}}$ will vary as $\beta\varrho br$. So that when $\varrho = \infty$, or is infinite, we must have $\text{F} = rb\dot{t}$ (333), and when $r = \infty$, we have $\text{F} = \varrho\beta\dot{t}$: let us, then, multiply $\beta\rho br$ by such a quantity $\frac{1}{\beta\varrho + br}$ as will satisfy these two conditions, and there will result $\frac{\text{F}}{\dot{t}} = \frac{\beta\varrho br}{\beta\varrho + br}$. And if $\frac{\text{F}}{\dot{t}}$, or the force of percussion, be denoted by P, we have $\text{P} = \frac{\beta\varrho br}{\beta\varrho + br}$; an expression for the percussive force in any instant whatever.

335. *The fluxions of the impressions are reciprocally as the degrees of hardness.* The fluxions $b\dot{x}$, $\beta\dot{z}$, of the impressions are produced at every instant by the same power $\text{P}\dot{t}$ acting equally on the two bodies; and to produce these elementary impres-

sions the force $\mathrm{P}\dot{t}$ has to overcome the powers $r\dot{t}$, and $\rho\dot{t}$; thus the effects $b\dot{x}$ and $\beta\dot{z}$ are the greater as the powers $r\dot{t}$ and $\rho\dot{t}$ are the less; and as these effects are produced by the same cause, they will be in the inverse ratio of the obstacles $r\dot{t}$ and $\rho\dot{t}$, which oppose them; we have, therefore, $b\dot{x} : \beta\dot{z} :: \rho\dot{t} : r\dot{t} :: \rho : r$; whence $br\dot{x} = \beta\rho\dot{z}$.

336. Since the body m follows the body μ, and by the impact causes impressions in both whose depths are x and z, the space s run over by the body m must be equal to the space σ described by the body μ *plus* the sum $x + z$ of the depths of impression: hence $s = \sigma + x + z$.

Elastic bodies assume again at the end of the shock the figures which they had at its commencement; so that in this case the impressions vanish, and *at the end of the shock of elastic bodies* $s = \sigma$.

337. From the equation $s = \sigma + x + z$, or $s - \sigma = x + z$, we deduce $\dot{s} - \dot{\sigma} = \dot{x} + \dot{z}$: but the property of variable motion (232. I.) gives $u\dot{t} = \dot{s}$, and $v\dot{t} = \dot{\sigma}$; we have, therefore, $(u - v)\dot{t} = \dot{s} - \dot{\sigma} = \dot{x} + \dot{z}$, and $\dot{t} = \frac{\dot{x} + \dot{z}}{u - v}$, *the value of the element of the time.*

When the impressions arrive at their greatest value we have $x + z =$ a max. or $\dot{x} + \dot{z} = 0$, and the equation $u - v = \frac{\dot{x} + \dot{z}}{\dot{t}}$ becomes $u - v = 0$: consequently, *in the instant when the greatest impressions are produced the velocities of the two bodies are equal.*

338. The accelerating forces which act upon the body m are f and P; and as P tends to retard its motion, the accelerating force resulting from the joint operation of the two will be $f - \mathrm{P}$: for the same reason, contrarily applied, the accelerating force of μ will be $\varphi + \mathrm{P}$. Recollecting, therefore, that f, φ, and P, may represent the products of masses and velocities, we have, by the nature of variable motion (art. 232.) $m\dot{u} = (f - \mathrm{P})\dot{t}$, and $\mu\dot{v} = (\varphi + \mathrm{P})\dot{t}$; the sum of these equations is $(f + \varphi)\dot{t} = m\dot{u} + \mu\dot{v}$, the fluent of which is $(f + \varphi)t = mu + \mu v + \mathrm{C}$. To determine the constant quantity C, it may be observed that when $t = 0$, $u = \mathrm{W}$, and $v = w$; and then the equation reduces to $m\mathrm{W} + \mu w + \mathrm{C} = 0$; so that $\mathrm{C} = -m\mathrm{W} - \mu w$, and the corrected fluent is $(f + \varphi)t = m(u - \mathrm{W}) + \mu(v - w)$; whence it is easy to obtain $v = \frac{(f + \varphi)t + m\mathrm{W} + \mu w - mu}{\mu}$, *an equation which ex-*

presses the relation between the velocities, in any instant whatever of the shock of bodies, whether elastic or non-elastic.

339. If the bodies are not elastic the shock ceases at the instant when the impression is greatest; but in this instant the velocities are equal (art. 337.). Therefore, in the preceding equation, we must make $u = v$, and there will arise $u = v = \frac{(f+\varphi)\dot{t}+m\mathrm{W}+\mu w}{m+\mu}$, an equation expressing *the common velocity after the shock of non-elastic bodies.*

340. In like manner we might deduce from the final equation in art. 338. the velocities of elastic bodies after the shock. But as we would state the question generally, we must determine the velocities of both bodies, whether they are elastic or not, in any instant of the shock.

The equations $(f-\mathrm{P})\dot{t} = m\dot{u}$ and $(\varphi+\mathrm{P})\dot{t} = \mu\dot{v}$ (art. 338.) give $\dot{u} = \frac{(f-\mathrm{P})\dot{t}}{m}$, and $\dot{v} = \frac{(\varphi+\mathrm{P})\dot{t}}{\mu}$. Taking the latter of these from the former, we have $\dot{u}-\dot{v} = \left(\frac{f-\mathrm{P}}{m} - \frac{\varphi+\mathrm{P}}{\mu}\right)\dot{t}$; and this, multiplied by $m\mu$, gives $m\mu\,(\dot{u}-\dot{v}) = (\mu f - m\varphi - \mu\mathrm{P} - m\mathrm{P})\dot{t}$. Substituting, in this equation, for $\dot{t}$ its value in art 337. and for P and $\dot{x}$ their values in art. 334. 335. it will be transformed to this, $m\mu\,(u-v)\cdot(\dot{u}-\dot{v}) = (\mu f - m\varphi)\frac{\beta\varrho+br}{br}\,\dot{z} - (m+\mu)\,\beta\varrho\dot{z}$. Taking the fluents, we have $\frac{1}{2}m\mu\,(\overline{u-v}^2 - \overline{\mathrm{W}-w}^2) = (\mu f - m\varphi)\int\frac{\beta\varrho+br}{br}\,\dot{z} - (m+\mu)\int\beta\varrho\dot{z}$. Now, $\int\frac{\beta\varrho+br}{br}\,\dot{z} = x + z$; for $\frac{\beta\varrho+br}{br}\,\dot{z} = \frac{\beta\varrho\dot{z}}{br} + \dot{z} = \dot{x} + \dot{z}$, because $\dot{x} = \frac{\beta\varrho\dot{z}}{br}$, from art. 335. Conseq. we have, by substitution, $\frac{1}{2}m\mu\,(\overline{u-v}^2 - \overline{\mathrm{W}-w}^2) = (\mu f - m\varphi)\cdot(x+z) - (m+\mu)\int\beta\varrho\dot{z}$. Hence $u - v = \pm\left(\overline{\mathrm{W}-w}^2 + \frac{2(\mu f - m\varphi)\cdot(x+z)}{m\mu} - \frac{2(m+\mu)\int\beta\varrho\dot{z}}{m\mu}\right)^{\frac{1}{2}}$. Here the positive sign obtains previous to the instant of the greatest impression, and the negative sign after that instant: for at the instant of the greatest impression we have $u = v$ (art. 337.); and after that instant the re-establishing of the compressed parts accelerates the motion of μ and retards that of m, and consequently makes u smaller than v.

If the value of v in art. 338. be substituted for it in the preceding equation, we shall, after a little reduction, find

$$u = \frac{mW + \mu w + (f+\varphi)t}{m+\mu} \pm \frac{\mu}{m+\mu}\left(\overline{W-w}^2 + \frac{2(\mu f - m\varphi)\cdot(x+z)}{m\mu} - \frac{2(m+\mu)\int\beta\rho\dot{z}}{m\mu}\right)^{\frac{1}{2}}.$$

And, by a similar mode of proceeding, we find

$$v = \frac{mW + \mu w + (f+\varphi)t}{m+\mu} \mp \frac{m}{m+\mu}\left(\overline{W-w}^2 + \frac{2(\mu f - m\varphi)\cdot(x+z)}{m\mu} - \frac{2(m+\mu)\int\beta\rho\dot{z}}{m\mu}\right)^{\frac{1}{2}}.$$

In these two equations for the velocities the superior sign serves for any instant of the shock before that of the greatest impression, and the inferior sign after that instant. Thus, at the moment of the greatest impression the quantity affected with the double sign ± or ∓ vanishes, because it is then passing from positive to negative, or from negative to affirmative. These equations manifestly apply for any instant of the shock, whether the bodies be elastic or not.

In non-elastic bodies the values of u and v reduce to the common value $\frac{mW + \mu w + (f+\varphi)t}{m+\mu}$, which is the same as we found in the preceding article.

341. Bodies perfectly elastic, on the contrary, re-establishing themselves in their primitive state at the end of the shock, we have for that instant x, z, and $\int\beta\rho\dot{z}$, each equal to *zero*, and thus the values of u and v for *elastic bodies* become

$$u = \frac{mW + \mu w + (f+\varphi)t}{m+\mu} - \frac{m}{m+\mu}(W - w) = \frac{(m-\mu)W + 2\mu w + (f+\varphi)t}{m+\mu}.$$

$$v = \frac{mW + \mu w + (f+\varphi)t}{m+\mu} + \frac{m}{m+\mu}(W - w) = \frac{(\mu-m)w + 2mW + (f+\varphi)t}{m+\mu}.$$

If we suppose $f + \varphi = 0$, we may easily deduce from these values of u and v that of $mu^2 + \mu v^2$; for, by carrying through the calculus, we have $mu^2 + \mu v^2 = mW^2 + \mu w^2$. Hence, *in perfectly elastic bodies the sums of the products of the masses into the squares of their respective velocities, before and after impact, are equal.*

This equality was denominated by Bernoulli *Conservatio Virium Vivarum*, and was considered as a *general* law; but it is obvious from what is done here that it holds only in the case of perfectly elastic bodies. And, indeed, it is a consequence in itself easily deducible from the third law of motion, according to the Newtonian measure of momentum.

342. If the bodies are not perfectly elastic they will not restore themselves entirely to their first state, but there will remain an impression at the end of the shock: to find the final velocities of such sorts of bodies we must substitute for t, x, z,

β, and ρ, their proper values. Of these, the last four may be ascertained by experiment and the calculus; but the value of t seems more difficult to appreciate; nevertheless, this value is not incalculable, as will soon be seen. But it may be observed, beforehand, that whenever f and ϕ are not extremely great, or are not modifications of gravity, the value of t being very small indeed, the quantity $(f + \phi)\,t$ may be neglected without sensible error. This quantity was made to enter the preceding formulæ in order to give them all possible strictness, and that none of the physical circumstances of the motion might be neglected. Now, dropping $(f + \phi)\,t$ from the expression for the final velocity of the body, we have, for non-elastic bodies $u = \frac{m\text{W} + \mu w}{m + \mu}$; and for bodies perfectly elastic $u = \frac{(m - \mu)\text{W} + 2\mu w}{m + \mu}$, and $v = \frac{(\mu - m)w + 2m\text{W}}{m + \mu}$. These are the values of the final velocities which are commonly given by writers on mechanics, and from which the usual theorems readily flow.

343. The general theorem for non-elastic bodies may be thus expressed:

$$u = \frac{m\text{W} \pm \mu w}{m + \mu}.$$

In this we consider $w = 0$ for the case where the body struck is at rest: the superior sign must be taken for the case in which the bodies both move the same way, and the inferior sign for the case in which they meet each other.

Hence we deduce these conclusions:

I. If we suppose μ to be at rest, the equation becomes $u = \frac{m\text{W}}{m + \mu}$, in which it is obvious u will decrease as μ increases; so that when $\mu = \infty$, infinity, we have $u = 0$.

II. If bodies of equal masses are moving in opposite directions, our equation becomes $u = \frac{\text{W} - w}{2}$: thus *the two bodies move on together in the direction of that which is moving most swiftly, with a velocity equal to half the difference of the primitive velocities.*

III. If the velocities of the two bodies are equal, we have $u = \text{W} \cdot \frac{m \pm \mu}{m + \mu}$. In the case where both bodies move in the same direction, the superior sign being taken, we have $u = \text{W}$; as is evident enough, since there can be no shock.

IV. The centre of gravity of a system of two globules, or physical points, which strike each other, has an interesting property of great use in various mechanical disquisitions. The distance x of this centre from any one of the points of the line described is $\mathrm{x} = \frac{md \pm \mu\delta}{m+\mu}$ (art. 108.), *d* and *δ* denoting the respective distances of the moveables from that point. Now when these bodies are animated by the velocities W, *w*, we have for these velocities (art. 232.) the corresponding values $\frac{\dot{d}}{\dot{t}}$ and $\frac{\dot{\delta}}{\dot{t}}$; consequently that of the centre of gravity will be

$$\frac{\dot{\mathrm{x}}}{\dot{t}} = \frac{m\dot{d} \div \dot{t} \pm \mu\dot{\delta} \div \dot{t}}{m+\mu} = \frac{m\mathrm{W} \pm \mu w}{m+\mu}.$$

This expression being constant, justifies the conclusion that the centre of gravity of a system of two bodies striking each other has, both before and after the shock, a uniform motion. This answers to what the foreign mathematicians call *the conservation of the centre of gravity*. And it is certainly of importance to know that, *whether bodies mutually attract each other*, (295. cor. 2.), *or whether they impinge upon each other, their common centre of gravity will, if in motion, continue to move on uniformly in a right line.*

344. The general equations for perfectly elastic bodies may be stated thus:

$$u = \frac{(m-\mu)\,\mathrm{W} \pm 2\mu w}{m+\mu} \quad . \; . \; . \; . \quad v = \frac{\mp (m-\mu)\,w + 2m\mathrm{W}}{m+\mu}$$

The most important consequences they furnish are these.

I. *If one of the two bodies is at rest*, then making $w = 0$ we find,

$$u = \mathrm{W} \cdot \frac{m-\mu}{m+\mu} \quad . \; . \; . \; . \quad v = \frac{2m\mathrm{W}}{m+\mu}.$$

1. *If the bodies are equal, that which impinges remains at rest, and the body struck moves with the velocity which the other had.* For $m = \mu$ gives $u = 0$, and $v = \mathrm{W}$.

COR. Hence it follows, that if several perfectly elastic bodies A, B, C, D, &c. of equal masses, are placed in juxta-position and at rest, and if a body M of equal mass is made to strike A, all the bodies remain at rest except the last, which goes off with the velocity of M: for the body A transfers all its velocity to B, B its to C, and so on.

2. *If the bodies decrease in magnitude, each will go forward after the stroke.* For if $m \urcorner \mu$, u and v are positive.

3. *If they increase in magnitude each will be reflected back, except the last, and the quantity of motion will continually increase.* For $m \angle \mu$ renders u negative, and the last v positive.

4. Here too we might shew that the increase of motion will be a maximum when the bodies increase in geometrical proportion; and if the number of bodies be increased without limit,

the ultimate ratio of the velocities of A and Z (the first and last) will be that of $\sqrt{\text{Z}}$ to $\sqrt{\text{A}}$. But such theorems, though undoubtedly curious, are of too little utility to tempt us to digress widely from the main purpose of this chapter*.

II. *If the bodies both move the same way*, then we conclude from the theorems:

1. *The body struck moves with a greater velocity after the stroke.*

2. *If the masses are equal the bodies change their velocities during the shock, and continue to move in the same direction.* For $m = \mu$ gives $u = w$, and $v = \mathrm{w}$.

III. *If the bodies move in contrary directions* the inferior signs must be taken, and then

1. *If the masses and velocities are equal the bodies go back with the same velocities that each had before the shock.* For $m = \mu$, and $\mathrm{w} = w$, give $u = -\mathrm{w}$, and $v = \mathrm{w}$.

2. If the *velocities are equal* we have $u = \frac{m-3\mu}{m+\mu}\mathrm{w}$, and $v = \frac{3m-\mu}{m+\mu}\mathrm{w}$. Whence we conclude *that μ will stop a body of a triple mass, but will recoil with a double velocity:* For if $m = 3\mu$, $u = 0$, and $v = 2\mathrm{w}$. But if $\mu \; 7 \; 3m$, the mass m does not stop, but recoils.

3. *If the masses are equal the bodies will both recoil, after having exchanged velocities.* For then $u = -w$, and $v = \mathrm{w}$.

4. *The impinging body is stopped, continues its route, or recoils, according as* $\mathrm{w}\,(m - \mu)$ *is equal, greater, or less, than* $2\mu w$. And the same thing holds with regard to the body struck.

Many other theorems might readily be deduced from the equations at the head of this article; but it is time to return to the general investigation which we have brought down to art. 342.

345. When, in the equation given in art. 340. namely,

$$\tfrac{1}{2}m\mu(\overline{u-v}^2 - \overline{\mathrm{w}-w}^2 = (\mu f - m\varphi)\cdot(x + z) - (m + \mu)\int\beta\varrho\dot{z}$$

we suppose the hardness ϱ invariable, we thence find

$$\int\beta\dot{z} = \frac{\tfrac{1}{2}m\mu(\overline{\mathrm{w}-w}^2 - \overline{u-v}^2) + (\mu f - m\varphi)\cdot(x+z)}{\varrho(m+\mu)}.$$

But $\beta\dot{z}$ being the element of the impression, $\int\beta\dot{z}$ will be the total impression of which this equation will furnish the value for any instant whatever of the shock, the hardness being constant.

In this equation making $u = v$, we have

$$\int\beta\dot{z} = \frac{\tfrac{1}{2}m\mu(\mathrm{w}-w)^2 + (uf - m\varphi)\cdot(x+z)}{\varrho(m+\mu)}.$$

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* Many of them are elegantly exhibited in Bridge's Mathematical Principles of Natural Philosophy. Lect. 5.

And this value of $\int \beta \dot{z}$ answers to the instant of the greatest impression.

When the *body struck is immoveable*, we have only to suppose $\mu = \infty$, $w = 0$, and $\varphi = 0$, and the preceding equation will become $\int \beta \dot{z} = \frac{\frac{1}{2} m \mathrm{w}^2 + f(x+z)}{\varrho}$.

346. Since f is the quantity of motion which the accelerating force will produce in the body m in the unit of time, if we suppose that f is gravity, and the second is the unit of time, the English foot being that of space, we shall have $f = 32\frac{1}{6}m$, and the latter equation will become

$$\int \beta \dot{z} = \frac{\frac{1}{2} m \mathrm{w}^2 + 32\frac{1}{6} m (x+z)}{\varrho} = \frac{m}{\varrho} \left[\tfrac{1}{2} \mathrm{w}^2 + 32\tfrac{1}{6} (x+z)\right].$$

If h be the height due to the velocity w, we have $\mathrm{w}^2 = 64\frac{1}{3} h$, and

$$\int \beta \dot{z} = 32\tfrac{1}{6} \frac{m}{\varrho} (h + x + z).$$

When $x+z$ is very small with respect to w^2 or to h, which commonly happens, even in bodies which are not very hard, the value of $\int \beta \dot{z}$ will be sensibly proportional to w^2 or to h; so that *the impressions will be nearly as the squares of the velocities, or as the heights due to those velocities.* This will be still nearer the truth when $x = 0$ and $\int \beta \dot{z} = \frac{m}{\varrho} (\frac{1}{2} \mathrm{w}^2 + 32\frac{1}{6} z) = 32\frac{1}{6} \frac{m}{\varrho} (h + z)$: whence we conclude, that *when a hard body falls on another body which is immoveable, the impression made in the latter is inversely as its hardness of resisting force, and directly as the height due to the velocity and the mass of the impinging body conjointly.*

This conclusion corresponds with those deduced from very different processes, at p. 180. of Dr. Hutton's *Select Exercises*, and p. 52 of Mr. Atwood's *Treatise on the Motion of Bodies.*

From the equation $\int \beta \dot{z} = \frac{m}{\varrho} (\frac{1}{2} \mathrm{w}^2 + 32\frac{1}{6} z)$ it appears that however small the initial velocity w may be, provided it be a finite quantity, it will always make an impression, unless ϱ is infinite. It will not be the same if $\mathrm{w} = 0$: for from thence will result $\int \beta \dot{z} = \frac{32\frac{1}{6} m z}{\varrho}$ or $\frac{\int \beta \dot{z}}{z} = 32\frac{1}{6} \frac{m}{\varrho} = \frac{f}{\varrho}$; when $\int \beta \dot{z} = 0$, the depth of the impression will be also nothing, and the equation will become $\frac{0}{0} = 32\frac{1}{6} \frac{m}{\varrho} = \frac{f}{\varrho}$: now $\frac{0}{0}$ being equal to what we please, the impression may consequently be $= 0$, while m, f, and ϱ, are any quantities whatever.

In fact, we may see, *à priori*, that f and ϱ being quantities of the same nature, the effect of each being a pressure, may destroy each other at the first instant; and that it is not the same with regard to w, which, occasioning a shock, can only be destroyed after it has produced an impression.

The preceding formulæ accord very accurately with experiments; as will be sufficiently obvious to those who will compare them with the experiments described in s'Gravesande's *Physices Elementa Mathematica*, § 833, &c., and Mr. Atwood's Treatise just referred to, *passim*.

If the two striking masses are equal, and during the shock are not subjected to the action of any power (but their mutual resistances only) we have $m=\mu$, $f=0$, $\varphi=0$, and $\int\beta\dot{z}=\frac{m}{4\varrho}(\mathrm{w}-w)^2$; and when $w=0$, $\int\beta\dot{z}=\frac{m\mathrm{w}^2}{4\varrho}$. Which again is confirmed by many experiments.

Now since experiments correspond with these formulæ, not only in the case of soft bodies, as clay, but even in those of bodies imperfectly hard and elastic, when the hardness ϱ is supposed constant; it is evident that in these cases the hardness is very nearly constant, and that we may safely suppose it such, as we shall henceforth do.

347. The hypothesis of invariable hardness transforms the equation in art. 340. repeated at the beginning of art. 345. to this:

$$\tfrac{1}{2}m\mu\left(\overline{u-v}^2-\overline{\mathrm{w}-w}^2\right)=(\mu f-m\varphi)\cdot(x+z)-\varrho(m+\mu)\int\beta\dot{z};$$

whence we get $\varrho=\dfrac{\frac{1}{2}m\mu\left(\overline{\mathrm{w}-w}^2-\overline{u-v}^2\right)+(\mu f-m\varphi)\cdot(x+z)}{(m+\mu)\int\beta\dot{z}}$;

and, in the case of the greatest impression,

$$\varrho=\frac{\frac{1}{2}m\mu(\mathrm{w}-w)^2+(\mu f-m\varphi)\cdot(x+z)}{(m+\mu)\int\beta\dot{z}}.$$

Though the first of these equations contains variables which do not enter the second, they will, notwithstanding, give always the same result, provided there be substituted the proper values of u, v, x, z, and $\int\beta\dot{z}$, at any instant of the shock. This naturally results from the hypothesis of invariable hardness, which we see so much reason to adopt.

348. If *the body struck be immoveable*, then $\mu=\infty$, and $w=0$, and the preceding value of ϱ will become $\varrho=\dfrac{\frac{1}{2}m\mathrm{w}^2+f(x+z)}{\int\beta\dot{z}}$. Now if h be, as before, the height due to the velocity w, we have $\mathrm{w}^2=64\frac{1}{3}h=2gh$ (art. 243.) g being $=32\frac{1}{6}$. Substituting

this value of w^2 for it, we have $\varrho = \frac{gmh+f(x+z)}{\int\beta\dot z}$; and if the body fall vertically, then is $f = gm$; whence arises $\varrho = \frac{gm(h+x+z)}{\int\beta\dot z} = \frac{f(h+x+z)}{\int\beta\dot z}$: thus *the hardness or resisting force* ϱ *will be to gravity* f, as $h+x+z$ to $\int\beta\dot z$, or as 1 to $\frac{\int\beta\dot z}{h+x+z}$.
Here, however, we must not lose sight of the circumstance that it is not the relation of these powers considered in their nature but solely in their effects, that may be ascertained; and that f is the product of a mass and velocity.

After we have found the value of ϱ, we may readily find that of r, from the equation $br\dot x = \beta\varrho\dot z$ (art. 335.): for this gives $r\int b\dot x = \varrho\int\beta\dot z$, and consequently $r = \varrho \cdot \frac{\int\beta\dot z}{\int\beta\dot x} = \varrho\int\frac{\beta\dot z}{\beta\dot x}$.

Formulæ such as these may be rendered very useful in determining the relative hardness or resisting force of different bodies. A table of the relative hardness of materials united with a table of their relative densities (or specific gravities) would be of great utility; and particularly in the construction of arches, sluices, &c.

We have all along taken m as the *capacity* of a sphere, because in bodies of the same matter the capacities vary as the weights: in bodies of different matter in order to obtain a constant value of ϱ, we must make $m \propto$ cube of diameter × density.

349. At the instant of the greatest impression the fluxion of P or its equal $\frac{\beta\varrho br}{\beta\varrho+br}$ (art. 334.) is equal to nothing; that is, $\frac{r\varrho(b\dot\beta+\beta\dot b)\cdot(\beta\varrho+br)-br\beta\varrho(\varrho\dot\beta+r\dot b)}{(\beta\varrho+br)^2} = 0$. Performing the multiplications at length, suppressing the quantities which destroy each other, and divided by $\frac{r\varrho}{(\beta\varrho+br)^2}$, we have $\varrho\beta^2\dot b + rb^2\dot\beta = 0$; now this quantity cannot $=0$ except the fluxions $\dot b$ and $\dot\beta$ are likewise $=0$, that is, unless the amplitudes b and β, and consequently the impressions, have received all the augmentation of which they are susceptible. Therefore, *the greatest force of percussion is that which obtains at the moment when the impression is a maximum.*

350. The force of percussion in any instant of the shock is $P = \frac{\beta\varrho br}{\beta\varrho+br}$. Making, then, $\int\beta\dot x = I$, and $\int\beta\dot z = i$, which will make

r (art. 349.) $= \rho \frac{I}{i}$; and introducing this value of r for it in the value of P, it becomes $P = \frac{\rho^2 b\beta i}{I(\beta\rho + b\rho i \div I)} = \frac{\rho b\beta i}{\beta I + bi}$. Lastly, substituting for ρ, in this equation, its final value in art. 347. we thence obtain

$$P = \frac{b\beta}{\beta I + bi} \cdot \frac{\frac{1}{2} m\mu (W - w)^2 + (\mu f - m\phi) \cdot (x + z)}{m + \mu}.$$

Here it must be remarked, that although we have used that value of ρ which suits the greatest impression, yet as ρ is constant, the value of P would refer not less to any other instant of the shock. As x and z are introduced by the substitution of ρ, they are constant, and express the greatest depths of impression: thus there are no other variable quantities in the value of P than β and b.

351. When the body struck is immoveable, we have $\mu = \infty$ $w = 0$, and $\phi = 0$. And if we suppose, moreover, that the body struck is sufficiently hard to receive no impression, then will $I = 0$, and $x = 0$; whence we shall find in this case

$$P = \frac{\beta}{i} \left(\tfrac{1}{2} m W^2 + fz\right).$$

352. Here also, taking h the height due to the velocity W, or $W^2 = 2gh$, and f for the gravity, or $f = gm$; then will the value of P become $P = \frac{\beta}{i} gm (h + z) = \frac{g\beta m}{i} (h + z)$: whence flows this proportion, $P : f :: \frac{g\beta m}{i} (h + z) : gm :: \frac{\beta}{i} (h + z) : 1$. *And this is the relation of the percussive to the gravitating force.*

353. Retaining always the immobility of the body struck, if the impinging body is susceptible of impression, the equation which furnishes the value of the force of percussion will be

$$P = gm \frac{b\beta}{\beta I + bi} (h + x + z)$$

whence it appears that $f : P :: 1 : \frac{b\beta}{\beta I + bi} (h + x + z)$.

If when the body struck is immoveable we have very nearly $b = \beta$, $I = i$, and $x = z$, then will $P = \frac{\beta}{2i} \left(\tfrac{1}{2} m W^2 + 2fz\right)$; and when f is gravity, $P = \frac{g\beta m}{2i} (h + 2z)$: so that, in this case, the percussion will be to gravity as $\frac{\beta}{2i} (h + 2z)$ to unity.

354. When two very hard bodies, as two of iron, for example, are made to strike one another, the impression i which is made in each is nearly infinitely small with respect to $\frac{1}{2}\beta$

$(h+2z)$; therefore, in this case, the force of percussion is almost infinite with respect to gravity. Take, for example, *the stroke of a hammer on an anvil:* since i represents the magnitude of the impression, which must vary as the product of β into a quantity proportional to its depth, which we know by experience to be extremely small; we may suppose $i=\beta\frac{1}{\omega}$, ω expressing any number whatever, such that $\frac{1}{\omega}$ is less than the depth of the impression, supposed less extended at the bottom than at the surface. This depth when it is at the greatest cannot fairly exceed the $\frac{1}{13000}$ or $\frac{1}{12000}$ of a foot. Substituting, therefore, for i, its value $\frac{\beta}{\omega}$, the force of gravity will be to that of percussion as $1 : \beta(h+2z) \div \frac{2\beta}{\omega}$, or as $1 : \frac{\omega}{2}\ (h+2z)$, or finally (neglecting $2z$ because of its extreme minuteness), as $1 : \frac{1}{2}\omega h$. Now, if the velocity of the hammer be equivalent to that which would be acquired by falling freely 10 feet, we shall have $h=10$, and making $\omega=12000$ only, there will result $\frac{1}{2}\omega h=60000$; that is, the weight of the hammer will be to the force of percussion as 1 to 60000. Thus the effect of the hammer is at least equivalent to what would be caused if at the same point where the blow was directed *a weight were laid sixty thousand times as great as that of the hammer.* This will suffice to show the prodigious effects of the force of percussion, even with so moderate a velocity as that due to a free descent though 10 feet.

355. *The duration of the shock* is an interesting particular, to the determination of which we may next proceed: Now we found (art. 337.) $\dot{t}=\frac{\dot{x}+\dot{z}}{u-v}$, and (art. 340.) the equation $u-v=\pm\left((\mathrm{W}-w)^2+\frac{2(\mu f-m\varphi)\cdot(x+z)}{m\mu}-\frac{2(m+\mu)\varrho\int\beta\dot{z}}{m\mu}\right)^{\frac{1}{2}}$. And if we put for $u-v$ in the first equation its value in the second, at the same time making $\frac{2\varrho(m+\mu)}{m\mu}=\varepsilon$, $(\mathrm{W}-w)^2=\gamma$, and $\frac{2(\mu f-m\varphi)}{m\mu}=\delta$, we shall have $\dot{t}=\frac{\dot{x}+\dot{z}}{\pm[\gamma+\delta(x+z)-\varepsilon\int\beta\dot{z}]^{\frac{1}{2}}}$; an equation the integration of which will depend upon the value of β, and the relation of x and z.

We may at first suppose $x=0$, and $\dot{x}=0$; then will the equation become $\dot{t}=\frac{\dot{z}}{\pm(\gamma+\delta z-\varepsilon\int\beta\dot{z})^{\frac{1}{2}}}$. We may also imagine $x=z$, and then will $\dot{t} = \frac{2\dot{z}}{\pm(\gamma + 2\delta z - \varepsilon\int\beta\dot{z})^{\frac{1}{2}}}$. Now it is manifest that by finding the fluents in this second hypothesis solely we may readily infer the value of t in the first; it will merely be necessary to take half the term which contains δ, and after that to take half the value of t.

The value of β depends on the figure, the disposition, and the reciprocal hardness of the striking bodies. So that we may conceive $\int\beta\dot{z}$ equal to any function of z with constant quantities; for although it is not possible that this supposition can suit all bodies, yet we may always determine to what bodies it may be accommodated.

Supposing, then, $\int\beta\dot{z}^2 = \mathrm{c}z^2$, c being a constant quantity, we have $\dot{t} = \frac{2\dot{z}}{\pm(\gamma + 2\delta z - \varepsilon\mathrm{c}z^2)^{\frac{1}{2}}} = \frac{2\dot{z} \div \sqrt{\varepsilon\mathrm{c}}}{\pm\left(\frac{\gamma}{\varepsilon\mathrm{c}} + \frac{\delta^2}{\varepsilon^2\mathrm{c}^2} - \frac{\delta^2}{\varepsilon^2\mathrm{c}^2} + \frac{2\delta z}{\varepsilon\mathrm{c}} - z^2\right)^{\frac{1}{2}}}$: in which we have added $\frac{\delta^2}{\varepsilon^2\mathrm{c}^2} - \frac{\delta^2}{\varepsilon^2\mathrm{c}^2} = 0$, to the denominator, in order that its three last terms with their signs changed should form a complete square. Making $\frac{\gamma}{\varepsilon\mathrm{c}} + \frac{\delta^2}{\varepsilon^2\mathrm{c}^2} = \mathrm{R}^2$, we obtain after a little reduction $\dot{t} = \frac{2}{\mathrm{R}\sqrt{\varepsilon\mathrm{c}}} \cdot \frac{\mathrm{R}\dot{z}}{\pm\left[\mathrm{R}^2 - (z - \frac{\delta}{\varepsilon\mathrm{c}})^2\right]^{\frac{1}{2}}} = \frac{2}{\sqrt{\varepsilon\mathrm{c}}} \cdot \frac{\dot{z} \div \mathrm{R}}{\pm\left[1 - (\frac{z}{\mathrm{R}} - \frac{\delta}{\mathrm{R}\varepsilon\mathrm{c}})^2\right]^{\frac{1}{2}}}$.

Now it is known that if x be an arc of a circle whose radius is unity, and s its sine, we have universally $\dot{x} = \frac{\dot{s}}{\sqrt{1 - s^2}}$; and that the fluxion of the supplement of x is equal to $\frac{\dot{s}}{-\sqrt{1 - s^2}}$. Hence the last value of $\dot{t}$ expresses the product of $\frac{2}{\sqrt{\varepsilon\mathrm{c}}}$ into the fluxion of an arc of a circle whose radius $= 1$, and sine $= \frac{z}{\mathrm{R}} - \frac{\delta}{\mathrm{R}\varepsilon\mathrm{c}}$, or into the fluxion of the supplement of that arc, according as the denominator of the second member has the positive or the negative sign. Thus then, the fluents of this equation will be $t = \frac{2}{\sqrt{\varepsilon\mathrm{c}}} \text{arc } \sin\left(\frac{z}{\mathrm{R}} - \frac{\delta}{\mathrm{R}\varepsilon\mathrm{c}}\right) + \mathrm{M}$.

To determine the constant quantity M it must be remarked that when $t=0$ the depth of impression $z=0$ also; in this case, therefore, observing that negative sines answer to negative arcs, we have $\frac{-2}{\sqrt{\varepsilon\mathrm{C}}}$ arc sin $\frac{\delta}{\mathrm{R}\varepsilon\mathrm{C}}+\mathrm{M}=0$, or $\mathrm{M}=\frac{2}{\sqrt{\varepsilon\mathrm{C}}}$ arc sin $\frac{\delta}{\mathrm{R}\varepsilon\mathrm{C}}$.

Thus the value of t is

$$t=\frac{1}{\frac{1}{2}\sqrt{\varepsilon\mathrm{C}}}\left[\text{arc sin }\left(\frac{z}{\mathrm{R}}-\frac{\delta}{\mathrm{R}\varepsilon\mathrm{C}}\right)+\text{arc sin }\frac{\delta}{\mathrm{R}\varepsilon\mathrm{C}}\right].$$

Hence, restoring the values of γ, δ, and ε, we have

$$t=\sqrt{\frac{2m\mu}{\mathrm{C}\varrho(m+\mu)}}\left[\text{arc sin }\left(\frac{z}{\mathrm{R}}-\frac{\mu f-m\varphi}{\mathrm{RC}\varrho(m+\mu)}\right)+\text{arc sin }\frac{\mu f-m\varphi}{\mathrm{RC}\varrho(m+\mu)}\right].$$

This is the value of t when the positive sign is taken to the denominator of the second member of the preceding fluxional equation; and it then expresses any time elapsed from the commencement of the shock to any instant previous to the greatest impression. To have the time which elapses from the commencement of the shock up to the instant *after* the greatest impression, we must take, instead of the expression arc sin $\left(\frac{z}{\mathrm{R}}-\frac{\mu f-m\varphi}{\mathrm{RC}\varrho(m+\mu)}\right)$, its supplement, that is, $\pi-$ arc sin $\left(\frac{z}{\mathrm{R}}-\frac{\mu f-\mu\varphi}{\mathrm{RC}\varrho(m+\mu)}\right)$, π representing, as heretofore, 3·141593 the semicircumference to radius 1. Hence, for the latter case, we have $t=$

$$\sqrt{\frac{2m\mu}{\mathrm{C}\varrho(m+\mu)}}\left[\pi-\text{arc sin }\left(\frac{z}{\mathrm{R}}-\frac{\mu f-m\varphi}{\mathrm{RC}\varrho(m+\mu)}\right)+\text{arc sin }\frac{\mu f-m\varphi}{\mathrm{RC}\varrho(m+\mu)}\right].$$

These two expressions for the time ought to be equal for the instant of the greatest impression: we have, therefore, for that instant.... $\frac{1}{2}\pi=$ arc sin $\left(\frac{z}{\mathrm{R}}-\frac{\mu f-m\varphi}{\mathrm{RC}\varrho(m+\mu)}\right)$.

Substituting this value in either of the preceding equations for t, we shall have for the *time from the commencement of the shock to the instant of greatest impression*

$$t=\sqrt{\frac{2m\mu}{\mathrm{C}\varrho(m+\mu)}}\left(\tfrac{1}{2}\pi+\text{arc sin }\frac{\mu f-m\varphi}{\mathrm{RC}\varrho(m+\mu)}\right).$$

In bodies perfectly elastic the impression diminishes till $z=0$; substituting, therefore, this value of z in the second equation for the time, we have for the *whole duration of the shock* in such bodies $t=\sqrt{\frac{2m\mu}{\mathrm{C}\varrho(m+\mu)}}\left(\pi+2\text{ arc sin }\frac{\mu f-m\varphi}{\mathrm{RC}\varrho(m+\mu)}\right)$.

Whence it appears that the whole time is *double* that which elapses from the commencement of the shock to the instant of greatest impression: thus shewing the exactness of the last condition in the definition of perfect elasticity (art. 329.)

In bodies *void of elasticity*, the total duration of the shock is equal to the time which elapses between the commencement of the shock and the instant of greatest impression.

If the bodies are not subjected to the operation of any active power, we have $f=0$, and $\varphi=0$: whence the *whole duration* of the shock for *non-elastic bodies* will be $t=\frac{1}{2}\pi\sqrt{\frac{2m\mu}{c\rho(m+\mu)}}$; and for *elastic bodies* $t=\pi\sqrt{\frac{2m\mu}{c\rho(m+\mu)}}$.

The quantity R which contains the initial velocities, not entering either of these expressions, we see that, when no active power animates the bodies, the time of *duration of the shock is always the same* (cæt. par.), *whatever are the initial velocities.*

The value of R is, as we have seen,

$$R=\sqrt{\left(\frac{\gamma}{\rho c}+\frac{\delta^2}{\rho^2c^2}\right)}=\left(\frac{\frac{1}{2}m\mu(W-w)^2}{c\rho(m+\mu)}+\left(\frac{\mu f-m\varphi}{c\rho(m+\mu)}\right)^2\right)^{\frac{1}{2}}$$

Now if W be $=0$, and $w=0$, that is, if the bodies are merely actuated by powers, the value of R will become $R=\frac{\mu f-m\varphi}{c\rho(m+\mu)}$

Substituting this value in the expressions for the whole time, they will become,

For non-elastic bodies $t=\sqrt{\frac{2m\mu}{c\rho(m+\mu)}}\left(\frac{1}{2}\pi+\text{arc sin }(1)\right)=\pi\sqrt{\frac{2m\mu}{c\rho(m+\mu)}}$:

For elastic bodies $t=2\pi\sqrt{\frac{2m\mu}{c\rho(m+\mu)}}$.

Here it will be observed that sin (1) = radius, and arc sin (1) $=90°$. Hence it follows, 1st, That the duration of the shocks when the bodies are only actuated by powers, or accelerating forces, without initial velocities, is double the duration of the shock when they only move with certain velocities. 2dly, That the value of these powers or accelerating forces, not entering the above equations, the duration of the shock is the same, whatever are the powers or accelerating forces which act upon the bodies. It must be recollected, though, that the accelerating forces are supposed constant.

356. The quantities c and ρ which enter the preceding equations being embarrassing in the computation, it will not be improper to find values of t which do not contain those quantities, and by means of which we may deduce the value of t immediately from experiments.

Let x be the greatest depth of impression; then, because $\int\beta\dot{z}=cz^2$, we have $cx^2=i$; and from the hypothesis of $x=z$

we have $\varrho = \frac{\frac{1}{2}m\mu(\mathrm{W}+w)^2 + (\mu f - \mu\Phi)2\mathrm{X}}{(m+\mu)\mathrm{C}\mathrm{X}^2}$; whence we find $(m+\mu)\varrho\mathrm{C} = \frac{\frac{1}{2}m\mu(\mathrm{W}-w)^2 + (\mu f - m\varphi)\,2\mathrm{X}}{\mathrm{X}^2}$. We have also

$$\mathrm{R}^2 = \frac{\frac{1}{2}m\mu(\mathrm{W}-w)^2}{\mathrm{C}\varrho(m+\mu)} + \left(\frac{\mu f - m\varphi}{\mathrm{C}\varrho(m+\mu)}\right)^2.$$

Multiplying the two terms by $[\mathrm{C}\varrho\,(m+\mu)]^2$, substituting for $\mathrm{C}\varrho\,(m+\mu)$ its value, reducing the whole to a fraction, and exracting the roots, we shall have at length

$$\mathrm{RC}\varrho\,(m+\mu)\frac{\frac{1}{2}m\mu(\mathrm{W}-w)^2 + \mathrm{X}(\mu f - m\varphi)}{\mathrm{X}}.$$

Substituting these values of $\mathrm{RC}\varrho$, and of $\mathrm{C}\varrho$, in the preceding values of t, we have, for *non-elastic bodies*,

$$t = \sqrt{\frac{2m\mu\mathrm{X}^2}{\frac{1}{2}m\mu(\mathrm{W}-w)^2 + 2\mathrm{X}(\mu f - m\Phi)}}\left(\tfrac{1}{2}\pi + \text{arc sin}\ \frac{\mathrm{X}(\mu f - m\varphi)}{\frac{1}{2}m\mu(\mathrm{W}-w)^2 + \mathrm{X}(\mu f - m\varphi)}\right).$$

And for *perfectly elastic bodies*,

$$t = \sqrt{\frac{2m\mu\mathrm{X}^2}{\frac{1}{2}m\mu(\mathrm{W}-w)^2 + 2\mathrm{X}(\mu f - m\varphi)}}\left(\pi + 2\ \text{arc sin}\ \frac{\mathrm{X}(\mu f - m\phi)}{\frac{1}{2}m\mu(\mathrm{W}-w)^2 + \mathrm{X}(\mu f - m\phi)}\right).$$

In these equations neither C, ϱ, nor R, is to be found; so that they comprise merely such quantities as result at once from experience.

When $f=0$, and $\phi=0$, the preceding equations become, for *non-elastic bodies* $t = \frac{\pi\mathrm{X}}{\mathrm{W}-w}$; for *elastic bodies* $t = \frac{2\pi\mathrm{X}}{\mathrm{W}-w}$.

It must be observed that in the equation $t = \frac{\pi\mathrm{X}}{\mathrm{W}-w}$ the difference $\mathrm{W}-w$ may remain the same, though the absolute values of W and w vary; so that this equation by no means contradicts the observation made in the preceding article, that; when the accelerating forces vanish, the times are independent of the initial velocities.

If the bodies are only animated by powers, or only act by mutual pressure, then are $\mathrm{W}=0$ and $w=0$; and the values of t become $t=\pi\sqrt{\frac{m\mu\mathrm{X}}{\mu f - m\varphi}}$, and $t=2\pi\sqrt{\frac{m\mu\mathrm{X}}{\mu f - m\varphi}}$: if the body struck is immoveable, these equations become $t=\pi\sqrt{\frac{m\mathrm{X}}{f}}$, and $t=2\pi\sqrt{\frac{m\mathrm{X}}{f}}$. If we suppose f to be gravity, then $f=gm$, and $t=\pi\sqrt{\frac{\mathrm{x}}{g}}$, or $t=2\pi\sqrt{\frac{\mathrm{x}}{g}}$, in the respective cases of non-elastic and elastic bodies; g denoting as usual $32\frac{1}{6}$.

Retaining the hypothesis of the immobility of the body struck, we have $\mu=\infty$, and of consequence $w=0$; then will the expression for the time of greatest impression become

$$t=\sqrt{\frac{2m\mathrm{x}^2}{\frac{1}{2}m\mathrm{w}^2+2f\mathrm{x}}}\left(\tfrac{1}{2}\pi+\text{arc sin }\frac{f\mathrm{x}}{\frac{1}{2}m\mathrm{w}^2+f\mathrm{x}}\right).$$

If the striking body is moved by gravity, h being the height due to the velocity w, we have $\mathrm{w}^2=2gh$, and $f=gm$. Substituting these values of w^2 and f for them, there will result

$$t=\frac{\mathrm{x}}{\sqrt{\frac{1}{2}g\,(h+2\,\mathrm{x})}}\left(\tfrac{1}{2}\pi+\text{arc sin}\frac{\mathrm{x}}{h+\mathrm{x}}\right).$$

When x is extremely small with respect to h, this equation will become $t=\frac{\pi\mathrm{x}}{\frac{1}{4}g\sqrt{h}}=\cdot 39079\frac{1}{2}\,\mathrm{x}\sqrt{h}$. This theorem would apply to the case of a hammer striking on an anvil.

357. In a similar manner we may find the time in which the greatest impressions are formed, on the supposition that $x=0$, and $\dot{x}=0$: for, we have seen (art. 355.) that in order to this we must take the half of the term multiplied by $\delta=\frac{\mu f-m\varphi}{\frac{1}{2}\mu m}$, and then take half the value of t; thus we shall have

$$t=\sqrt{\frac{m\mu\mathrm{x}^2}{m\mu(\mathrm{w}-w)^2+2(\mu f-m\varphi)\mathrm{x}}}\left(\tfrac{1}{2}\pi+\text{arc sin}\right.$$

$$\left.\frac{(\mu f-m\varphi)\mathrm{x}}{m\mu(\mathrm{w}-w)^2+(\mu f-m\varphi)\,\mathrm{x}}\right).$$

The method will be the same for every other case, by finding the fluents of the general equation,

$$\dot{t}=\frac{\dot{x}+\dot{z}}{\pm\left[(\mathrm{w}-w)^2+\frac{(\mu f-m\varphi)x+z)-(m+\mu)\varrho\int\beta\dot{z}}{\frac{1}{2}m\mu}\right]^{\frac{1}{2}}}.$$

We have found (art. 340.) the equation

$$m\mu(\dot{u}-\dot{v})=[\mu f-m\varphi-(m+\mu)\,\mathrm{P}]\,\dot{t};$$

if in this equation we suppose P constant, its fluent will be $m\mu(u-v)=[\mu f-m\varphi-(m+\mu)t+\mathrm{F}$; when $\mathrm{P}t=0$, $u=\mathrm{w}$, and $v=w$, whence we have then $m\mu\,(\mathrm{w}-w)=\mathrm{F}$, and consequently $m\mu[(u-v)-(\mathrm{w}-w)]=[\mu f-m\varphi-(m+\mu)\mathrm{P}]\,t$, or, by division, $t=\frac{m\mu[(u-v)-[\mathrm{w}-w)]}{(\mu f-m\varphi)-(m+\mu)\mathrm{P}}$. At the instant of the greatest impression $u-v=0$, and the equation becomes $t=\frac{m\mu(\mathrm{w}-w)}{(m+\mu)\,\mathrm{P}-(\mu f-m\varphi)}$.

If the body struck is immoveable, $\varphi = \infty$, $w = 0$, and $\varphi = 0$; wherefore $t = \frac{m\mathrm{W}}{\mathrm{P} - f}$.

When $f = 0$, and $\varphi = 0$, then $t = \frac{m\mu(\mathrm{W} - w)}{(m + \mu)\mathrm{P}}$: and if $\mu = \infty$, and $w = 0$, $t = \frac{m\mathrm{W}}{\mathrm{P}}$.

In these equations the value of P may be found, as in arts. 350—355.

358. The application of the preceding calculus depends principally on the determination of the impression: and this determination is sometimes attended with difficulties. For, notwithstanding that many authors have supposed that the figure of the impression is generally the same as that of the impinging body, it is manifest that this supposition will not be accurate with regard to hard and tenacious bodies. In a great number of these latter the amplitude of the impression is always much the largest; for the parts contiguous to the surface of contact, being not easily detached from the neighbouring parts, yield to the impulsion at the same time that this surface carries before it those which touch, and so on successively. Hence the diameter of the actual impression is always greater than the diameter of the surface of contact; so that it is difficult to obtain an exact measure of the real impression.

This remark, however, true as it is, cannot be applied to all cases without some modification; because there are some forms of the impinging body, such as the spherical, and convex forms in general, which greatly diminish this excess of the diameter of the impression over that of the surface of contact: indeed this excess may often be considered as entirely vanished, when the body struck is not of an extreme hardness and tenacity.

359. It will appear, from the perusal of this chapter, that the theory is made to flow from facts adverted to in art. 329. and serves for their explication. We hence see that when we know the physical circumstances of a percussion we may always assign an equivalent pressure, which will put it in equilibrio. It is not, therefore, absurd to say that we can weigh or balance the blow of a mass, or the stroke of a hammer; it is solely necessary to observe that the stroke of a hammer or other mass is not an absolute weight, but that such weight depends on the hardness, the form, and the nature of the bodies striking and struck. Thus, in general, when a resistance of pressure is equivalent to a very considerable weight, it will resist the shock of a mass whose weight is small in comparison with

the former, and of which the velocity is that due to a moderate height.

360. It has been all along supposed that the direction of the stroke is the same as that of the motion: but this is far from being always the case; and when it is not, the *impact* is called *oblique*. To consider this in its utmost extent would carry us very far indeed: but we have not room for a particular investigation. We shall, therefore, just mention two or three general facts already proved; and from them deduce a few easy cases. These facts are: 1st, That the actions of bodies on each other depend upon their relative motions. 2dly, That the motion of the common centre is not changed by the collision. By these we can reduce all to the case of a body in motion striking another at rest. Now the relative motion may be determined by the construction in art. 219. and to this must be superadded the common motion which changes the relative into the true motions.

If, for example, two bodies A B, (fig. 2, pl. XV.) describe the lines AD, BD, and meet in D, the collision is the same as if B had remained at rest, and A had struck it with the direction and velocity AB. In the mean time, the common centre of inertia has described CD with a uniform velocity, which at the end of an equal portion of time will carry it to *c*, CD being =D*c*. If the bodies are non-elastic they remain together, and will proceed along D*c*; their common velocity being represented by D*c*, while AD, BD, represent the individual velocities of A and B previous to the impact. If the bodies are imperfectly elastic, draw through *c* the line *ab* parallel to AB; make AC to *ac*, as the force of compression to the force of restitution, with regard to the body A; and make BC to *bc* as the force of compression to the restitutive force, relatively to B; join D*a*, D*b*, and those lines will be the paths of the bodies after collision. If the bodies are perfectly elastic, make *cb*=CB, *ac*=AC; and D*a*, D*b*, will be the paths of the bodies. If AB be perpendicular to C*c*, then will angle ADC=*a*DC, and *b*DC=*b*DC. And in a similar way it naturally follows, that if a body A, perfectly elastic, impinge upon a perfectly hard plane C*c*, in the direction AD, it will rebound from D in such a line D*a*, that *the angles of incidence and reflection* ADC, *a*DC, *shall be equal.*

361. On this latter principle depends the solution of the problems often proposed relative to the game of billiards: the reader may take the following as a specimen.

To find in what points C, D, *on the two sides* KL, LI, *of a rectangular billiard table, an elastic ball placed at* B *must strike, so that it may hit a ball placed at* A, *after a double reflexion.*

From the point A (fig. 3. pl. XV.) draw AH perpendicular to IL, and produce it till IH = IA : parallel to LI draw HF, and make it equal to 2GH, the point G being in KL produced. Join the points F, B, by a right line cutting KL in C, and join the points C, H, by a right line cutting LI in D; then shall C and D be the points required. For the angles ADI, CDL, are each equal to IDH, and the angles FCG, GCH, each equal to BCK; whence the truth of the construction is obvious.

After methods slightly varied might several entertaining problems be solved: but such great simplicity is not to be expected in many of the cases of oblique collision which actually occur. We shall, therefore, conclude this chapter with a general proposition (from Dr. Robison), by means of which the different motions, whether progressive or rotatory, may in general be determined without much trouble; and which, in fact, requires merely a combination of some of the principles already exhibited in the preceding and present chapters.

362. *Let the body* A (fig. 4. pl. XV.) *moving with the velocity* V *in the direction* AD, *strike the body* B *at rest*; *to determine the circumstances resulting from the shock.*

Let F be the point of mutual contact, and *h*FH a plane touching both bodies in F. Draw AFP perpendicular to this tangent plane, and through G, the centre of position of B, draw PGC perpendicular to FP, and GI parallel to FP. Let C, in the line PG, be the spontaneous centre of conversion corresponding to the point of percussion F. Join CF. Let the direction cut the tangent plane in H, and PF in A; and let AH represent the velocity V.

The impulse is made at the point F, in the direction AF or FP; and the centre of inertia of the body B will advance in the direction GI, parallel to FP, the direction of the effective impulse. But, because this does not pass through the centre G, the body will advance, and will also turn round an axis passing through G, perpendicular to the plane of the lines GP, PF, and the spontaneous axis of conversion will pass through some point C of the line PG, and will also be perpendicular to the same plane. Complete the parallelogram AFHE. It is plain that the motion AH is equivalent to AE and AF. By the motion AE, A only slides along the surface of B without pressing it, or causing any tendency to motion in that direction, except perhaps a little arising from friction. It is by the motion AF alone that the impulse is made. Therefore let v be $= V \frac{AF}{AH}$; and then $A \cdot v$ may be called the *efficient impulse* of the body A in the present circumstances, and v the *efficient ve-*

locity. This will be diminished by the collision. Let x be the unknown velocity remaining in A after the collision, or rather, in the instant of the greatest compression and common motion of the touching points of A and B, estimated in the direction FP. The effective momentum lost by A must therefore be A $(v-x)$: but the same must be gained by B, and its centre G must move in the direction GI, parallel to FP, with this momentum; and therefore with the velocity $\frac{A(v-x)}{B}$. That this may be the case, the point of percussion F must yield with the velocity x, because the bodies are in contact. But because C is the spontaneous axis of conversion, every particle is *beginning* to describe an arch of a circle round this axis. Therefore F is beginning to move in the direction Fg, perpendicular to the momentary radius vector CF. Let Fg be a very minute arch, described in a moment of time. Draw gf perpendicular to FP. Then Ff is the motion Fg reduced to the direction FP, and will express the yielding of B in the direction of the impulse, while G describes a space equal to $\frac{A(v-x)}{B}$, and A describes a space x. Therefore Fg will express x. Let Pp be the space described in the same time that Fg is described. Draw pC, cutting GK in the point I. GI is the yielding of the body B to the impulse, and must therefore be equal to $\frac{A(v-x)}{B}$.

The triangles Ffg and CPF are similar; for the angle CFP is the complement of fFg to a right angle: it is also the complement of PCF to a right angle. Therefore Fg : Ff : : FC : CP. But Fg : Pp : : FC : CP; because the little arches Fg, Pp, have the same angle at C. Therefore Pp = Ff, = x. It is plain that CG : CP : : GI : Pp. Therefore CG ; CP : $\frac{A(v-x)}{B}$: x, and $x = \frac{A(v-x)CP}{B \cdot CG}$, or $x = v\frac{A \cdot CP}{B \cdot CG} - x\frac{A \cdot CP}{B \cdot CG}$; wherefore $x \cdot B \cdot CG + x \cdot A \cdot CP = v \cdot A \cdot CP$, and $x\ (B \cdot CG + A \cdot CP) = v \cdot A \cdot CP$, and $x = v\frac{A \cdot CP}{B \cdot CG + A \cdot CP}$, = the velocity remaining in A, estimated in the direction FP.

And u, the velocity with which G will advance, is $x\frac{CG}{CP}$; for CP : CG : : Pp : GI : : x : u. It is evident that A will change its direction by the collision: for in the instant of greatest compression it was reacted on by a force = A $(v-x)$ in the direction FA. This must be compounded with A·V, in the direction AH, in order to obtain the new motion of A; or it may be found by compounding x, which is retained by A, with FH, which has suffered no change by the collision. *The bodies will, therefore, separate, although they be unelastic.* If they are perfectly elastic we *double* these changes in each.

If B were also in motion before the collision, the motion of A must be resolved into two, one of which is equal and parallel to the motion of B; and the other must be employed in the same manner as AH in the preceding part of this article.

CHAPTER VI.

On the Motion of Machines, and their Maximum Effects.

363. When forces acting in contrary directions, or in any such directions as produce contrary effects, are applied to machines, there is, with respect to every simple machine (and of consequence with respect to every combination of simple machines) a certain relation between the powers and the distances at which they act, which, if subsisting in any such machine when at rest, will always keep it in a state of rest, or of *statical* equilibrium (art. 28.); and for this reason, because the efforts of these powers, when thus related, with regard to magnitude and distance, being equal and opposite, annihilate each other, and have no tendency to change the state of the system to which they are applied. So also, if the same machine have been put into a state of *uniform* motion, whether rectilinear or rotatory, by the action of any power distinct from those we are now considering, and these two powers be made to act upon the machine in such motion in a similar manner to that in which they acted upon it when at rest, their simultaneous action will preserve it in that state of uniform motion, or of *dynamical* equilibrium (art. 28.); and this for the same reason as before, because their contrary effects destroy each other, and have therefore no tendency to change the *state* of the machine. But, if at the time a machine is in a state of balanced rest, any one of the opposite forces be increased while it continues to act at the same distance, this excess of force will disturb the statical equilibrium, and produce motion in the machine; and if the same excess of force continues to act in the same manner, it will, like every constant force, produce an accelerated motion; or, if it should undergo particular modifications when the machine is in different positions, it may occasion such variations in the motion as will render it alternately accelerated and retarded. Or the different species of resistance to which a moving machine is subjected, as the rigidity of ropes, friction, resistance

of the air, &c. may so modify a motion as to change a regular or irregular variable motion into one which is uniform.

364. Hence, then, the motion of machines may be considered as of *three* kinds. 1. That which is gradually accelerated, which obtains commonly in the first instants of the communication. 2. That which is entirely uniform. 3. That which is alternately accelerated and retarded. Pendulum clocks, and machines which are moved by a balance, are related to the third class. Most other machines, a short time after their motion is commenced, fall under the second. Now although the motion of a machine is alternately accelerated and retarded, it may, notwithstanding, be measured by a uniform motion, because of the periodical and regular repetition which may exist in the acceleration and retardation. Thus the motion of a second pendulum, considered in relation to a single oscillation, is accelerated during the first half second, and retarded during the second: but the same motion taken for many oscillations may be considered as uniform. Suppose, for example, that the extent of each oscillation is five inches, and that the pendulum has made ten oscillations: its total effect will be to have run over 50 inches in 10 seconds; and, as the space described in each second is the same, we may compare the effect to that produced by a moveable which moves for 10 seconds with a velocity of 5 inches per second. We see, therefore, that the theory of machines whose motions are uniform, conduces naturally to the estimation of the effects produced by machines whose motion is alternately accelerated and retarded: so that the problems comprised in this chapter will be directed to those machines whose motions fall under the first two heads; such problems being of far the greatest utility in practice.

365. DEFS. 1. When in a machine there is a system of forces or of powers mutually in opposition, those which produce or tend to produce a certain effect are called *movers*, or *moving powers;* and those which produce or tend to produce an effect which opposes those of the moving powers are called *resistances.* If various movers act at the same time, their equivalent (found by the theorems in book I. chap. i.) is called individually *the moving force;* and, in like manner, the resultant of all the resistances reduced to some one point, *the resistance.* This reduction in all cases simplifies the investigation.

2. The *impelled point* of a machine is that to which the action of the moving power may be considered as immediately applied; and the *working point* is that where the resistance arising from the work to be performed immediately acts, or to which it ought all to be reduced. Thus in the wheel and axle (fig. 6. pl. V.) where the moving power P is to overcome the

weight or resistance W, by the application of the cords to the wheel and to the axle, A is the impelled point, and B the working point.

3. The *velocity of the moving power* is the same as the velocity of the impelled point; the *velocity of resistance* the same as that of the working point.

4. The *performance* or *effect* of a machine, or the *work done*, is measured by the product of the resistance into the velocity of the working point; the *momentum of impulse* is measured by the product of the moving force into the velocity of the impelled point.

These definitions being established, we may now exhibit a few of the most useful problems, giving as much variety in their solutions as may render one or other of the methods of easy application to any other cases which may occur.

366. **Prop.** *If* R *and* r *be the distances of the power* P, *and the weight or resistance* W *from the fulcrum* F *of a straight lever* (fig. 1. pl. IV.), *then will the velocity of the power and of the weight at the end of any time* t *be* $\frac{R^2P - RrW}{R^2P + r^2W}gt$, *and* $\frac{RrP - r^2W}{R^2P + r^2W}gt$, *respectively, the weight and inertia of the lever itself not being considered.*

If the effort of the power balanced that of the resistance, P would be equal to $\frac{rW}{R}$. Consequently, the difference between this value of P and its actual value, or $P - \frac{r}{R}W$ will be the force which tends to move the lever. And because this power applied to the point A accelerates the masses P and W, the mass to be substituted for W in the point A must be $\frac{r^2}{R^2}W$ (art. 310. cor. 4.) in order that this mass at the distance R may be equally accelerated with the mass W at the distance R. Hence the power $P - \frac{r}{R}W$ will accelerate the quantity of matter $P + \frac{r^2}{R^2}W$; and the accelerating force $F = (P - \frac{r}{R}W) \div (P + \frac{r^2}{R^2}W) = \frac{PR^2 - RrW}{PR^2 + r^2W}$. But (art. 228.) $v \propto Ft$ or is $= gtF$; which in this case $= \frac{R^2P - RrW}{R^2P + r^2W}gt$, the velocity of P. And, because veloc. of P : veloc. of W :: R : r, we have veloc. of W $= \frac{r}{R}$ veloc. of P $= \frac{r}{R} \cdot \frac{R^2P - RrW}{R^2P + r^2W}gt = \frac{RrP - r^2W}{R^2P + r^2W}gt$.

Cor. 1. The space described by the power in the time t will be $= \frac{R^2P - RrW}{R^2P + r^2W} \cdot \frac{1}{2}gt^2$; the space described by W in the same time will be $= \frac{RrP - r^2W}{R^2P + r^2W} \cdot \frac{1}{2}gt^2$.

COR. 2. If $R : r :: n : 1$, then will the force which accelerates A be $= \frac{Pn^2 - Wn}{Pn^2 + W}$.

COR. 3. If at the same time the inertia of the moving force P be $= 0$, as in muscular action, the force accelerating A will be $= \frac{Pn^2 - Wn}{W}$, obviously greater than the former.

COR. 4. If the mass moved have no weight, but possess inertia only, as when a body is moved along a horizontal plane, the force which accelerates A will be $= \frac{Pn^2}{Pn^2 + W}$. And either of these values may be readily introduced into the investigation.

COR. 5. The work done in the time *t*, if we retain the original notation, will be $= \frac{RrP - r^2W}{R^2P + r^2W} gt \times W = \frac{RrPW - r^2W^2}{R^2P + r^2W} gt$.

COR. 6. When the work done is to be a maximum, and we wish to know the weight when P is given, we must make the fluxion of the last expression $= 0$. Then we shall have $rR^3P^2 - 2r^2R^2PW - r^4W^2 = 0$ and $W = P\left(\sqrt{\left(\frac{R^4}{r^4} + \frac{R^3}{r^3}\right)} - \frac{R^2}{r^2}\right)$.

COR. 7. If $R : r :: n : 1$, the preceding expression will become $W = P\left[\sqrt{(n^4 + n^3)} - n^2\right]$.

COR. 8. When the arms of the lever are equal in length, that is, when $n = 1$, then is $W = P(\sqrt{2} - 1) = \cdot 414214\,P$, or nearly $\frac{5}{12}$ of the moving force.

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367. If we compare the values of *s* and *v* in this proposition and first corollary with those in the fourth example, art. 267. which relates to motion on the axis in peritrochio, it will be seen, as ought evidently to be the case, that the expressions correspond exactly. Hence it follows, that when it is required to proportion the power and weight so as to obtain a maximum effect on the wheel and axle (the weight of the machinery not being considered), we may adopt the conclusions of cors. 6 and 7 of this prop. And in the extreme case where the wheel and axle becomes a pulley, the expression in cor. 8. may be adopted. The like conclusions may be applied to machines in general, if R and *r* represent the distances of the impelled and working points from the axis of motion; and if the various kinds of resistance arising from friction, stiffness of ropes, &c. be pro-

perly reduced to their equivalents at the working points, so as to be comprehended in the character W for resistance overcome.

367. A. If instead of being required to find the greatest possible *momentum* by means of a *fixed pulley* in a given time t, it had been required to find the *greatest load*, which can be raised by a *given weight* through a given space s in a *given time* t; then the portion to be raised each time may be thus found. Let t' the time of one ascent, n the number of ascents, then $t = nt'$, or $n = \frac{t}{t'}$: P, as above, the given weight, and W the weight to be raised. Then, by the theorems for accelerating forces, we have $t' = \sqrt{\frac{s}{\frac{1}{2}gF}}$; and consequently $n = \frac{t}{t'} = t \div \sqrt{\frac{s}{\frac{1}{2}gF}} = t\sqrt{\frac{gF}{2s}} = t\sqrt{\frac{g}{2s}} \times \sqrt{\frac{P-W}{P+W}}$, because $\frac{P-W}{P+W}$ is manifestly equal to F. But the *whole load* is equal to nP; hence $t\sqrt{\frac{g}{2s}} \times \sqrt{\frac{P-W}{P+W}} \times W$ is to be a max. or, striking out the constant quantities, and squaring, $\frac{PW^2 - W^3}{P+W}$ = a max. In fluxions, $(2PW\dot{W} - 3W^2\dot{W})(P + W) - \dot{W}(PW^2 - W^3) = 0$. Hence $W^2 + PW - P^2 = 0$, and $W = \frac{1}{2}P(\sqrt{5} - 1)$, or $P : W :: 8 : 5$, very nearly. Consequently the greatest *quantity* of materials which will be raised in a given time under the specified conditions, will be when the quantity raised at each ascent will be about $\frac{5}{8}$ths of the given weight. This corresponds with the result of a more general proposition treated in art. 375.

368. PROP. *Given* R *and* r, *the arms of a straight lever* M *and* m, *their respective weights, and* P *the power acting at the extremity of the arm* R, *to find the weight raised at the extremity of the other arm when the effect is a maximum.*

In this case $\frac{m}{2}$ is the weight of the shorter end reduced to B (fig. 1. pl. IV.), and consequently $\frac{mr}{2R}$ is the weight which, applied at A, would balance the shorter end: therefore, $\frac{mr}{2R} + \frac{r}{R}W$, would sustain both the shorter end and the weight W in equilibrio. But $P + \frac{1}{2}M$ is the power really acting at the longer end of the lever; consequently $P + \frac{1}{2}M - \left(\frac{mr}{2R} + \frac{r}{R}W\right)$, is the absolute *moving* power. Now by art. 312. the distance of the centre of gyration of the beam from F is $= \sqrt{\frac{R^3 + r^3}{3(R+r)}}$, which let be denoted by ϱ; then (by cor. 4. art. 310.) $\frac{\varrho^2}{R^2}\cdot(M+m)$ will represent the mass equivalent to the beam or lever when reduced to the point A; while the weight equivalent to W, when referred to that point, will be $\frac{r^2}{R^2}W$. Hence, proceeding as in the last prop. we shall have $\frac{\varrho^2}{R^2}\cdot(M+m)+P+\frac{r^2}{R^2}W$ for the inertia to be overcome; and $(P+\frac{1}{2}M-\frac{mr}{2R}-\frac{r}{R}W)\div\frac{\varrho^2}{R^2}(M+m)+P+\frac{r^2}{R^2}W=$ the accelerating force of P, or of W reduced to A. Multiply this by W, and, for the sake of simplifying the process, put q for $P+\frac{1}{2}M-\frac{rm}{2R}$, and n for $P+\frac{\varrho^2}{R^2}(M+m)$ then we shall have $\dfrac{qW-\frac{rW^2}{R}}{n+\frac{r^2}{R^2}W}$, a quantity which varies as the effect varies, and which, indeed, when multiplied by gt, denotes the effect itself. Putting the fluxion of this equal to nothing, and reducing, we at length find $W=\frac{R}{r}\sqrt{\left(\frac{nqR}{r}+\frac{n^2R^2}{r^2}\right)}-\frac{nR^2}{r^2}$.

COR. When $R=r$, and $M=m$, if we restore the values of n and q the expression will become $W=\sqrt{(2P^2+2mP+\frac{4}{9}m^2)}-(P+\frac{2}{3}m)$.

369. PROP. *Given the length* l *and angle* e *of elevation of an inclined plane* BC (fig. 5. pl. XV.), *to find the length* L *of another inclined plane* AC, *along which a given weight* W *shall be raised from the horizontal line* AB *to the point* C, *in the least time possible, by means of another given weight* P *descending along the given plane* CB: *the two weights being connected by an inextensible thread* PCW *running always parallel to the two planes.*

Let the angle of elevation CAD be denoted by E, and let the expression for the space passed over in a given time, found art. 267. II. be accommodated to the present notation; it will then be $s=\frac{P\sin e-W\sin E}{P+W}\,\frac{1}{2}gt^2$. This gives $t^2=\frac{s(P+W)}{\frac{1}{2}g(P\sin e-W\sin e)}$. But in the triangle ABC we have AC : BC :: sin B : sin A, that is, L : l :: sin e : sin E; hence $\frac{1}{m}L=\sin e$, and $\frac{1}{m}l=\sin E$; m being a constant quantity always determinable from the data given. And t^2 becomes $\frac{s(P+W)}{\frac{1}{2}g\frac{1}{m}(PL-Wl)}$. Now when any quantity, as t, is a minimum, its square is manifestly a minimum: so that, substituting for s its equal L, and striking out the constant factors, we have $\frac{L^2}{PL-Wl}$ = a min. or its fluxion $\frac{2L\dot{L}(PL-Wl)PL^2\dot{L}}{(PL-Wl)^2}=0$. Here, as in all similar cases, since the fraction vanishes, its numerator must be equal to 0; consequently $2PL^2 - 2WlL - PL^2 = 0$, $PL = 2Wl$, or $L : l :: 2W : P$.

Cor. 1. Since neither sin e nor sin E enters the final equation, it follows, that if the elevation of the plane BC is not given the problem is unlimited.

Cor. 2. When sin $e = 1$, BC coincides with the perpendicular CD, and the power P acts with all its intensity upon the weight W. This is the case of the present problem which has commonly been considered.

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370. This proposition admits of a neat *geometrical* demonstration. Thus, let CE (fig. 5, pl. XV.) be the plane upon which, if W were placed, it would be sustained in equilibrio by the power P upon the plane CB, or the power P′ hanging freely in the vertical CD; then (art. 154. cor. 5, 6.) BC : CD : CE :: P : P′ : W. But W is to the force with which it tends to descend along the plane CA as CA to CD; consequently, the weight P′ is to that force as CA : CE; or the weight P upon the plane BC is to the same force in the same ratio; because either of these weights in their respective positions would sustain W on CE. Therefore the excess of P above that force (which excess is the power accelerating the motions of P and W) is to P as CA − CE to CA; or, taking CH = CA, as EH to CA. Now, the motion being uniformly accelerated, we have $S \propto FT^2$, or $T^2 \propto \frac{S}{F}$: consequently, the square of the time in which AC is described by W will be as AC directly, and as $\frac{EH}{AC}$ inversely; and will be least when $\frac{CA^2}{EH}$ is a *minimum;* that is, when $\frac{CE^2}{EH} + EH + 2CE$, or (because 2CE is invariable) when $\frac{CE^2}{EH} + EH$ is a minimum. Now, as when the sum of two quantities is given their product is a *maximum* when they are equal to each other; so it is manifest that when their product is given their sum must be a *minimum* when they are equal. But the product of $\frac{CE^2}{EH}$ and EH is CE^2, and consequently given; therefore, the sum of $\frac{EC^2}{EH}$, and EH is least when those parts are equal; that is, when

EH = CE, or CA = 2CE. So that the length of the plane CA is double the length of that on which the weight W would be kept in equilibrio by P acting along CB.

When CD and CB coincide the case becomes the same as that considered by that admirable mathematician Mr. Maclaurin, in his *View of Newton's Philosophical Discoveries,* pa. 183, 8vo. ed. Whence, with a very slight variation, the method in this scholium was deduced.

371. PROP. *Let the given weight* P (fig. 5.) *descend along* CB, *and by means of the thread* PCW *(running parallel to the planes) draw a weight* W *up the plane* AC: *it is required to find the value of* W, *when its momentum is a maximum, the lengths and positions of the planes being given.*

The general expression for the velocity is $v = \frac{P \sin e - W \sin E}{P+W} gt$ (art. 267. ex. II.); which, by substituting $\frac{1}{m}L$ for $\sin e$, and $\frac{1}{m}l$ for $\sin E$, becomes $v = \frac{\frac{1}{m}(PL - Wl)}{P+W} gt$. This multiplied into W gives $\frac{\frac{1}{m}(PWL - W^2 l)}{P+W} gt$; which, by the prop. is to be a maximum. Or, striking out the constant factors, $\frac{1}{m}$, gt, we have $\frac{WL - W^2 l}{P+W}$ = a max. Putting this into fluxions, and reducing, we have $P^2 L - 2PWl - W^2 l = 0$, or $W = P\sqrt{\left(\frac{L}{l} + 1\right)} - P$.

COR. When the inclinations of the planes are equal, L and l are equal, and $W = P\sqrt{2} - P = P(\sqrt{2} - 1) = \cdot 4142\,P$; agreeing with the conclusion of the lever of equal arms, or the extreme case of the wheel and axle, *i. e.* the pulley. Art. 366, cor. 7. art. 367.

372. PROP. *Given the radius* R *of a wheel, and the radius* r *of its axle, the weight of both,* w, *and the distance of the centre of gyration from the axis of motion,* ρ; *also a given power* P *acting at the circumference of the wheel; to find the weight* W *raised by a cord folding about the axle, so that its momentum shall be a maximum.*

The force which absolutely impels the point A (fig. 6. pl. V.) is P, while W acts in a direction contrary to P, with a force $= \frac{rW}{R}$; this, therefore, subducted from P, leaves $P - \frac{rW}{R} = \frac{RP - rW}{R}$, for the reduced force impelling the point A. And the inertia which resists the communication of motion to the point A will be the same as if the mass $\frac{\rho^2 w + r^2 W + R^2 P}{R^2}$ were concentrated in the point A (art. 310. cor. 4. 5.) If the former of these be divided by the latter, the quotient $\frac{R(RP - r^2W)}{\rho^2 w + r^2W + R^2P}$ is the force accelerating A: multiplying this by $\frac{r}{R}$, we have $\frac{RrP - r^2W}{\rho^3 w + r^2W + R^2P}$ for the force which accelerates the weight W in its ascent. Consequently the velocity of W will be $= \frac{RrP - r^2W^2}{\rho^2 w + r^2W + R^2P} gt$: which multiplied into W gives $\frac{RrPW - r^2W^2}{\rho^2 w + r^2W + R^2P}$ gt for the momentum. As this is to be a maximum, its fluxion will $= 0$; whence we shall obtain W $=$

$$\frac{\sqrt{(R^4P^2 + 2R^2P\rho^2 w + \rho^4 w^2 + PwRr\rho^2 + P^2R^3r)} - R^2P - \rho^2 w}{r^2}.$$

Cor. 1. When $R = r$, as in the case of the single fixed pulley, then $W = \sqrt{\left(2P^2R^3 + 2RP\rho^2 w + \frac{\rho^4}{R} w^2 + PwR\rho^2\right)} - \frac{\rho^2}{R^2} w - P$.

Cor. 2. When the pulley is a cylinder of uniform matter $\rho^2 = \frac{1}{2}R^2$, and the expression becomes $W = \sqrt{[R^3(2P^2 + \frac{3}{2}Pw + \frac{1}{4}w^2)]} - \frac{1}{2}w - P$.

Cor. 3. If in the first general expression for the momentum of W we put $Q = R^2P + \rho^2 w$, we shall have $\frac{RrPW - r^2W^2}{Q + r^2W} =$ a max. Which fluxed and reduced gives $W = \frac{1}{r^2}\sqrt{[Q \cdot (Q + RrP)]} - \frac{1}{r^2} Q$.

Cor. 4. If the moving force be destitute of inertia, then will $Q = \rho^2 w$, and W, as in the last corollary.

373. *Let a given power* P *be applied to the circumference a wheel, its radius* R, *to raise a weight* W *at its axle, whose radius is* r, *it is required to find the ratio of* R *and* r *when* W *is raised with the greatest momentum; the characters* w *and* ρ *denoting the same as in the last proposition.*

Here we suppose r to vary in the expression for the momentum of W, $\frac{WRrP - r^2W^2}{\rho^2 w + r^2W + R^2P} gt$. And we suppose that by the conditions of any specified instance we can ascertain what quantity of matter q shall make $r^2q = \rho^2 w$, which, in fact, may always be done as soon as we can determine ρ. The expression for the work will then become $\frac{RrPW - r^2W^2}{R^2P + r^2(q + W)} gt$. The fluxion of which being made $= 0$, gives, after a little reduction,

$$r = \frac{R\sqrt{[P^2W^2 + P^3(q + W)]} - PW}{P(q + W)}.$$

Cor. When the inertia of the machine is evanescent, with respect to that of P + W, then is $r = R\sqrt{\left(1 + \frac{P}{W}\right)} - 1$.

374. **PROP.** *In any machine whose motion accelerates, the weight will be moved with the greatest velocity when the velocity of the power is to that of the weight as* $1 + \text{P}\sqrt{\left(1 + \frac{\text{P}}{\text{W}}\right)}$ *to* 1; *the inertia of the machine being disregarded.*

For any such machine may be considered as reduced to a lever, or to a wheel and axle whose radii are R and r; in which the velocity of the weight $\frac{\text{R}r\text{P} - r^2\text{W}}{\text{R}^2\text{P} + r^2\text{W}}\, gt$ (art. 366.) is to be a maximum, r being considered as variable. Hence, then, following the usual rules, we shall find $\text{PR} = r\,[\text{W} + \sqrt{(\text{W}^2 + \text{PW})}]$. From which, since the velocities of the power and weight are respectively as R and r, the ratio in the proposition immediately flows.

COR. 1. *When the weight moved is equal to the power, then is* $\text{R} : r :: 1 + \sqrt{2} : 1 :: 2{\cdot}4142 : 1$ *nearly.*

COR. 2. When the weight is *double, triple, quadruple,* &c. the power, the preceding ratio becomes $2 + \sqrt{6}$, $3 + \sqrt{12}$, $4 + \sqrt{20}$, &c. respectively to 1.

375. **PROP.** *If in any machine whose motion accelerates, the descent of one weight causes another to ascend, and the descending weight be given, the operation being supposed continually repeated, the effect will be greatest in a given time when the ascending weight is to the descending weight, as* 1 *to* 1·618, *in the case of equal heights; and in other cases, when it is to the exact counterpoise in a ratio which is always between* 1 *to* 1½ *and* 1 *to* 2.

Let the space descended be 1, that ascended s; the descending weight 1, the ascending weight $\frac{1}{w}$: then would the equilibrium require $w = s$ (art. 363.); and $1 - \frac{s}{w}$ will be the force acting on 1. Now the mass $\frac{1}{w}$ reduced to the point at which the mass 1 acts will be $= \frac{1}{w}s^2 = \frac{s^2}{w}$; consequently the whole mass moved is equivalent to $1 + \frac{s^2}{w}$, and the relative force is $(1 - \frac{s}{w}) \div (1 + \frac{s^2}{w}) = \frac{w-s}{w+s^2}$. But, the space being given, the time is as the root of the accelerating force inversely, that is, as $\sqrt{\frac{w+s^2}{w-s}}$: and the whole effect in a given time being directly as the weight raised, and inversely as the time of ascent, will be as $\frac{1}{w}\sqrt{\frac{w-s}{w+s^2}}$; which must be a maximum. Consequently its square $\frac{w-s}{u^3+s^2w^2}$ must be a maximum likewise. This latter expression fluxed and reduced gives $w = \frac{1}{4}\ (\sqrt{}(s^2 + 10s + 9) - a + 3)$.

Here if $s = 1$, $w = \frac{1+\sqrt{5}}{2}$: but if s be diminished without limit, $w = \frac{3}{2}s$; if it be augmented without limit, then will $\sqrt{}(s^2 + 10s + 9)$ approach indefinitely near to $s + 5$, and consequently $w = 2s$. Whence the truth of the proposition is manifest.

376. Prop. *Let φ denote the absolute effort of any moving force, when it has no velocity, and suppose it not capable of any effort when the velocity is* W; *let* F *be the effort answering to the velocity* V, *then, if the force be uniform,* F *will be* $= \varphi\ (1 - \frac{V}{W})^2$.

For it is the difference between the velocities W and V which is efficient, and the action being constant, will vary as the square of the efficient velocity. Hence we shall have this analogy, $\varphi : F :: (W - 0)^2 : (W - V)^2$: consequently, $F = \varphi \left(\frac{W-V}{W}\right)^2 = \varphi \left(1 - \frac{V}{W}\right)^2$.

Although the pressure of an animal is not actually uniform during the whole time of its action, yet it is nearly so: so that in general we may adopt this hypothesis in order to approximate to the true nature of animal action. On which supposition the preceding prop. as well as the remaining one, in this chapter, will apply to animal exertion.

Cor. Retaining the same notation we have $W = \frac{V\sqrt{\varphi}}{\sqrt{\varphi} - \sqrt{F}}$. This, applied to the motion of animals, gives this theorem: *The utmost velocity with which an animal not impeded can move, is to the velocity with which it moves when impeded by a given resistance, as the square root of its absolute force, to the difference of the square roots of its absolute and efficient forces.*

377. Prop. *To investigate expressions by means of which the maximum effect, in machines whose motion is uniform, may be determined.*

I. It follows, from the observations made in art. 363. and the definitions in art. 365. that when a machine, whether simple or compound, is put into motion, the velocities of the impelled and working points are inversely as the forces which are in equilibrio when applied to those points in the direction of their motion. Consequently, if f denote the resistance when reduced to the working point, and v its velocity; while F and

V denote the force acting at the impelled point, and its velocity; we shall have $\mathrm{FV} = fv$, or introducing t the time, $\mathrm{FV}t = fvt$. Hence, *in all working machines which have acquired a uniform motion, the performance of the machine is equal to the momentum of impulse.*

II. Let F be the effort of a force upon the impelled point of a machine when it moves with the velocity V, the velocity being W when $\mathrm{F} = 0$, and let the relative velocity $\mathrm{W} - \mathrm{V} = u$. Then since (art. 376.) $\mathrm{F} = \phi\left(\frac{\mathrm{W}-\mathrm{V}}{\mathrm{W}}\right)^2$, the momentum of impulse FV will become $\mathrm{V}\phi\left(\frac{u}{\mathrm{W}}\right)^2 = \phi\cdot\frac{u^2}{\mathrm{W}^2}(\mathrm{W} - u)$; because $\mathrm{V} = \mathrm{W} - u$. Making this expression for FV a maximum, or, suppressing the constant quantities, and making $u^2(\mathrm{W} - u)$ a max. or its fluxion $= 0$, when u is variable, we find $2\mathrm{W} = 3u$, or $u = \frac{2}{3}\mathrm{W}$. Whence $\mathrm{V} = \mathrm{W} - u = \mathrm{W} - \frac{2}{3}\mathrm{W} = \frac{1}{3}\mathrm{W}$.

Consequently, *when the ratio of* V *to* v *is given by the construction of the machine, and the resistance is susceptible of variation, we must load the machine more or less till the velocity of the impelled point is one-third of the greatest velocity of the force; then will the work done be a maximum.*

Or, the work done by an animal is greatest when the velocity with which it moves is one-third of the greatest velocity with which it is capable of moving when not impeded.

III. Since $\mathrm{F} = \phi\frac{u^2}{\mathrm{W}^2} = \phi\left(\frac{\frac{4}{9}\mathrm{W}^2}{\mathrm{W}^2}\right) = \frac{4}{9}\phi$, in the case of the maximum, we have $\mathrm{FV} = \frac{4}{9}\phi\mathrm{V} = \frac{4}{9}\phi\cdot\frac{1}{3}\mathrm{W} = \frac{4}{27}\phi\mathrm{W}$, for the momentum of impulse, or for the work done, when the machine is in its best state. *Consequently when the resistance is a given quantity we must make* $\mathrm{V} : v :: 9f : 4\phi$; *and this structure of the machine will give the maximum effect* $= \frac{4}{27}\phi\mathrm{W}$.

IV. If we enquire the greatest effect on the supposition that ϕ only is variable, we must make it infinite in the above expression for the work done, which would then become $\mathrm{W\,F}$, or $\mathrm{W}\,\frac{\mathrm{V}}{v}f$, or $\mathrm{W}\,\frac{\mathrm{V}}{v}ft$, including the time in the formula. Hence we see, *that the sum of the agents employed to move a machine may be infinite, while the effect is finite:* for the variations of ϕ which are proportional to this sum, do not influence the above expression for the effect.

SCHOLIUM.

378. The propositions now delivered contain, it is hoped, the most material principles in the theory of machines. The manner of applying several of them is very obvious: the application of some, being less manifest, may be briefly illustrated, and the chapter concluded with two or three observations.

The last theorem may be applied to the action of men and of horses, with more accuracy than might at first be supposed. Observations have been made on men and horses drawing a lighter along a canal, and working several days together. The force exerted was measured by the curvature and weight of the track-rope, and afterwards by a spring steelyard. The product of the force thus ascertained, into the velocity per hour, was considered as the momentum. In this way the action of *men* was found to be very nearly as $(w - v)^2$: the action of horses loaded so as not to be able to trot was nearly as $(w - v)^{1 \cdot 7}$, or as $(w - v)^{\frac{9}{5}}$. Hence the hypothesis we have adopted may in many cases be safely assumed.

According to the best observations, the force of a man at rest is on the average about 70 pounds; and the utmost velocity with which he can walk is about 6 feet per second, taken at a medium. Hence, in our theorems, $\phi = 70$, and $w = 6$. Consequently $F = \frac{4}{9}\phi = 31\frac{1}{9}$ lbs. the greatest force a man can exert when in motion: and he will then move at the rate of $\frac{1}{3}w$, or 2 feet per second, or rather less than a mile and a half per hour.

The strength of a horse is generally reckoned about six times that of a man; that is, nearly 420 lbs. at a dead pull. His utmost walking velocity is about 10 feet per second. Therefore his maximum action will be $\frac{4}{9}$ of $420 = 186\frac{2}{3}$ lbs. and he will then move at the rate of $\frac{1}{3}$ of 10, or $3\frac{1}{3}$ feet, per second, or nearly $2\frac{1}{3}$ miles per hour. In both these instances we suppose the force to be exerted in drawing a weight along a horizontal plane; or by raising a weight by a cord running over a pulley which makes its direction horizontal. Other methods of exerting animal force will be spoken of in the second volume, art. 66, &c.

379. The theorems just given may serve to shew in what points of view machines ought to be considered by those who would labour beneficially for their improvement.

The first object of the utility of machines consists in furnishing the means of *giving to the moving force the most commodious direction:* and, when it can be done, of causing its action to be applied immediately to the body to be moved. These can rarely be united: but the former can be accomplished in most instances; of which the use of the simple lever, pulley, and wheel and axle, furnish many examples. Sometimes local circumstances do not permit the employment of that machine which, considered in itself, would be most proper to accomplish the desired effect; yet, with correct theory and a little practice, it will scarcely ever be difficult to fix upon the machine which in the given state of things will be most advantageous in practice. The second object gained by the use of machines is *an accommodation of the velocity of the work to be performed to the velocity with which alone a natural power can act.* Thus, whenever the natural power acts with a certain velocity which cannot be changed, and the work must be performed with a greater velocity, a machine is interposed moveable round a fixed support, and the distances of the impelled and working points are taken in the proportion of the two given velocities.

But the essential advantage of machines, that, in fact, which properly appertains to the *theory* of mechanics, consists in augmenting, or rather in modifying, the energy of the moving power, in such manner that it may produce effects of which it would have been otherwise incapable. Thus a man might carry up a flight of steps twenty pieces of stone, each weighing 30 pounds (one by one) in as small a time as he could (with the same labour) raise them all together by a piece of machinery, that would have the velocities of the impelled and working points as 20 to 1; and, in this case, the instrument would furnish no real advantage, except that of saving his steps. But if a large block of 20 times 30, or 600 lbs. weight, were to be raised to the same height, it would far surpass the utmost efforts of the man, without the intervention of some such contrivance.

The same purpose may be illustrated somewhat differently; confining the attention all along to machines whose motion is uniform. The product fv represents, during the unit of time, the effect which results from the motion of the resistance; this motion being produced in any manner whatever. If it be produced by applying the moving force immediately to the resistance, it is necessary not only that the products FV and fv should be equal; but that at the same time $F = f$, and $V = v$: if, therefore, as most frequently happens, f be greater than F, it will be absolutely impossible to put the resistance in motion by applying the moving force immediately to it. Now machines furnish the means of disposing the product FV in such a manner that it may always be equal to fv, however much the factors of FV may differ from the analogous factors in fv; and, consequently, of putting the system in motion, whatever is the excess of f over F.

Or, generally, as M. Prony remarks (Archi. Hydraul. art. 504.), machines enable us to dispose the factors of FVt in such a manner, that while that product continues the same its factors may have to each other any ratio we desire. Thus, to give another example: Suppose that a man exerting his strength immediately upon a mass of 25 lbs. can raise it vertically with a velocity of 4 feet per second; the same man acting upon a mass of 1000 lbs. cannot give it any vertical motion though he exerts his utmost strength, unless he has recourse to some machine. Now he is capable of producing an effect equal to $25 \times 4 \times t$: the letter t being introduced because if the labour is continued the value of t will not be indefinite, but comprised within assignable limits. Thus we have $25 \times 4 \times t = 1000 \times v \times t$; and consequently $v = \frac{1}{10}$ of a foot. This man may, therefore, with a machine, as a lever, or axis in peritrochio, cause a mass of 1000 lbs. to rise $\frac{1}{10}$ of a foot, in the same time that he could raise 25 lbs. four feet without a machine; or he may raise the greater weight as far as the less, by employing 40 times as much time.

From what has been said on the extent of the effects which may be attained by machines, it will be seen that so long as a moving force exercises a determinate effort with a velocity likewise determinate, or so long as the product of these is constant, the effect of the machine will remain the same: thus, under this point of view, supposing the preponderance of the effort of the moving power, and abstracting from inertia and friction of materials, the convenience of application, &c. all machines are equally perfect. But, from what has been shewn, (arts. 376. 377.) a moving force may, by diminishing its velocity, augment its effort, and reciprocally. There is, therefore, a certain effort of the moving force, such that its product by the velocity which comports to that effort is the greatest possible. Admitting the truth of the law assumed in the articles just referred to, we have, when the effect is a *maximum*, $\text{V} = \frac{1}{3}\text{W}$, or $\text{F} = \frac{4}{9}\phi$; and these two values obtaining together, their product $\frac{4}{27}\phi\text{W}$ expresses the value of the greatest effect with respect to the unit of time. In practice it will always be advisable to approach as nearly to these values as circumstances will admit; for it cannot be expected that they can always be exactly attained. But a small variation will not be of much consequence: for, by a well-known property of those quantities which admit of a proper maximum and minimum, a value assumed at a moderate distance from either of these extremes will produce no sensible change in the effect.

If the relation of F to V followed any other law than that which we have assumed, we should find from the expression of *that law* values of F, V, &c. different from the preceding. The general method, however, would be nearly the same.

With respect to practice, the grand object in all cases should be to procure a *uniform motion*, because it is that from which (*cæteris paribus*) the greatest effect always results. Every irregularity in the motion wastes some of the impelling power; and it is the *greatest* only of the varying velocities which is equal to that which the machine would acquire if it moved uniformly throughout: for, while the motion accelerates, the impelling force is greater than what balances the resistance at that time opposed to it, and the velocity is less than what the machine would acquire if moving uniformly; and when the machine attains its greatest velocity, it attains it because the power is not then acting against the whole resistance. In both these situations, therefore, the performance of the machine is less than if the power and resistance were exactly balanced, in which case it would move uniformly (art. 363.). Besides this, when the motion of a machine, and particularly a very ponderous one, is irregular, there are continual repetitions of strains and jolts which soon derange and ultimately destroy the whole structure. Every attention should, therefore, be paid to the removal of all causes of irregularity. Some of the most successful methods of ensuring a uniformity of motion will be given in the second volume. We must now turn to other subjects.

HYDROSTATICS.

BOOK III.

Introductory Definitions and Remarks.

380. ACCORDING to the general division marked out in art. 20, we now proceed to the subject of HYDROSTATICS, which comprises the doctrine of the pressure and the equilibrium of non-elastic fluids, as water, mercury, &c. and that of the weight and pressure of solids immersed in them.

DEF. A *fluid* is a body whose parts are very minute, yield to any force impressed upon it (however small), and by so yielding are easily moved among themselves.

This is nearly the same as the definition given by NEWTON, in the *Principia*, book 2. sect. 5. and is adopted here because, in conjunction with two or three established facts, it may serve as a basis for all which distinguishes the doctrines of hydrostatics from those of pure mechanics. The writers on the continent, however, though they admit that the minuteness of fluid moleculæ, and their excessive mobility, are characteristics common to all such bodies, yet they have recourse to a different definition. Thus the celebrated Euler in the *New Commentaries of St. Petersburgh*, vol. 13. takes for the basis of his analysis the following consideration: "The distinguishing nature of fluids consists in this property, namely, that when it is subjected to any pressure whatever that pressure is so distributed throughout the mass, that while it remains in equilibrio all its parts are equally pressed." And M. D'Alembert in his *Traité de l'Equilibre et du Mouvement des Fluides,* as well as M. Prony in his *Architecture Hydraulique*, adopt the same property as a definition. It is strictly consistent with experiment (though, as will soon be seen, it is rather a proposition, capable of proof, than a definition)and furnishes a natural foundation for an *algebraical* calculus, by which the whole doctrine of hydrostatics may be exhibited in a few equations. But this method, though it possesses some advantages, is not entirely pursued here, from a firm conviction that a judicious combination of the geometrical and algebraical methods is far more likely to convey distinct ideas to the student than the modern analysis merely.

Perfect fluidity, according to the Newtonian system, arises from a want of any sensible cohesion between the constituent particles of the fluid; and this want of cohesion is commonly attributed to the spherical figure of the particles. The nature of this work does not require that we should enter into minute disquisitions on the formal cause of fluidity. We shall merely state that the late Dr. Black of Edinburgh speaks of fluidity as an effect of heat: and before him Boerhaave pleaded strenuously for the same opinion. According to this view of the matter, fluidity may be caused by a certain degree of fire, which, when employed for this purpose, seldom manifests itself by any other perceptible effect: not dilating the volume, but resisting the particular attachment of the parts. Some strive to give mechanical ideas of a fluid body, by comparing it to a heap of sand: but the impossibility of giving fluidity by any kind of mechanical comminution will appear by considering two of the circumstances necessary to constitute a fluid body: 1. That the parts, notwithstanding any compression, *may* be moved in relation to each other, with the smallest conceivable force, or will give no *sensible resistance* to motion within the mass in any direction. 2. That the parts shall gravitate to each other, whereby there is a constant tendency to arrange themselves about a common centre, and form a spherical body; which, as the parts do not resist motion, is easily executed in small bodies. Hence the appearance of drops always takes place when a fluid is in proper circumstances. It is obvious that a body of sand can by no means conform to these circumstances.

Different fluids have different degrees of fluidity, according to the facility with which the particles may be moved amongst each other. Water and mercury are classed among the most perfect fluids. Many fluids have a very sensible degree of tenacity, and are thereforce called *viscous* or *imperfect* fluids.

381. Def. Fluids may be divided into *compressible* and *incompressible*, or *elastic* and *non-elastic* fluids. A *compressible* or *elastic* fluid is one whose apparent magnitude is diminished as the pressure upon it is increased, and increased by a diminution of pressure. Such is air, and the different vapours. An *incompressible* or *non-elastic* fluid is one whose dimensions are not, at least as to sense, affected by any augmentation of pressure. Water, mercury, wine, &c. are generally ranged under this class.

It is not unusual to apply the term *fluid* to that class solely which are elastic, and *liquid* to such as are non-elastic; thus making *air* a fluid and *water* a liquid: but the distinction, though it may have some advantages in chemical nomenclature, need not be insisted upon here.

Although the use of that well-known instrument the *thermometer* is founded upon the circumstance of different degrees of heat and cold causing a corresponding dilatation or condensation in spirits of wine, mercury, and some other fluids; a fact which it might be supposed would have led to the opinion that such fluids were compressible by other means; yet has it been universally believed and asserted till within the last half century, that after the fluid was freed from all air no art or violence could press it into less space. This opinion has been grounded chiefly, if not altogether, on a gross and inadequate experiment made by the *Açademi del Cimento*, at Florence; in which water when violently squeezed made its way through the fine pores of a globe of gold, rather than yield to the compression. Even so lately as 1790, so skilful a mathematician and philosopher as M. Prony speaks of the incompressibility of water with an obvious allusion to the Florentine experiment, as though he were not at all conscious of its insufficiency, and seeming quite ignorant of any contrary experiments. For, says he, " Si une quantité d'eau est renfermée dans un vase de capacité et de forme quelconque, et qu'on l'y comprime avec toute la force qu'on voudra, jamais on ne pourra la reduire à occuper un espace moindre que celui qu'elle occupoit d'abord. Tout le monde connoît les experiences qu'on a faites pour constater cette propriété; on sait que l'eau etant renfermée dans des globes de métal, quelque percussion ou quelque pression qu'on emploie pour le faire diminuer de volume, on n'y parvient jamais, et que lorsque la résistance qu'elle oppose à de pareils efforts ne lui fait pas briser, son enveloppe, elle se fait jour à travers les pores du métal, d'où elle sort en forme de rosée."

But our ingenious countryman, Mr. Canton, attentively considering this experiment, found that it was not sufficiently accurate to justify the conclusion which had always been drawn from it; since the Florentine philosophers had no method of determining that the alteration of figure in their globe of gold occasioned such a diminution of its internal capacity as was exactly equal to the quantity of water forced into its pores. To bring this matter therefore to a more accurate and decisive trial, he procured a small glass tube of about two feet long, with a ball at one end, of an inch and a quarter in diameter. Having filled the ball and part of the tube with mercury, and brought it exactly to the heat of 50° of Fahrenheit's thermometer, he marked the place where the mercury stood in the tube, which was about six inches and a half above the ball; he then raised the mercury by heat to the top of the tube, and there sealed the tube hermetically; then upon reducing the mercury to the same degree of heat as before, it stood in the tube $\frac{32}{100}$ of an inch higher than the mark. The same experiment was repeated with water exhausted of air, instead of mercury, and the water stood in the tube $\frac{43}{100}$ of an inch above the mark. Since the weight of the atmosphere on the outside of the ball, without any counterbalance from within, will compress the ball, and equally raise both the mercury and water, it appears that the water expands $\frac{11}{100}$ of an inch more than the mercury by removing the weight of the atmosphere. Having thus determined that water is really compressible, he proceeded to estimate the degree of compression corresponding to any given weight. For this purpose he prepared another ball, with a tube joined to it; and finding that the mercury in $\frac{23}{100}$ of an inch of the tube was the hundred thousandth part of that contained in the ball, he divided the tube accordingly. He then filled the ball and part of the tube with water exhausted of air; and leaving the tube open, placed his apparatus under the receiver of an air-pump, and observed the degree of expansion of the water answering to any degree of rarefaction of the air: and again by putting it into the glass receiver of a condensing engine, he noted the degree of compression of the water corresponding to any degree of condensation of the air. He thus found by repeated trials, that, in a temperature of 50°, and when the mercury has been at its mean height in the barometer, the water expands one part in 21740; and is as much compressed by the weight of an additional atmosphere; or the compression of water by twice the weight of the atmosphere, is one part in 10870 of its whole bulk. Should it be objected that the compressibility of the water was owing to any air which it might be supposed to contain, he answers, that more air would make it more compressible; he therefore let into the ball a bubble of air, and found that the water was not more compressed by the same weight than before.

In some further experiments of the same kind, Mr. Canton found that water is more compressible in winter than in summer; but he observed the contrary in spirit of wine, and oil of olives.

The following table was formed, when the barometer was at 29 inches and a half, and the thermometer at 50 degrees.

Compression of	Millionth parts.	Spec. grav.
Spirit of wine	66	846
Oil of olives	48	918
Rain water	46	1000
Sea water	40	1028
Mercury	3	13595

See *Phil. Transac.* for 1762 and 1764.

Indeed it seems reasonable to conclude, independent of all experiments, that no fluids are absolutely incompressible: for all bodies being porous, their parts may be brought nearer to each other: and a liquid being an assemblage of solid bodies, should, therefore, be compressible. Hence, then, the usual distinction of fluids into compressible and incompressible is, strictly speaking, inaccurate. Nevertheless, as the compression of the liquids in the preceding table is *very small* compared with their mass, it may safely be neglected in most practical cases, so that the fluids usually considered as incompressible may still be reckoned so in the investigations we are about to enter upon; and the consideration of air, and other easily compressible and elastic fluids, may be properly referred to the separate head of Aerostatics or Pneumatics.

382. We know so little of the essential nature and constitution of fluids, that it would be by no means advisable to apply to them the principles of equilibrium and of motion, as they have been stated in the preceding part of this work, without first enquiring whether there is not some other general law which appertains to fluids only, and from which, in conjunction with the principles just adverted to, the doctrines of hydrostatics may readily be deduced. For the actions of fluids upon each other differ so essentially in some particulars from the mutual actions of solid bodies, that some distinct principle must be sought, to account for such varying effects. The parts of a solid are so connected together as to form but one and the same whole; their effort is, according to its nature, concentrated into one point, as the centre of gravity, centre of gyration, &c.; which is by no means the case with fluids, their particles being extremely moveable, and entirely independent of each other. Again: no statical equilibrium can take place between two bodies of different weights, unless the lighter body acts at some mechanical advantage; whereas a very small weight of fluid may, without acting in so advantageous a position, be made to balance any weight however large. Solid bodies, again, when left to themselves, press only in the direction of gravity; while fluids press equally in all directions. This property indeed is one of the most remarkable which we meet with in fluids, and from it most of the other properties may be readily inferred; on which account the continental philosophers assume it as a kind of definition. The Newtonian definition is more simple, and naturally leads to this property, which can only be conceived to arise from the extreme freedom with which the particles move against each other. But the most satisfactory proof results from experiment; to which it is proper to have recourse in the establishment of the first principles of Hydro-

statics, and which will at once furnish the general law necessary to be combined with the received principles of proper mechanics.

383. Def. The *specific gravity* of any solid or fluid body is the absolute weight of a known volume of that substance, namely, of that which we take for unity in measuring the capacities of bodies.

Comparing this definition with that of density (art. 10.) it will appear that the two terms *density* and *specific gravity* express the same thing under different aspects; the former being more accurately restrained to the greater or less vicinity of particles, the latter to a greater or less weight in a given volume; hence as weight depends upon the closeness of particles, the density varies as the specific gravity, and the terms may in most cases be indiscriminately used. The specific gravities of fluids are usually considered without any regard to the empty spaces between the particles, though if the particles of fluids are spherical, the vacuities make at least $\frac{1}{4}$ of the whole bulk. But it is sufficient that we know precisely in what sense the specific gravity of fluids is understood.

CHAPTER I.

On the Pressure of non-elastic Fluids.

384. Prop. *The upper surface of a homogeneous heavy fluid in any vessel, or any system of communicating vessels, is horizontal.*

This is a matter of universal experience; and, as it is easily observed, may be taken for the distinguishing property of fluids. Thus, if ABCDEF (fig. 7. pl. XV.) be a vessel in which the branches CDH, EFG, have a free communication with the part AB; then, if water, or mercury, or wine, or any other fluid commonly reckoned non-elastic, be poured in, either at A, C, or E, and when the whole is at rest, the surface of the fluid stands at IK in the larger trunk; if the line LIKM be drawn parallel to the horizon, the surface of the fluid will stand at L in the branch EF, and at M in the branch CD; and this whatever are the inclinations of those branches, or the angles at F and D, G and H.

Remark. This is usually explained by saying, that, since the parts of a fluid are easily moveable in any direction, the higher particles will descend by reason of their superior gravity, and raise the lower parts till the whole comes to rest in a horizontal plane. Now what is called the horizontal plane is, in fact, a portion of a spherical surface whose centre is the centre of the earth: hence it will follow, that *if a fluid gravitate towards any centre it will dispose itself into a spherical figure, the centre of which is the centre of force.*

385. Prop. *If a fluid, considered without weight, be contained in any vessel whatever, and an orifice being made in the vessel, any pressure whatever be applied thereto, that pressure will be distributed equally in all directions.*

Through any point N (fig. 7.) taken at pleasure below the surface of the fluid LIKM, imagine the horizontal plane PNOQ to pass. It is obvious the *weight* of the fluid contained in the vessel below PNOQ contributes nothing to the support of the columns LP, IO, MQ; so that the equilibrium would obtain in like manner if the fluid contained in that part of the vessel below

PNOQ had lost its weight entirely. We may, therefore, regard this fluid as being solely a mean of communication between the columns LP, IO, and MQ; in such manner that it will transmit the pressure resulting from the columns LP, MQ, to the column IO, and reciprocally. If, now, instead of the columns LP, IO, MQ, of the fluid, pistons were applied to the surfaces P, NO, and Q, and were separately urged by pressures respectively equal to the pressures of the columns LP, IO, MQ, the equilibrium would manifestly obtain in like manner. Or, if a pressure equal to that of the column MQ be applied at Q, while the columns LP, IO, remain, the equilibrium will still obtain; and this whatever are the directions of the several branches, and their sinuosities at D, F, &c. whence the proposition is evident.

Cor. 1. Not only is the pressure transmitted equally in all directions, but *it acts perpendicularly upon every point of the surface of the vessel which contains the fluid.*

For if the pressure which acts upon the surface were not exerted perpendicularly, it is easy to see that it could not be entirely annihilated by the reaction of that surface; the surplus of force would, therefore, occasion fresh action upon the particles of the fluid, which must of consequence be transmitted in all directions, and thus necessarily occasion a motion in the fluid: that is, the fluid could not be at rest in the vessel, which is contrary to experience.

Cor. 2. Hence, also, *if the parts of a fluid contained in any vessel* ABCD (fig. 6. plate XV.), *open towards the part* AB, *are solicited by any forces whatever, and remain notwithstanding in equilibrio, these forces must be perpendicular to the surface* AB. For the equilibrium would obtain in like manner if a cover or a piston of the same figure as the surface AB were applied to it; and it is manifest that, in this latter case, the forces which act at the surface, or their resultant, must be perpendicular to that surface.

Cor. 3. If, therefore, the forces which act upon the particles of the fluid are those of gravity, we shall see that the direction of gravity is necessarily perpendicular to the surface of a tranquil fluid; consequently, *the surface of a heavy fluid must be horizontal to be in equilibrio, whatever may be the figure of the vessel in which it is contained.*

Thus, again, we see the reason of the fact stated in art. 383.

Cor. 4. If a vessel, as ABCD (fig. 6. pl. XV.) closed throughout, except at a small orifice O, be full of a fluid without weight; then if any pressure be applied at O, the resulting pressure on the plane surface or bottom CD will neither depend upon the quantity of fluid in the vessel nor on its shape; but, since the pressure applied at O is transmitted equally in all directions, the actual pressure upon CD will be to the pressure at O, as the area of CD to that of the orifice.

COR. 5. In the same manner will the pressure applied at O be exerted in raising the top AB of the vessel; so that if the top be a plane, of which O forms a part, the vertical pressure tending to force AB upwards will be to the force applied at O, as the surface AB to the area O.

386. PROP. *The pressure of a fluid on the horizontal base of a vessel in which it is contained is as the base and perpendicular altitude, whatever be the figure of the vessel that contains it: the upper surface of the fluid being supposed horizontal.*

Let any horizontal plane GH (fig. 10. pl. XV.) be supposed drawn, and conceive the fluid contained in the part GCDH of the vessel to be void of weight; then it is evident from cor. 3. of the foregoing proposition, that any vertical filament whatever, EI of the heavy fluid ABHG, exerts at the point I a pressure which is distributed equally through the fluid GCDH; and that this pressure acts equally upwards, to oppose the action of each of the other filaments which stand vertically above GH; therefore, the filament EI alone keeps in equilibrium all the other filaments of the mass AGHB: consequently the mass GCDH being still supposed without weight, there will not result any other pressure on the bottom CD than that of a single filament EI, which, being transmitted equally to all the points of CD, will make the pressure upon CD to that upon the base I of the filament EI as the area CD to the area I. If, therefore, we imagine (fig. 9. pl. XV.) a heavy fluid contained in ACB to be divided into horizontal laminæ, the upper lamina will communicate to the bottom CD no other action than would be communicated by the single filament *ab*; and the same thing obtaining with respect to each lamina, the bottom, therefore, is pressed in the same degree as it would be by the combined operation of the filaments *ab*, *bc*, *cd*, &c. Whence, as this pressure is transmitted equally to all the points of CD, it will be equal to the product of CD into the sum of the pressures which the filaments *ab*, *bc*, *cd*, are capable of exercising on the same point, or it will be proportional to $CD \times (ab + bc + cd + \&c.)$

COR. 1. Hence, *if the fluid contained in the vessel* ABDC *be homogeneous, the pressure on the bottom* CD *will be expressed by* $CD \times EC$; *and will be measured by the weight of the prism or cylinder whose base is* CD *and height* EC.

COR. 2. Hence, also, *when the heights are equal, the pressures (of the same fluid) are as the bases: when the bases are equal the pressures are as the heights: when both heights and* *bases are equal the pressures on the horizontal bottoms are equal in all, however irregular the shape and different the capacities of the vessels may be.*

Cor. 3. *In different vessels containing different fluids, the pressures are as the areas of the bottoms × depths × specific gravities.*

Cor. 4. *If the laminæ* AH, GK, &c. *be of different densities, or specific gravities,* D, d, δ, &c. *then will the pressure on the bottom* CD *be equal to* $\mathrm{CD} \times (ab \cdot \mathrm{D} + bc \cdot d + cd \cdot \delta + \&c.)$.

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387. Upon the two principles that fluids press equally in all directions, and in proportion to their perpendicular depths, depends the explanation of the peculiarity known by the title of the *hydrostatic paradox,* which is this: *any quantity of water or other fluid, how small soever, may be made to balance and support any quantity or any weight, however great:* a circumstance which has been converted to a useful purpose in the construction of some machines. (See **Bramah's** *Machine,* vol. ii.) A well-known contrivance to illustrate this principle is the *hydrostatic bellows.* It consists of two thick boards EF, CD, (fig. 8. pl. XV.) about sixteen or eighteen inches diameter, covered or connected firmly with pliable leather round the edges, to open and shut like common bellows, but without valves; a pipe AB about three feet high being fixed into the bellows at B. Now let water be poured into the pipe at A, and it will run into the bellows, gradually separating the boards by raising the upper one. Then, if several weights (three hundred weights, for instance) be laid upon the upper board, the water being poured in at the pipe till it be full, will sustain all the weights, though the water in the pipe should not weigh a quarter of a pound. For the narrower the pipe the better (beyond certain limits), provided we make it long enough, the proportion being always this:

As the area of the orifice or section of the pipe,
To the area of the bellows board, FE:
So is the weight of water in the pipe, AG,
To the weight it will sustain on the board.

For the fluid at B, the bottom of the tube, is pressed with a force varying as its altitude AB: and this pressure is communicated horizontally to all the particles in the space FE, and then distributed equally throughout the fluid in the bellows: consequently, the pressure upward at FE is equal to the weight of a cylinder of the fluid whose base is FE and altitude AB; while the actual weight of water borne up is only that of the cylinder whose base is FE and height BG ; and hence no weights laid upon CD that do not exceed the weight of a cylinder of the fluid whose base is EF and altitude AG will disturb the equilibrium.

388. PROP. *If two immisceable fluids are included in a bent tube, and balance each other, their perpendicular altitudes, estimated from an horizontal plane drawn through the common surface where they are in contact, will be reciprocally as their specific gravities.*

Let ABCD (fig. 1. pl. XVI.) be such a bent tube, its form and dimensions being arbitrary; and let the common surface of the two immisceable fluids be GH; one fluid occupying the space EFHG, the other the space GHBCKI. Let the specific gravity of the fluid in EFHG be *s*, that of the other S. Through the surface GH draw the horizontal plane GHLOM, then it is manifest (art. 384.) that the part GHBCML is naturally in equilibrio: in order, therefore, that the equilibrium may exist in the whole, the pressures exerted upon GH by the fluids contained in EFHG, IKML, must be equal. Now (art. 386. cor. 3.) the former of these pressures is denoted by GH·FH·*s*, and the latter by GH·NO·S. Consequently GH·FH·*s* = GH·NO·S, or FH·*s* = NO·S; whence flows the proposition, *i.e.* FH : NQ :: S : *s*.

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389. Before we commence the investigation of the pressure of fluids on oblique and curvilinear surfaces, we may just remark with respect to pressures upon the horizontal bottoms of vessels, that it is necessary to distinguish between the pressure which the plane CD (fig. 6. pl. XV.) would sustain as arising from the fluid, and that which it would have to sustain if it carried the vessel. If the bottom CD were detached from the vessel, in order to prevent the escape of the water there, the bottom CD must be pressed upwards with a force equal to the weight of the cylinder CDEF of the fluid: but if we would support the vessel, it will require a force equal to the weight both of the vessel and the fluid it contains. Thus, when the vessel is narrowest at bottom it will require more force to support the vessel than to keep its bottom from falling: while, if the vessel is widest at bottom, it may be supported with a less effort than would be necessary to prevent the bottom from separating from the sides of the vessel. But the pressure of the fluid on the bottom of an upright prismatic vessel is *equal* to its weight.

390. PROP. *Any plane surface immersed in a heavy fluid, of which the upper surface is horizontal, is perpendicularly pressed with a force equal to the weight of a column of that fluid, having the surface pressed for its base, and the depth of*

its centre of gravity under the surface of the fluid for its altitude.

Let ABCD (fig. 10, 11. pl. XV.) be a vertical section of a vessel terminated by surfaces either plane or curved, and any way inclined to the horizon; and let the vessel be filled with a fluid whose upper surface intersects the section ABCD in the horizontal line AB. If GH*hg* be an indefinitely thin lamina of the fluid, we may consider it abstractedly from its weight, and then conceive this lamina as pressed by the superior fluid. Now this pressure is distributed equally through all the particles of the lamina, and acts perpendicularly and equally upon all the points of the faces G*g*, H*h*: hence, because this force is the same as would be occasioned by the filament EI alone, the pressure which is exerted perpendicularly upon G*g* will be expressed by G*g* · EI : and the same will manifestly obtain, if, instead of regarding G*g* as an evanescent right line, we consider it as an evanescent surface. Therefore, in general, the pressure which is exerted perpendicularly upon any evanescent surface, by a heavy homogeneous fluid, is estimated by the continual product of that surface, its distance from the horizontal surface, and the specific gravity of the fluid.

Hence it will follow, that the total pressure exerted upon any plane surface whatever, whether vertical or oblique, is equal to the product of the specific gravity into the sum of the products of the evanescent parts of this surface into their respective distances from the upper surface of the fluid: but by the nature of the centre of gravity (art. 108.) the sum of these latter products is equal to the product of the whole surface into the distance of its centre of gravity from the horizontal surface of the fluid: so that the whole pressure will be denoted by the continual product of the surface pressed, the distance of its centre of gravity from the upper surface, and the specific gravity of the fluid; which is the proposition in other words.

Cor. 1. *The entire lateral pressure of a vessel whose sides are perpendicular to the base, is equal to the weight of the fluid contained in a rectangular prism, whose altitude is that of the fluid, and base is a parallelogram, one side of which is equal to the altitude of the fluid, and the other to the semiperimeter of the vessel.*

Cor. 2. *The pressure against one side of a cubical vessel filled with a fluid is equal to half the pressure against the bottom. And the whole pressure against the sides and bottom is equal to three times the weight of the fluid in the vessel.*

Cor. 3. *If* ABCD, CDEF (fig. 2. pl. XVI.), *are two rectangles whose common breadth is* CD, *standing vertically in a fluid,* *whose upper surface is* SS′, *then will the pressures upon the rectangles,* ABCD *and* CDEF *be as* AC^2 *and* $AE^2 - AC^2$.

For if G and *g* be the respective centres of gravity of the two rectangles, we shall have pressure upon ABCD : pressure upon CDEF $:: ABCD \cdot IG : CDEF \cdot Ig :: AC \cdot \frac{1}{2} AC : CE \cdot (AC + \frac{1}{2} CE) :: AC \cdot \frac{1}{2} AC : (AE - AC) \cdot \frac{1}{2}(AE + AC) :: AC^2 : AE^2 - AC^2$.

COR. 4. *Hence, if* AE *be to* AC *as* $\surd 2$ *to* 1, *the pressures upon* ABCD *and* CDEF *will be equal.*

391. DEF. The *centre of pressure* is that point of a surface against which any fluid presses, through which the resultant of all the individual pressures passes, or to which, if a force equal to the whole pressure were applied in a contrary direction, it would keep the surface at rest.

392. PROP. *If a plane surface which is pressed by a fluid be produced to the horizontal surface of it, and their common intersection be regarded as the axis of suspension, the centre of percussion will be the centre of pressure.*

Let ABCD (fig. 3. pl. XVI.) be the horizontal surface of the fluid which presses upon the plane EIF: produce this plane till it meets the surface of the fluid in the line MN; and let O be the centre of pressure. From any point *p* of the surface pressed draw the vertical *pm*, meeting the horizontal surface in *m*; and in the plane CB draw from *m* the line *m*M perpendicular to MN. The pressure upon *p* (art. 390.) is as $p \cdot pm$, and its effect to turn the plane about MN is as $p \cdot pm \cdot p\text{M}$, by the nature of the lever: also, its effect to turn the plane about NI is as $p \cdot pm \cdot \text{MN}$. In like manner, if the plane EIF be supposed to revolve about the axis MN, and to strike an obstacle at O, the percussive force of the particle *p*, by which it endeavours to move the plane about MN, will be as $p \cdot pm$, or as $p \cdot pm \cdot p\text{M}$; and its force to turn the plane about NI will be as $p \cdot p\text{M} \cdot \text{MN}$, or as $p \cdot pm \cdot \text{MN}$. And the like correspondence between the percussive and the pressive forces, of any other particles in the plane EF, may be shewn in the same manner. Consequently, the percussive forces of the whole of the particles, whereby they endeavour to move the plane in the two directions, have the same relation as the forces of pressure, and therefore the centres of pressure and percussion are coincident.

COR. 1. Hence, the theorems given for the centre of percussion in arts. 318, 319. may be applied to the determination of the centre of pressure.

COR. 2. Hence also appears the mistake of those who assert that the centres of percussion and of pressure do not coincide. They are the centres of *oscillation* and of *pressure* which do not coincide universally. See art. 320.

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393. To adopt the general formulæ for the centre of percussion to the instance of the centre of pressure, it will be proper to make a slight change in the notation. Let d be the distance from MN of any particle, or of any horizontal lamina of the fluid in contact with the plane EIF; let l be the length of such lamina, and $\dot{d}$ its depth (being considered as evanescent), then will $l\dot{d}$ be its area; also, let δ be the distance of the centre of gravity of the plane EIF from the line MN, where that plane intersects the surface; and let the horizontal distance of $l\dot{d}$ from the line NI be denoted by h: then, with respect to the line MN the formula (art. 318.) will become $\frac{\int ld^2\dot{d}}{\int ld\dot{d}}$ or $\frac{\int ld^2\dot{d}}{\text{EIF}\cdot\delta}$; and with respect to the line NI, the formulæ (art. 319.) will become $\frac{\int lhd\dot{d}}{\int ld\dot{d}}$ or $\frac{\int lhd\dot{d}}{\text{EIF}\cdot\delta}$.

A few examples are here added to illustrate the use of these theorems.

I. Let a reservoir which contains water, or any other fluid (its specific gravity being s), have one of its sides plane and vertical: if we imagine a right line drawn vertically upon this plane, its length being λ, and the distance of its centre of gravity from the surface of the fluid δ, the pressure exerted upon this line will be $s\,\delta\,\lambda$ (art. 390.). Let the distance of the superior extremity of the line λ from the surface of the fluid be a, and make $a+\lambda=e$, so shall e be the distance of the lower extremity of the line λ from the horizontal surface of the fluid. Then, to find the centre of pressure of the line λ, we take the complete fluent of the expression $\frac{\int ld^2\dot{d}}{\int ld\dot{d}}$ (l being constant) which is $\frac{2(d^3-b^3)}{3(d^2-a^2)}$; or, when the variable d becomes equal to e we have $\frac{2(e^3-a^3)}{3(e^2-a^2)}$, for the distance of the centre of pressure from the horizontal surface of the fluid. When one of the extremities of the line λ coincides with this surface we have $a=0$, and $e=\lambda$, and the distance of the centre of pressure becomes $\frac{2}{3}\lambda$.

II. If upon the vertical line λ we construct a rectangle, of which the horizontal base is h, the whole pressure upon it will be $sh\delta\lambda$, and the distance of the centre of pressure from the surface of the fluid will be $\frac{2(e^3-a^3)}{3(e^2-a^2)}$ the same as we have just found. And this centre must evidently be found upon the vertical line which divides the parallelogram into two equal parts.

If the upper horizontal side of the parallelogram coincide with the surface of the fluid (as the side AB of the parallelogram ABCD, fig. 2. pl. XVI.) its tendency to turn about its base will be $\frac{1}{2}sh\lambda^2\cdot\frac{1}{3}\lambda=\frac{1}{6}sh\lambda^3$, and its tendency to turn about one of its vertical sides will be $\frac{1}{2}sh\lambda^2\cdot\frac{1}{2}h=\frac{1}{4}sh^2\lambda^2$: thus the first of these efforts will be to the second as $\frac{1}{6}sh\lambda^3:\frac{1}{4}sh^2\lambda^2$, or as $2\lambda:3h$; which reduces to 2 : 3, when the rectangle becomes a square.

III. To determine the centre of pressure in the triangle CAB (fig. 4. pl. XVI.) whose side AB is horizontal, and which is placed vertically in a fluid whose horizontal surface is SS′. Through C, the summit of the triangle, draw the vertical line SCP, also the line CQ bisecting the base AB, and any line TR parallel to AB. Make $CP=\lambda$, $AB=h$, $CS=a$, $SP=a+\lambda=e$, the distance of the horizontal line in which lies the centre of gravity of the triangle from $S=\delta$ (that is, if $Cg=\frac{2}{3}CQ$, the distance between SS′ and $g=\delta$), the angle $PCQ=k$, $SM=d$, $TR=l$, $CM=c=d-a$.

The whole pressure upon this triangle will be represented by $s\cdot\frac{1}{2}h\lambda\cdot\delta=\frac{1}{2}s\delta\lambda h$. And to find the depth of the centre of pressure below SS′ we must find the fluent of the expression $\frac{\int ld^2\dot{d}}{ABC\cdot\delta}$, or $\frac{\int ld^2\dot{d}}{\frac{1}{2}\delta\lambda h}$. In order to this we have $CP:AB::CM:TR$, or $\lambda:h::c:l=\frac{ch}{\lambda}=\frac{h}{\lambda}(d-a)$, which substituted for l in $\int ld^2\dot{d}$, gives $\frac{h}{\lambda}\int(d^3-ad^2)\dot{d}=\frac{h}{\lambda}(\frac{1}{4}d^4-\frac{1}{3}ad^3)+C$. The constant quantity C may be determined by considering that the fluent must vanish at the point C, that is, when $l=0$, or when $d-a=0$, that is, when $d=a$. Hence, making the substitution, we have

$$\frac{h}{\lambda}(\tfrac{1}{4}a^4-\tfrac{1}{3}a^4)+C=0,\text{ whence } C=\frac{h}{12\lambda}a^4$$

and the correct fluent is $\frac{h}{\lambda}(\frac{1}{4}d^4-\frac{1}{3}ad^3+\frac{1}{12}a^4)$.

Hence then, for the whole triangle, in which $SM=SP$, or $d=e$, we substitute e for d in the fluent, and divide by the denominator $\frac{1}{2}\delta\lambda h$ of the general expression, which gives for the depth of the centre of pressure of the whole triangle the value $\frac{\frac{1}{4}e^4-\frac{1}{3}ae^3+\frac{1}{12}a^4}{\frac{1}{2}\delta\lambda^2}$, which is evidently independent of the base h of the triangle.

When the vertex of the triangle is at the horizontal surface of the fluid $e = \lambda$, $a = 0$, $\delta = \frac{2}{3}\lambda$, and the expression becomes $\frac{3}{4}\lambda$.

IV. To determine the distance of the centre of pressure from the vertical line CP. Here we must take the formula $\frac{\int l\mathrm{H}d\dot{d}}{\int ld\dot{d}}$, in which H represents MN $= c \tan k = (d-a) \tan k$. Substituting this value of H, and $\frac{h}{\lambda}(d-a)$ for l, its value found above, we have $\int l\mathrm{H}d\dot{d} = \frac{h \tan k}{\lambda}\int(d^3 - 2ad^2 + a^2d)\dot{d} = \frac{h \tan k}{\lambda}(\frac{1}{4}d^4 - \frac{2}{3}ad^3 + \frac{1}{2}a^2d^2) + \mathrm{C}$. In this case, also, the fluent vanishes when $d = a$; whence we have $\frac{h \tan k}{12\lambda}a^4 + \mathrm{C} = 0$, and $\mathrm{C} = -\frac{h \tan k}{12\lambda}a^4$:

so that the correct fluent is $\frac{h \tan k}{\lambda}(\frac{1}{4}d^4 - \frac{2}{3}ad^3 + \frac{1}{2}a^2d^2 - \frac{1}{12}a^4)$. Hence, making $d = e$, and dividing by $\int ld\dot{d} = \frac{1}{2}\delta\lambda h$, we have for the distance sought, $\tan k \cdot \frac{\frac{1}{4}e^4 - \frac{2}{3}ae^3 + \frac{1}{2}a^2e^2 - \frac{1}{12}a^4}{\frac{1}{2}\delta\lambda^2}$.

1. When the triangle is isosceles the angle $k = 0$, and the preceding value vanishes, as it obviously ought to do; for in that case the triangle is symmetrical with respect to CP, and the pressures are in equilibrio about that line.

2. When the triangle is right-angled, and has its base or horizontal side AB $= h$ one of the sides about the right-angle, that is, when CA coincides with CP, then is $h = 2\lambda \tan k$, or $\tan k = \frac{h}{2\lambda}$, which transforms the expression for the centre of pressure's distance from SP

$$\text{to } \frac{h}{\delta\lambda^3}(\tfrac{1}{4}e^4 - \tfrac{2}{3}ae^3 + \tfrac{1}{2}a^2e^2 - \tfrac{1}{12}a^4).$$

3. Finally, when the vertex of the triangle is at the surface SS′ of the fluid, $a = 0$, $e = \lambda$, $\delta = \frac{2}{3}\lambda$, and the expression becomes $\frac{3}{4}\lambda \tan k$; which, for the right-angled triangle, reduces to $\frac{3}{8}h$.

4. When the triangle has its vertex at the horizontal surface of the fluid, the tendency to turn about the base is to that to turn about the perpendicular let fall from the vertex upon the base as 1 to $3 \tan k$; and, in the case of the right-angled triangle, as 2λ to $3h$: which, when the legs of the triangle are equal, reduces to the ratio of 2 to 3, As in the case of the rectangle and square.

If C and AB lie on different sides of SS′, that is, if part of the triangle is out of the fluid, no other change will be necessary in the preceding expressions than a change of signs in those terms which contain *uneven* powers of a. So that this simple transformation will accommodate the preceding general theorems to the case of trapezoids.

V. If the radius of a circle be r, and δ the distance of its centre below the surface of the fluid in the plane of the circle, then is the distance of the centre of pressure from the upper surface, in the same plane, expressed by $\delta + \frac{r^2}{4\delta}$; which when the upper part of the circle just touches the surface becomes $\frac{5}{4}r$. The investigation of this is left for the student's exercise.

It is hardly necessary to remark, that the results of the operations in this scholium may be safely applied in the analogous enquiries relative to the centre of percussion.

394. Prop. *To enquire generally into the results of all the pressures, upon any surface plane or curved, regular or irregular, both in the vertical and the horizontal direction.*

This is usually performed by foreign authors, by means of the calculus of partial differences. But another mode of investigation is pursued here, which is likely to carry more conviction to the mind of a learner. Had the vertical pressure alone been the object of investigation, it might be determined far more concisely.

I. Whatever the figure of a body may be, we may always imagine it to consist of an assemblage of an infinite number of indefinitely small laminæ respectively parallel, and the surface of each lamina as an assemblage of many trapezoids, their number indeed being infinite likewise, when the surface in contact with the fluid is curved. Hence, to estimate the result of the pressures of a fluid, whether upon the interior surface of a vessel which contains it, or upon the exterior surface of a body immersed in it, we must first estimate the result of the pressure upon the surface of a trapezoid whose height is evanescent.

Conceive, therefore, *abcd* (fig. 5. pl. XVI.) a trapezoid whose two parallel sides are *ab*, *cd*, and whose height HK is infinitely small with regard to those sides. To resist the pressure upon this surface we must apply to the centre of gravity *g* of the trapezoid a force P perpendicularly to its plane, the value of which is expressed by the product of the surface of the trapezoid into the distance G*g* of its centre of gravity from the horizontal surface ABCD of the fluid.

To determine the effect of this force P both in the vertical and the horizontal direction, conceive a vertical plane *cd* FE to pass through the line *cd*, and a horizontal plane *ab* EF through the line *ab*; the common intersection of these planes being EF; then, having drawn the vertical lines *c*E, *d*F, meeting the horizontal plane in E and F, join *b*E and *a*F: again, through the direction P*g* of the force P, conceive a plane KIH perpendicular to *cd*, and of which H*g*K and HI are the intersections, with the two planes *abcd* and FE*cd*: this plane will be perpendicular to the planes *abcd* and FE*cd*, because *cd* is their common intersection: finally, from the point K, where *ab* and H*g* meet, draw KI perpendicular to the plane FE*cd*, this line must necessarily be perpendicular to HI.

The construction effected, resolve the force P (represented by *g*N) into two others which are also in the plane KIH, of which the horizontal one is *g*L, and the vertical one *g*M. Calling these component forces L and M, we have, by the nature of the parallelogram of forces $P : L : M :: gN : gL : gM :: gN : gL : LN :: HK : HI : IK$, the triangles *g*LN, HIK, being evidently similar. Multiplying the three latter terms by $\frac{ab+cd}{2} \cdot Gg$, which will not change the ratio, we shall have $P : L : M :: HK \cdot \frac{ab+cd}{2} \cdot Gg : HI \cdot \frac{ab+cd}{2} \cdot Gg : IK \cdot \frac{ab+cd}{2} \cdot Gg$.

Now it may be observed, 1st. That $HK \cdot \frac{ab+cd}{2}$ is the surface of the trapezoid *abcd*. 2dly. That since *c*E and *d*F are parallel, as likewise *cd* and EF, we have $cd = FE$, therefore $IK \cdot \frac{ab+cd}{2} = IK \times \frac{ab+EF}{2}$, which, of consequence, is the surface of the trapezoid *ab*EF. 3dly. That, because the height of the trapezoid *abcd* is evanescent with respect to the sides *ab* and *cd*, EF, which is equal to *cd*, may be taken both for *cd* and for *ab*; so that $HI \cdot \frac{ab+cd}{2}$ reduces to $HI \cdot EF$, which is the surface of the rectangle E*cd*F. We have, therefore, $P : L : M :: abcd \cdot Gg : EcdF \cdot Gg : aFEb \cdot Gg$. But we have supposed that the force P is expressed by $abcd \cdot Gg$; consequently, the horizontal force L is denoted by $EcdF \cdot Gg$, and the vertical force M by $aFEb \cdot Gg$.

As the triangle may be considered as a trapezoid, of which one of the parallel sides vanishes, the same thing, therefore, obtains for any evanescent triangle.

Conceiving, now, that from the angles *a*, *d*, *c*, *b*, lines are drawn to fall perpendicularly upon the plane ABCD, these perpendiculars will be the edges of a prismatic frustrum, of which the horizontal base is equal to *a*FE*b*, and the inclined base *abcd*; or, as *ab* and *cd* are supposed indefinitely near, the solidity of the prismatic frustrum will not differ sensibly from that of the prism which has the same horizontal base, and whose height is G*g*: but this latter is equal to *a*FE*b* · G*g*, which is precisely the expression above found for the vertical force M. Hence it appears, that *this force is equal to the weight of a prismatic frustrum of the fluid whose inclined base is* abcd, *and horizontal base the projection of* abcd *upon the horizontal surface* ABCD.

II. Let us next consider any solid whatever cut into an indefinite number of horizontal laminæ, such as ABDE*abde* (fig. 6. pl. XVI.), and that perpendicularly to the centre of gravity of the surface of each trapezoid into which the contour of the laminæ is divided, forces are applied, each represented by the product of the surface of the corresponding trapezoid into the distance of its centre of gravity from the horizontal surface A′D′. These forces are the pressures of a heavy fluid, sustained by the interior surface of the laminæ ABDE*abde* of a vessel which contains it; they are also the pressures of such a fluid which would be sustained by the exterior surface of a solid whose contour is the same, and which is immersed to the same depth. But it is manifest that if each of those forces P, P′, P″, &c. were decomposed into two others, the one vertical, the other horizontal, each vertical force would be represented by the weight of a prismatic frustrum of the fluid whose inclined base is one of the trapezoids in the contour of the lamina, and its horizontal base the projection of that trapezoid upon the upper surface of the fluid. Therefore the sum of these vertical forces, or the single vertical resulting force, will be represented by the sum of the weights of all those prismatic frustrums: and the same property may obviously be extended to every other horizontal lamina, we may conclude, then,

1. That, *if a vessel, of any figure, be full of a fluid, and have over every part of the sides and bottom a perpendicular column of the fluid reaching to the surface, the whole vertical pressure of the fluid upon the bottom and sides of that vessel will be equal to the weight of the whole fluid.*

2. That, *if a body, as* AEDBM (fig. 7. pl. XVI.) *of which* AIBF *is the greatest horizontal section, is immersed in a fluid to any depth whatever, and if we drop the consideration of the pressure sustained by the upper part* AMB, *the vertical effort of the fluid to raise the body is equal to the weight of the volume of fluid which is comprised between the surface* A′D′, *the surface* AIBFE, *and the convex surface formed by perpendiculars let fall from all the points of the perimeter* AIBF *upon the plane* A′D′; *that is, equal to the sum of the weights of fluid in the prism* aibfAIBF, *and the space* AIBFCE.

III. If we would now estimate the pressure sustained by the superior surface AMBFI of the body, we shall see, by the same kind of reasoning, that the result in the vertical direction tending to force the body downwards, is an effort equal to the weight of the volume comprised between the horizontal projection *aibf*, and the upper surface AMBFI of the body. If, then, from the first of these efforts we deduct the second, it will appear that the body is pushed vertically upwards, with an effort equal to the weight of a volume of the fluid equal to that of the body immersed. We conclude therefore that, *if a body is immersed in any fluid whatever, it will lose (relatively) as much of its weight as is equal to the weight of the quantity of fluid it displaces.*

IV. With regard to the resultant of all the vertical forces whose magnitude we have just determined, it is easy to see that it must pass through the centre of gravity of the volume of fluid displaced. For, if we conceive this volume decomposed into an infinite number of evanescent vertical filaments, the effort made by the fluid to push each filament vertically will be expressed by the weight of a quantity of fluid equal to that filament. Therefore, to obtain the distance of the resultant from any vertical plane whatever, we must multiply the mass of each filament (considered as of the same nature with the fluid) by its distance from this plane, and divide the sum of the products by the sum of the filaments; which is precisely the rule that must be followed to find the centre of gravity of the volume displaced. Therefore, universally, *a body immersed either wholly or in part in a heavy fluid, and at rest, receives from the fluid pressures which are together equivalent to a vertical force directed upwards through the centre of gravity of the fluid displaced by the body, and equal to the weight of a quantity of the fluid so displaced by the immersed part of the body.*

Indeed we may readily assign a reason, *à priori*, of this: for, supposing a force acting on a body without heaviness retains it in equilibrio when immersed either wholly or partly in a heavy fluid; if we substitute for the immersed part of the body, that is, for the fluid it displaces, an equal and similar portion of the same fluid become solid (as ice, and the density unchanged), the equilibrium will still obviously subsist: consequently, the pressure of the fluid upon the immersed body will be altogether equal and directly opposed to the weight of this solid; and must, therefore, pass through its centre of gravity in order to sustain it in equilibrium.

V. It now remains for us to consider how the horizontal forces are disposed of.—If we take any one of the horizontal laminæ into which either the fluid, or the solid immersed in the fluid, may be imagined to be divided, and through the sides *ab*, *bc*, *cd*, &c. (fig. 6. pl. XVI.) of the inferior section conceive vertical planes to pass, and to be terminated by the superior section; these planes will form the contour of a prism whose height is that of the lamina, and each face of the prism will have (I) the measure of its surface proportional to the value of the horizontal force to which it is perpendicular. But, as all these faces are of the same altitude, their surfaces are proportional to their bases *ab*, *bc*, &c. and consequently the horizontal forces are respectively in the ratio of the sides *ab*, *bc*, &c. And as the altitudes of these faces are evanescent, we may regard all these forces as applied in the same horizontal plane *abcdef*, and to be each respectively proportional to the length of the side, on the middle of which it acts perpendicularly. Now it has been shewn (art. 53.), that if any number of forces represented in magnitude and direction by the sides of a polygon taken in order, act simultaneously upon the same point, they will be mutually destroyed, and the point continue at rest: also (art. 85.), that when any number of forces are in equilibrio when applied to different points of a body, they are the same as would be in equilibrio about a single point; and, since the directions of the several forces P, P′, P″, &c. in the present case would, if produced, form a polygon similar to *abcdef*, the consequences just referred to will apply to them likewise: and, in like manner, to the pressures upon any other horizontal laminæ. Consequently, *the efforts which result in the horizontal direction, from the pressure of a heavy fluid upon the surface of any body immersed in it, are mutually destroyed.*

SCHOLIUM.

395. From the preceding doctrine of the pressure of fluids, an important practical maxim may be deduced. We have seen that in any vessel containing a heavy fluid, the parts that are deepest below the surface sustain a proportionally greater pressure. If, therefore, we have to construct an assemblage of vertical pipes or tubes, to elevate water or any other fluid, we may run into a superfluous expense, by giving the same thickness to the material in every part. For, if the substance be uniformly thick, and the lower parts are sufficiently strong, the upper parts are, of consequence, much thicker than necessary. The method suggested by theory is, while we give to the whole assemblage the same interior diameter, to give a safe and sufficient thickness to the material at the lowest part, and let it gradually diminish to the top, in the same ratio nearly as the diminution in the depth of the fluid. The same maxim may also find an application in the construction of sluice-gates, dams, banks, &c. And in all such cases it is adviseable to de-

termine, first, the adequate strength to resist the pressure at the greatest depth; as, by this means, safety may always be ensured without any waste of materials.

To take a simple example, suppose the half figure ACD (fig. 10. pl. III.), to be the profile of a bank, whose summit A is on a level with the surface of the water, AC being the vertical face of the bank, PM and CD two horizontal sections; and let the water be supposed to act upon the face AMD by its hydrostatic pressure. The part APM may be moved from its place either by turning upon the point P, or by sliding along the line PM; in either case separating from the lower part PCDM. The question is reduced to the finding such a curve AMD, that every portion of it as APM, may of itself be in equilibrio upon its base PM. Putting AP $= z$, PM $= y$, $g =$ the specific gravity of water, g' that of the bank; then the horizontal thrust of the water with respect to the point P is $\frac{1}{6}gz^3$, and the moment of the weight of the profile APM with respect to the same point P is equal to $g'\int y\dot{z}$ multiplied by the horizontal distance of the centre of gravity of the profile from the point A, which distance is $\frac{\int y^2\dot{z}}{2\int y\dot{z}}$. To favour the stability of the bank, let us drop the consideration of the adhesion of the profile APM to the line PM, and of the vertical pressure of the water, and there will result the equation

$$\tfrac{1}{6}gz^3 = g'\int y\dot{z}\,\frac{\int y^2\dot{z}}{2\int y\dot{z}} = \tfrac{1}{2}g'\int y^2\dot{z}.$$

Solving this in the usual way, we have $z = y\sqrt{\frac{g'}{g}}$. Thus the face AMC is the hypothenuse of a right-angled triangle, whose base CD is to its altitude CA, as $\sqrt{g}$ to $\sqrt{g'}$.

The conditions of the stability necessary to prevent the sliding of the profile APM along the base PM, will be found in the equation $\frac{1}{2}gz^2 = ng'\int y\dot{z}$; whence $gz = ng'y$, or $y : z :: g : ng'$; where n is a constant quantity to be determined by experiment, which multiplied into the weight of matter ponderating upon a given section, shall measure the counteraction arising from friction and adhesion to horizontal motion in the direction of that section.

In pipes employed for conveying liquids, the thickness should be every where, in the compound ratio of their internal diameters, the perpendicular height of the liquid, and its specific gravity, directly, and the tenacity or strength of the materials inversely; that is $t \propto \frac{adg}{s}$, or as $\frac{ad}{s}$ when g is constant.

Now, it has been found by experiment that a tube of lead of 16 inches diameter, and 6½ lines in thickness, sustained water having a head or charge of 50 feet. And that a tube of copper of 6 inches diameter, with a charge of water of 30 feet, required half a line in thickness. From this, suppose we wished to find the thickness of a leaden pipe of 12 inches diameter, to bear a head of 40 feet, we should have from the standard experiment $s \propto \frac{ad}{t} \propto \frac{600 \times 16}{6\frac{1}{2}} \propto 1477$; and then $t \propto \frac{ad}{s} = \frac{480 \times 12}{1477} =$ 3·89, or less than 4 lines.

When the conduit pipes are horizontal and made of lead, their thickness, according to Bossut, should be 2½, 3, 4, 5, 6, 7, 8, lines, when their diameters are 1, 1½, 2, 3, 4½, 6, 7, inches. When the pipes are made of iron, their thickness should be 1, 3, 4, 5, 6, 7, 8, lines, when their diameters are 1, 2, 4, 6, 8, 10, 12, inches, respectively.

In the farther application of these principles, reference may be made to the table of the relative tenacities of bodies towards the end of chap. 5. book I.

CHAPTER II.

On the Determination of the Specific Gravities of Solid and Fluid Bodies.

396. The term *specific gravity* has already been defined in art. 383. The object of the present chapter is to explain the principles on which the different methods of ascertaining the specific gravities of solids and fluids are founded, and to give some account of the best of those methods. Previous to which it may be proper to make a few observations, naturally deduced from the definition of specific gravity and the nature of bodies in general.

1. The specific gravities of bodies are in the same proportion as their weights, when the magnitudes of the bodies are equal.
2. Where the weights of the bodies are equal, the specific gravities are inversely as their magnitudes.
3. When the specific gravities are equal, their weights are directly as their magnitudes.
4. When neither the magnitudes nor the specific gravities are equal, the weights of bodies are as their magnitudes and specific gravities conjointly.

To express these relations algebraically, let $\frac{W}{w}$ represent the ratio of the weights of two substances, $\frac{M}{m}$ the ratio of their magnitudes, and $\frac{S}{s}$ that of their specific gravities; then will the general relation of these quantities be expressed by the equation $\frac{W}{w} = \frac{M}{m} \cdot \frac{S}{s}$. In estimating the weights, magnitudes, and specific gravities of substances, some standard quantities are always assumed to which other bodies are referred: the letters *w*, *m*, and *s*, may represent these standards, and each of them might be assumed equal to 1; but such assumption would not correspond with the measures and weights now in use. It will, therefore, be more eligible that *m* should represent the magnitude of some known measure which may be assumed for unity, as a cubic inch, a cubic foot, &c. and s any convenient number in the geometrical progression 1, 10, 100, 1000, &c. Now it has been found, that the density of rain-water is more nearly uniform in different circumstances of time and place, than any other substance, whether solid or fluid; and by a fortunate coincidence it happens, that the weight of a cubic foot of rain-water is exactly 1000 ounces avoirdupoise. If, then, we make $w = 1000$, $m = 1$, and $s = 1000$, we shall obtain $\mathrm{W} = \mathrm{M} \times \mathrm{S}$, that is, *the weight of any body in avoirdupoise ounces will be equal to the product of the magnitude in cubic feet, into the specific gravity taken from that scale in which the specific gravity of rain-water is* 1000. Hence it appears, that a knowledge of the specific gravities of homogeneous bodies will enable us to determine their weight, without actually weighing them, provided we can ascertain their magnitudes: and conversely, however irregular the shape of bodies may be, if we know their weights and specific gravities, we may readily determine their magnitudes in feet, namely, *by dividing the weight in avoirdupoise ounces, by the specific gravity, or by the weight of a cubic foot in avoirdupoise ounces.*

But in philosophical subjects the weights of bodies being for the most part small, are estimated in Troy ounces or grains, the magnitudes being referred to a cubic inch as a standard. Now a Troy ounce is to an avoirdupoise ounce as 480 to 437½: consequently $\frac{1000}{1728} \times \frac{437\frac{1}{2}}{480} = \cdot 52746$ of an ounce Troy, or 253·181 grains, the weight of a cubic inch of water. And hence the magnitude of a solid, estimated in cubic inches, is $= \frac{\mathrm{W}}{253\cdot 181\,\mathrm{S}}$, when the weight W is in grains; or $= \frac{\mathrm{W}}{\cdot 52746\,\mathrm{S}}$, when the weight is known in Troy ounces. And conversely, the weight estimated in grains $= 253\cdot 181 \times \mathrm{M} \times \mathrm{S}$, when the magnitude M is in cubic inches; and, if estimated in Troy ounces, $\mathrm{W} = \cdot 52746 \times \mathrm{M} \times \mathrm{S}$. In all these cases S being the specific gravity, in terms corresponding to 1000, for that of rain-water.

Hence also we may shew how to determine very accurately the diameter D of any small sphere, whose specific gravity is S (to that of water 1000), its weight W being known in grains. For the content of a sphere whose diameter is 1 being ·523598, we have 1 : 0·523598 :: 253·181 gr. (the weight of a cubic inch of water) : 132·5648 gr. the weight of a sphere of water, whose diameter is 1 inch. Therefore, since the weights are as the magnitudes and specific gravities conjointly; and the magnitudes of spheres are as the cubes of their diameters, we have $132{\cdot}5648 \cdot D^3 \cdot \frac{s}{1000} = w$; whence we find $D = 1{\cdot}961208 \sqrt[3]{\frac{w}{s}}$.

After a manner not very widely different, may various other useful rules and theorems, applicable to the admeasurement of bodies either regular or irregular, be easily deduced. What is done above is intended chiefly as a specimen of the method; but it serves at the same time to shew the importance of an acquaintance with the specific gravities of different substances. We now proceed to exhibit the most useful propositions in this branch of our subject.

397. Prop. *A body immersed in a fluid will, when left to itself, sink, if its specific gravity be greater than that of the fluid: it will rise to the surface and float there, if its specific gravity be less than that of the fluid: but, if the specific gravities of the solid and fluid be equal, the body will rest in any part wherever it happens to be placed.*

1. For the body endeavours to descend by its own weight, and is supported by a force equivalent to the weight of an equal bulk of fluid, or of as much fluid as will fill the space occupied by the body. If, therefore, the body be heavier than the fluid, bulk for bulk, its weight will be greater than the upward pressure of the fluid which is to counteract it, and, consequently, this latter pressure is not sufficient to prevent the body from sinking.

2. If the body be specifically lighter than the fluid, its pressure downwards will be less than the upward pressure of the water at the same depth; consequently, in this case also the greater force will overcome the less, and the surplus cause the body to rise.

3. When the body and the fluid are of the same specific gravity, equal masses of each are of the same weight, and consequently the force with which the body tends to descend, and the force which opposes the descent, are equal to each other; and as they act in contrary directions, the body will rest between them, so as neither to sink by its own weight, nor to ascend by the upward pressure of the fluid.

Cor. 1. If by any contrivance the specific gravity of a solid can be so varied, as to be one while *greater*, another *less*, and then *equal* to the specific gravity of the fluid wherein it is immersed, the body will sink, or rise, or remain suspended, according to the variations of its specific gravity.

This is the case in the experiment with those little glass images (which some philosophers exhibit in their lectures) which are made to ascend or descend, or remain suspended, at pleasure.

COR. 2. If a solid specifically heavier than a fluid be immersed to a depth which is to its thickness, as the specific gravity of the solid to that of the fluid, and the pressure of the fluid from above be removed, the body will be sustained by the fluid. For the pressure from above being removed, the body is in the same state with respect to the contrary pressure, as though with the same weight it filled the whole space to the surface of the fluid; that is, as though its specific gravity and that of the fluid were equal.

This serves for the explication of the common experiment of making lead *swim*, in consequence of being fitted to the bottom of a glass tube.

COR. 3. Hence also we see the meaning of the proposition, that *all bodies when immersed in a fluid lose the weight of an equal bulk of that fluid* (art. 394. III.). The weight is not otherwise lost than as it is sustained by the action of a contrary force. And it therefore becomes obvious, why the weight of a bucket of water is not perceived while it is in the water; it is not because that weight is *destroyed*, but because it is *supported;* not because fluids do not gravitate when they are in fluids of the same sort, but because there is a pressure in a contrary direction which is precisely equal to their gravity.

COR. 4. The weights thus lost, by immerging the same body in different fluids, are as the specific gravities of the fluids.

COR. 5. Bodies of equal weight, but different bulk, lose in the same fluid weights which are reciprocally as the specific gravities of the bodies, or directly as their bulks.

COR. 6. The whole weight of a body which will float in a fluid, is equal to the weight of as much of the fluid as the immersed part of the body displaces.

COR. 7. Hence the magnitude of the whole body is to that of the part immersed, as the specific gravity of the fluid to that of the body. And, if the body be any prism with its base kept horizontal, the altitude of the prism will be to the depth immersed, as the specific gravity of the fluid to that of the body.

COR. 8. And because, when the weight of a body taken in a fluid is subtracted from its weight out of the fluid, the difference is the weight of an equal volume of the fluid; this, therefore, is to its weight in the air, as the specific gravity of the fluid is to that of the body.

Consequently, if W be the weight of a body in air,
W' its weight in water, or any other fluid;
S the specific gravity of the body,
s the specific gravity of the fluid,

we shall have $W - W' : W :: s : S$; whence

$$S = \frac{W}{W - W'}s, \text{ the specific gravity of the body,}$$

$$\text{and } s = \frac{W - W'}{W}S, \text{ the specific gravity of the fluid.}$$

So that *the specific gravities of bodies are as their weights in the air directly, and their loss in one and the same fluid inversely.*

Cor. 9. Hence, for two bodies connected together, or mixed together into one compound of different specific gravities, we may, supposing there is no *penetration of dimensions*, easily deduce the necessary equations.

Let the respective weights and specific gravities be denoted thus:

$$\left.\begin{array}{l} H = \text{weight of the heavier body in air,} \\ H' = \text{weight of the same in water,} \end{array}\right\} S = \text{its spec. gravity;}$$

$$\left.\begin{array}{l} L = \text{weight of the lighter body in air,} \\ L' = \text{weight of the same in water,} \end{array}\right\} S' = \text{its spec. gravity;}$$

$$\left.\begin{array}{l} C = \text{weight of the compound in air,} \\ C' = \text{weight of the same in water,} \end{array}\right\} S'' = \text{its spec. gravity;}$$

$s =$ the specific gravity of water. Then,

1st. $(H - H')S = Hs$.
2d. $(L - L')S' = Ls$.
3d. $(C - C')S'' = Cs$.
4th. $H + L = C$.
5th. $H' + L' = C'$.
6th. $\frac{H}{S} + \frac{L}{S'} = \frac{C}{S''}$.

From which equations any of the above quantities may be found in terms of the rest.

If the body L be of less specific gravity than water, then L' must be considered as negative, and to find its specific gravity we must have recourse to a compound mass, as C: thus, because from equa. 4 and 5, $L - L' = (C - C') - (H - H')$; and from equa. 2. $S' = \frac{Ls}{L - L'}$; consequently, substituting for $L - L'$ its value, we have $S' = \frac{Ls}{(C - C') - (H - H')}$. Or, if we deduce the value of S' from the last equa. we shall thence find $S' = \frac{sS''L}{Cs - HS''}$.

398. Prop. *If a vessel contain two immisceable fluids (such as water and mercury), and a solid of some intermediate specific gravity, be immersed under the surface of the lighter fluid and float on the heavier; the part of the solid immersed in the heavier fluid is to the whole solid, as the difference between the specific gravities of the solid and the lighter fluid to the difference between the specific gravities of the two fluids.*

Let the specific gravity of the heavier fluid be s, the part of the body immersed in it $= B$; the specific gravity of the lighter fluid $= s'$, the part of the body immersed in it $= B'$; and let the specific gravity of the body be S. Let also the area of the horizontal section of the solid coinciding with the contiguous surfaces of the two fluids be $= A$, and its perpendicular distance from the upper surface of the lighter fluid be $= d$. Then (art. 394.) the pressure against the section A from the lighter fluid will be $Ad - B's'$; which, added to the weight of the solid $(B + B')\,S$, will give the whole force by which this section of the solid is urged downwards. And the pressure upward against the same section is $Ad + Bs$. But as the solid is sustained in equilibrio by these contrary forces, they must be equal; that is, $Ad - B's' + (B + B')\,S = Ad + Bs$: whence we find $BS + B'S = Bs + B's'$, or $B : B' :: S - s' : s - S$, or lastly, $B : B + B' :: S - s' :: s - s'$.

COR. 1. The analogy $B : B' :: S - s' : s - S$, may be thus expressed in words: *As the part of the solid within the heavier fluid, to the part within the lighter; so is the difference between the specific gravities of the solid and lighter fluid to the difference between the specific gravities of the heavier fluid and the solid.*

COR. 2. When the specific gravity of the lighter fluid is very small, compared with that of the heavier or that of the solid, then we may, without any sensible error, use the proportion $B : B + B :: S : s$. Thus, if the lighter fluid be air, the heavier water, and the solid elm-wood, their specific gravities being $1\frac{2}{9}$, 1000, and 600 respectively; then would the ratio of $600 - 1\frac{2}{9} : 1000 - 1\frac{2}{9}$ or $588\frac{7}{9} : 988\frac{7}{9}$ be very nearly equal to that of 600 : 1000.

COR. 3. Hence also it appears, that the common rule for ascertaining the specific gravities of a fluid and a lighter solid by the ratio of the part immersed to the whole, is not *accurately*, though nearly, true; because the air is a heavy fluid, and therefore every solid floating on a fluid and in air, is in fact a solid of intermediate specific gravity floating between two immisceable fluids. We may, however, render the rule accurate, by *subducting the number expressing the specific gravity of air from the two numbers expressing the specific gravity of the solid and the fluid on which it floats;* the remainders will express the actual ratio between those specific gravities, and may be reduced to the usual standard by a simple and obvious analogy.

399. PROP. *To find the specific gravity of a body.*

This may be done generally by means of the *hydrostatic balance*, which is a kind of balance contrived for the exact and easy determination of the weight of bodies, either in the air, or when immersed in water; (see vol. II. for a description of this instrument). The problem may be divided into three cases, as below.

I. *When the body is heavier than water*: weigh it both in water, and out of water, and the difference of these weights will express the weight lost in water. Then if B represent the weight of the body out of water, B' its weight in water, s its specific gravity, and s the specific gravity of water, the first equation in cor. 9. art. 397, will become $(B - B')\,S = Bs$, whence we find $S = \frac{Bs}{B-B'}$; consequently the general rule in words at length is this:

As the weight lost in water
Is to the whole or absolute weight;
So is the specific gravity of water
To the specific gravity of the body.

II. *When the body will not sink in water, being specifically lighter.* In this case attach to it a piece of another body heavier than water, so that the mass compounded of the two may sink together. Weigh the denser body and the compound body separately, both out of the water and in it; and find how much each loses in water, by subtracting its weight in water from its weight in air; and subtract the less of these remainders from the greater. Then use this proportion:

As the last remainder
Is to the weight of the light body in air;
So is the specific gravity of water
To the specific gravity of the body.

This also follows from cor. 9. art. 397, where it was shewn that $S' = \frac{Ls}{(C-C')-(H-H')}$.

III. *When the specific gravity of a fluid is required.* Take a piece of some body of known specific gravity; weigh it both in and out of the fluid, and find the loss of weight by taking the difference of these two: then say,

As the whole or absolute weight
Is to the loss of weight;
So is the specific gravity of the solid
To the specific gravity of the fluid.

This rule flows evidently from cor. 4. art. 397.

On the same principle, likewise, depends the use of the hydrometer, as will soon be seen (art. 401.).

400. Prop. *To find the respective weights of two known ingredients in a given compound.*

If we adopt the notation in cor. 9. art. 397. and make use of the 4th and 6th equations there given, namely, $H + L = C$, and $\frac{H}{s}+\frac{L}{s'}=\frac{C}{s''}$, we shall thence find $H=\frac{(s''-s')s}{(s-s')s''}C$, and $L=\frac{(s-s'')s'}{(s-s')s''}$. From whence we deduce the following rule in words, at length:

Take the three differences of every pair of the three specific gravities, viz. the specific gravities of the compound, and of each ingredient; and multiply each specific gravity by the difference of the other two: then say,

As the greatest product
Is to the whole weight of the compound;
So is each of the other two products
To each respective weight of the two ingredients.

COR. If, instead of finding the weights, we were to find the magnitudes M and M' of the two ingredients, the specific gravities being as above; we should have the weight of $M = sM$, and the weight of $M' = s'M'$, while the weight of the compound would be $s''(M + M')$. Hence we should have $sM + s'M' = s''M + s''M'$, an equation similar to that in art. 398. and, consequently, similar analogies, viz. $M : M' :: s''-s' : s-s''$; and $M : M + M' :: s''-s' : s-s'$.

It is here supposed that the magnitude $M + M'$ of the compound is equal to the sum of the magnitudes of the two parts when separate. But it very frequently happens that the magnitude of the mixture is less than this sum; a circumstance which is probably occasioned by two causes, the different magnitudes of the constituent particles of the two bodies, and their chemical affinity. But this rule is, notwithstanding, of use in determining the quantity of penetration or rarefaction, by comparing the computed magnitudes or densities with those which are discovered by observation.

On the Hydrometer.

401. The *Hydrometer*, or *Areometer*, is an instrument contrived to measure the specific gravity of water and other fluids; and, indeed, has sometimes been made use of to determine the specific gravity of solid bodies. The general principle on which the construction and use of the hydrometer depends is contained in cor. 4. art. 397. from which it follows that a body specifically lighter than several fluids will serve to find out their specific gravities, because it will sink deepest in the fluids whose specific gravity is the least. Thus, if AB (fig. 9. pl. XVI.) be a small uniform glass tube, hermetically sealed, having a scale of equal divisions marked upon it, with a hollow ball of about an inch in diameter at bottom, and a smaller ball C under it,

communicating with the first; into the little ball is put mercury or small shot, before the tube is sealed, so that it may sink in water below the ball, and float or stand upright, the divisions on the stem shewing how far it sinks. If this instrument be dipped in common water, and sink to D, it will sink only to some lower point E in salt water; but in port wine it will sink to some higher point F, and in brandy perhaps to B.

It is evident that an hydrometer of this kind will only shew that one liquid is specifically heavier than another; but the true specific weight of any liquid cannot be determined without a calculation for this particular instrument, the tube of which should be truly cylindrical. Besides, these instruments will not serve for fluids whose densities are much different.

402. Mr. Clarke constructed a new hydrometer, shewing whether any spirits be proof, or above or below proof, and in what degree. This instrument was made of a ball of copper (because ivory imbibes spirituous liquors, and glass is apt to break), to which is soldered a brass wire about a quarter of an inch thick; upon this wire is marked the point to which it exactly sinks in proof spirits; as also two other marks, one above and one below the former, exactly answering to one-tenth above proof and one-tenth below proof. There are also a number of small weights made to add to it, so as to answer to the other degrees of strength besides those above, and for determining the specific gravities of different fluids.

403. Dr. Desaguliers contrived an hydrometer for determining the specific gravities of different waters to such a degree of nicety, that it would shew when one kind of water was but the 40,000 part heavier than another. It consists of a hollow glass ball of about 3 inches in diameter, charged with shot to a proper degree, and having fixed in it a long and very slender wire, of only the 40th part of an inch in diameter, and divided into tenths of inches, each tenth answering to the 40,000 part, as above. See his *Exper. Philos.* vol. 2. p. 234.

404. Mr. Nicholson has made an improvement by which the hydrometer is adapted to the general purpose of finding the specific gravity both of solids and fluids, (fig. 8. pl. XVI.). A is a hollow ball of copper; B a dish affixed to the ball by a short slender stem D; C is another dish affixed to the opposite side of the ball by a kind of stirrup. In the instrument actually made, the stem D is of hardened steel, $\frac{1}{40}$ of an inch in diameter, and the dish C is so heavy as in all cases to keep the stem vertical when the instrument is made to float in any liquid. The parts are so adjusted that the addition of 1000 grains in the upper dish B, will just sink it in distilled water (at the temperature of 60° of Fahrenheit's thermometer) so far that the surface

shall intersect the middle of the stem D. Let it now be required to find the specific gravity of any fluid. Immerse the instrument in it, and by placing weights in the dish B cause it to float, so that the middle of the stem D shall be cut by the surface of the fluid. Then, as the known weight of the instrument, added to 1000 grains, is to the same known weight added to the weights used in producing the last equilibrium, so is the weight of a quantity of distilled water displaced by the floating instrument, to the weight of an equal bulk, of the fluid under examination. And these weights are in the direct ratio of the specific gravities.

Again, let it be required to find the specific gravity of a solid body whose weight is less than 1000 grains. Place the instrument in distilled water, and put the body in the dish B. Make the adjustment of sinking the instrument to the middle of the stem, by adding weights in the same dish. Subtract those weights from 1000 grains, and the remainder will be the weight of the body. Place now the body in the lower dish C, and add more weight in the upper dish B, till the adjustment is again obtained. The weight last added will be the loss the solid sustains by immersion, and is the weight of an equal bulk of water. Consequently the specific gravity of the solid is to that of water, as the weight of the body to the loss occasioned by the immersion. Mr. Nicholson says, "This instrument was found to be sufficiently accurate to give weights true to less than one-twentieth of a grain." *Nicholson's Philosophy*, vol. II. p. 16. See also the art. *Gravimeter* in our second volume.

405. The areometer invented by *M. de Parcieux* in 1766, and presented to the Academy of Sciences, is very simple in its structure, and at the same time very convenient in application, and furnishes very accurate results. This instrument consists of a small glass phial, about two inches, or at most two inches and a half, in diameter, and seven or eight inches long. Its bottom must not be bent inwards (as bottles and phials commonly are), lest air should be lodged in the cavity when it is immersed in any liquid. The mouth is closed with a very tight cork stopper, into which is fixed, without passing through it, a very straight wire (of iron or of brass) about a line in diameter, and twenty-eight or thirty inches long. The bottle is then loaded in such a manner, by introducing into it small grains of shot, that the instrument when immersed in the lightest of the liquids to be compared, sinks so as to leave only the end of the wire above its surface, while in the heaviest the wire is immersed some inches. This may be properly regulated either by augmenting or diminishing the weight in the bottle; or by attaching a little dish or basin to the top of the wire, and changing the

weights in it; or by varying the thickness of the wire. To perform experiments of this kind it will be necessary to have a vessel for the reception of the fluid to be tried; a convenient size would be a cylinder of 3 or 4 inches in diameter, and as many feet in length: it may be made of tin plate; and a scale divided into inches and lines may be carried up above one side of the vessel, which will serve to measure the different depths of immersion in the fluid, by noting the division on the scale against which the top of the wire stands. M. de Parcieux recommends the use of *two* instruments of this kind at once, in order that two fluids may be compared at the same instant.

The areometer, when thus constructed, will exhibit very sensibly the least difference in the specific gravities of different liquors, or the changes which the same liquor may experience, in this respect, under different circumstances. Indeed M. Montucla says, "we have seen an instrument of this kind, the motion of which was so sensible, that when immersed in water cooled to the usual temperature, it sunk several inches while the rays of the sun fell upon the water, and immediately rose on the rays of that luminary being intercepted. A very small quantity of salt or sugar thrown into the water made it also rise some inches." *Hutton's Montucla's Recreations*, vol. ii. p. 26.

406. We here add a few articles respecting the theory of this areometer; observing, by the by, that the same theory may, with slight and obvious modifications, be applied to the examination of any other instrument of a nearly similar kind:

Let s denote the specific gravity of the fluid,
c the capacity or volume of the phial,
w the total weight of the areometer,
l the length of the part of the wire immersed,
r its semidiameter,
π the multiplier 3·141593.

Then will πr^2 represent a section of the wire parallel to its axis, and $\pi r^2 l$ the content of the wire. And when the areometer is in equilibrio, if we consider that it displaces a portion of the fluid whose weight is equal to its own, we shall have $w = s(c + \ r^2 l)$; whence we deduce

$$s = \frac{w}{c + \pi r^2 l}, \text{ and } l = \frac{w - cs}{\pi r^2 s}.$$

407. Let us first consider the equation $l = \frac{w - cs}{\pi r^3 s}$. Here the quantity $w - cs$ expresses the difference between the weight of the areometer, or that of the liquor displaced, and the weight of the quantity displaced by the phial only: it is therefore the weight of the fluid displaced by the part of the wire which is

immersed in it. But, considering the minuteness of the diameter, and consequently of the radius r, the difference $w-cs$ is a very small quantity, equal perhaps to a few grains; so that a very inconsiderable variation in s or in w must cause a great one in l, which is the length of the portion of the wire plunged in the fluid.

The variations of l are more especially sensible when r varies, because it is the square r^2 which is a divisor in its value: thus, then, we may see how it happens that this areometer is so well calculated to render manifest the slightest differences in specific gravities.

Suppose that when the specific gravity s becomes s', the length l of wire immersed is changed to l', and we shall have $l' = \frac{w-cs'}{\pi r^2 s'}$: subtracting the value of l' from that of l, and making the necessary deductions, we shall have $l-l' = \frac{w(s'-s)}{\pi s s' r^2}$; this is the difference in the depth immersed, occasioned by the difference of density, which, *cæteris paribus*, is proportional to w.

Thus we may augment the sensibility of the instrument, relative to the variation of specific gravity, either by augmenting w, the weight of fluid displaced by the areometer (which may be accomplished by enlarging the phial, or by putting in more shot, or by loading the dish at the top), or by diminishing the diameter of the wire; and, in general, all other things being equal, $\frac{w}{r^2}$ will express the sensibility of the instrument with regard to the specific gravity of the fluid.

408. But when the specific gravity remains the same, the sensibility of the instrument, the quantity by which it sinks by the addition of a little weight, *depends only upon that little weight and* r^2. To prove this, resume the equation $l = \frac{w-cs}{\pi r^2 s}$: if the areometer be charged with the little weight ω, l becomes l', and we have $l' = \frac{w+\omega-cs}{\pi r^2 s}$. Taking the first of these equations from the second, there will remain $l'-l = \frac{\omega}{\pi r^2 s}$: so that the variation in the immersion $l'-l$ is proportional to $\frac{\omega}{r^2}$ when s is constant *.

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* In using the areometer it is often requisite to know the diameter of the wire: to those who have not a micrometer M. *Prony* recommends the following method:

Measure the length of the wire exactly; weigh it both in water and

We see, therefore, that two areometers may be equally sensible when the specific gravity varies, and that the one of them may have need of an additional weight to descend an equal quantity when the specific gravity remains the same. Those, however, which have the little basin at top are preferable, because they are more solid, the wire being made thicker; but chiefly because by means of them we may ascertain more precisely the *difference* of specific gravities.

409. If the specific gravity s becomes s', and, in order to retain the areometer at the same depth, it be necessary that the weight w become $w+\omega$, we have (art. 406.) $s' = \frac{w+\omega}{c+\pi r^2 l}$. Dividing this equation by the equation $s = \frac{w}{c+\pi r^2 l}$, there re-

in the air, and take the difference of these two weights. Calling L the whole length of the wire, d the difference between its weight in air and in water, s the specific gravity of water, and D the diameter of the wire; then is $D = 2\sqrt{\frac{d}{\pi s L}}$. This may be done very expeditiously by means of logarithms.

To find the thickness of the wire in an areometer already constructed, we must charge the little basin at the top of the wire with a certain weight, and examine how far this causes the instrument to descend: then we must put in the formula $D = 2\sqrt{\frac{d}{\pi s L}}$, for d the little weight with which the areometer was charged, and for L the depth of immersion it occasioned. This follows from the equation $l'-l = \frac{\omega}{\pi r^2 s}$, which gives $2r = D = 2\sqrt{\frac{\omega}{\pi s(l'-l)}}$.

This may not be an improper place to describe a method of finding the radii of small cylinders (such as axles in fine machinery), first given, we believe, by Mr. Atwood; which is this.—Having fixed to the extremity of a very fine and flexible line a weight sufficient to keep it stretched, fasten the other extremity to the axle whose radius is required; the line being stretched by this weight, measure by a scale of equal parts any convenient length, 6 inches, for example, and mark the extremities of the length so measured. Then holding the axle horizontally, let the measured part of the line be wound round it in the form of a helix, the circumferences being every where contiguous. Count the number of revolutions, and parts of a revolution, and represent this number by n; also measure the length of the cylinder occupied by the helix, and let this $= a$; and the length of the helix, or line first mentioned $= \lambda$; then will the radius of the cylinder $= \frac{\sqrt{(\lambda^2 - a^2)}}{2\pi n}$. "The exactness of this method may be known," says Mr. Atwood, "by observing that, if the cylinder be truly made, and the process carefully repeated with different values of λ, n, and a, the radius deduced will, however, always come out the same, to the fourth or even the fifth decimal place."

sults $\frac{s'}{s}=\frac{w+\omega}{w}$, a formula expressing *the ratio of* the specific gravities.

This equation stated as an analogy gives $s' : s :: w + \omega : w$, whence $s' - s : s :: \omega : w$, and $s' - s = \frac{\omega s}{w}$, a formula which gives *the difference of the specific gravities.*

When this theorem is applied to different waters, we may always suppose one of them, as $s = 1000$ ounces averd. without apprehending any sensible error in the value of $s' - s$: for the quantity ω being only a few grains, s may undergo variations of 6 or 7 units, without the product $\frac{s}{w}\,\omega$ undergoing any such change as to affect the accuracy of the result. So that in finding the difference between the specific gravities of two waters it is not absolutely necessary to know one of them with great precision.

When we would make experiments with an areometer whose weight is exactly known, it will be proper, by means of the equations $\frac{s'}{s}=\frac{w+\omega}{w}$, and $s' - s = \frac{\omega s}{w}$, to construct a table which may contain the ratios and the differences of specific gravities, answering to the different values of ω between 1 and 40 or 50 grains: such a table, but little extended, will be easy to calculate, and will save the labour of a distinct computation for every individual experiment.

410. Previous to giving a table of medium specific gravities, we shall just remark, that all bodies of homogeneous or unorganised texture expand by heat, and contract by cooling. The expansion and contraction by the same change of temperature is very different in different bodies. Thus, water when heated from 60° to 100° increases its volume nearly $\frac{1}{167}$ of its bulk; mercury only $\frac{1}{243}$, and many substances much less. Hence it follows, that an experiment determines the specific gravity only in that very temperature in which the bodies are examined. It will therefore be proper always to note this temperature; and it will be convenient to adopt some very useful temperature for such trials in general: perhaps about 60° of Fahrenheit's thermometer is as convenient as any. It may always be procured in these climates without inconvenience. A temperature near to freezing would have some advantages, because water changes its bulk very little between the temperature 32° and 45°. But this temperature cannot always be obtained. It will much conduce to the facility of the comparison to know the variation which heat produces on pure water. The

following table, taken from the interesting observations of Dr. Blagden and Mr. Gilpin (Phil. Trans. 1792, or New Abridgement, vol. XVII. p. 263.), will answer this purpose.

Temperature of Water.	Bulk of Water.	Specific Gravity.
30		
35	99910	1·00090
40	99070	1·00094
45	99914	1·00086
50	99932	1·00068
55	99962	1·00038
60	100000	1·00000
65	100050	0·99950
70	100106	0·99894
75	100171	0·99830
80	100242	0·99759
85	100320	0·99681
90	100404	0·99598
95	100501	0·99502
100	100602	0·99402

Those gentlemen observed the expansion of water to be very anomalous between 32° and 45°, a fact which has been subsequently traced through all its gradations by numerous philosophers. This is distinctly seen during the gradual cooling of water to the point of freezing. It contracts for a while, and then suddenly expands. But we seldom have occasion to measure specific gravities in such temperatures.

Liquor-merchants often avail themselves of this circumstance of the expansion of fluids by heat, by contriving to make their chief purchases in the winter. M. *Homberg* and M. *Eisenschmid* found the absolute weight of a cubic inch of brandy to be 4 drs. 42 grs. in the winter, and only 4 drs. 32 grs. in the summer; and in spirits of nitre the difference was greater still. And, taking the average with respect to spirits in general, it is found that 32 gallons in the winter will expand to 33 gallons in the height of summer.

A TABLE

f the *Specific Gravities of different Bodies.*

Metals.

y, crude,	4064
-, glass of,	4946
-, molten,	6702
glass of, natural,	3594
molten,	5763
native orpiment,	5452
molten,	9823
, native,	9020
, ore of, in plumes,	4371
t, not hammered,	8396
to, wire-drawn,	8544
t, common,	7824
olten,	7812
lue, glass of,	2441
not hammered,	7788
he same wire-drawn,	8878
re of soft copper, or ral verdigr.	3572
re, of 24 carats, melted, ot hammered,	19258
same hammered,	19362
isian standard, 22 car. ammered,	17486
same hammered,	17589
nea, of Geo. II.	17150
nea of Geo. III.	17629
nish gold coin,	17655
lland ducats,	19352
ket standard, 20 car. ammered,	15709
same hammered,	15775
t,	7207
t at Carron,	7248
to, at Rotherham,	7157
, either hardened or not,	7788
ther tempered nor hard-,	7833
rdened, but not tem-d,	7840
npered and hardened,	7818
to not hardened,	7816
prismatic,	7355
to specular,	5218
to lenticular,	5012

Lead, molten,	11352
——, ore of, cubic,	7587
——, ditto horned,	6072
——, ore of black lead,	6745
——, ditto white lead,	4059
——, ditto ditto vitreous,	6558
——, ditto red lead,	6027
——, ditto saturnite,	5925
Manganese, striated,	4756
Molybdena,	4738
Mercury, solid or congealed	15632
———, fluent,	13568
———, natural calx of,	9230
———, precipitate, *per se*,	10871
———, precipitate red,	8399
———, brown cinabar,	10218
———, red cinabar,	6902
Nickel, molten,	7807
———, ore of, called Kupfer-nickel of Saxe	6648
———, Kupfer-nickel of Bohemia,	6607
Platina, crude, in grains,	15602
———, purified, not hammered,	19500
———, ditto hammered,	20337
———, ditto wire-drawn,	21042
———, ditto rolled,	22069
Silver virgin, 12 deniers, fine, not hammered,	10474
———, ditto hammered,	10511
———, Paris standard,	10175
———, shilling of Geo. II.	10000
———, shilling of Geo. III.	10534
———, French coin,	10408
Tin, pure Cornish, melted, and not hardened,	7291
——, the same hardened	7299
——, of Malacca, not hardened,	7296
——, the same hardened,	7307
——, ore of, red,	6935
——, ore of, black,	6901
——, ore of, white,	6008
Tungsten,	6066
Uranium,	6440
Wolfram,	7119
Zinc, molten,	7191

Precious Stones.

Beryl, or aqua-marine oriental,	3549
——, ditto occidental,	2723
Chrysolite, of the jewellers,	2782
——, of Brazil,	2692
Crystal, pure rock of Madagascar,	2653
——, of Brazil,	2653
——, European,	2655
——, rose coloured,	2670
——, yellow	2654
——, violet, or amethyst,	2654
——, white amethyst,	2651
——, Carthaginian,	2657
——, black	2654
Diamond, white oriental,	3521
——, rose-coloured orient.	3531
——, orange ditto,	3550
——, green ditto,	3524
——, blue ditto,	3525
——, Brazilian,	3444
——, yellow,	3519
Emerald, of Peru,	2775
Garnet, of Bohemia,	4189
——, of Syria,	4000
——, dodecaedral,	4063
——, volcanic, 24 faces,	2468
Girasol,	4000
Hyacinth, common,	3687
Jargon of Ceylon	4416
Quartz, crystallised,	2655
——, in the mass,	2647
——, brown crystallised,	2647
——, fragile,	2640
——, milky,	2652
——, fat, or greasy,	2646
Ruby, oriental,	4283
——, Spinell,	3760
——, Ballas,	3646
——, Brazilian,	3531
Sapphire, oriental,	3994
——, ditto white,	3991
——, of Puys,	4077
——, Brazilian,	3131
Spar, white parkling,	2595
——, red ditto,	2438
——, green ditto,	2704
——, blue sparkling,	2693
——, green and white ditto,	3105
——, transparent ditto,	2564
——, adamantine,	3873
Topaz, oriental,	4011
——, pistachio ditto,	4061
——, Brazilian,	3536
——, of Saxe,	3564
——, white ditto,	3554
Vermilion,	4230

Silicious Stones.

Agate, oriental,	2590
——, onyx,	2638
——, cloudy,	2625
——, speckled,	2607
——, veined,	2667
——, stained,	2632
Calcedony, common,	2616
——, transparent,	2664
——, veined,	2606
——, reddish,	2665
——, bluish,	2587
——. onyx,	2615
Carnelian, pale,	2630
——, speckled,	2612
——, veined,	2623
——, onyx,	2623
——, stalactite,	2591
——, simple,	2613
Flint, white,	2594
——, black,	2582
——, veined,	2612
——, Egyptian,	2565
Jade, white,	2950
——, green,	2966
——, olive,	2983
Jasper, clear green,	2359
——, brownish green,	2681
——, red,	2661
——, brown,	2691
——, yellow,	2710
——, violet,	2711
——, cloudy,	2735
——, veined,	2696
——, onyx,	2816
——, red and yellow,	2750
——, bloody,	2628
Opal,	2114
Pearl, virgin oriental,	2684
Pebble, onyx,	2664
——, of Rennes,	2654
——, English,	2609
——, veined,	2612
——, stained,	2587
Prasium,	2581
Sardonyx, pure,	2603
——, pale,	2606
——, speckled,	2692
——, veined,	2595
——, onyx,	2595
——, blackish,	2628
Schorl, black prism. hexaedral,	3364
——, octaedral,	3226
——, tourmalin of Ceylon,	3054
——, antique basaltes,	2923
——, Brazilian emerald,	3156
——, cruciform,	3286

Stone, paving,	2416
——, cutlers,	2111
——, grind,	2143
——, mill,	2484

Various Stones, Earths, &c.

Alabaster, oriental white,	2730
————, do. semi-transparent,	2762
————, yellow,	2699
————, stained brown,	2744
————, veined,	2691
————, of Piedmont,	2693
————, of Malta,	2699
————, Spanish saline,	2713
————, of Valencia,	2638
————, of Malaga,	2876
Amber, yellow transparent,	1078
Ambergris,	926
Amianthus, long,	909
————, short,	2313
Asbestos, ripe,	2578
————, starry,	3073
Basaltes from Giants' Causeway,	2864
Bitumen, of Judea,	1104
Brick,	2000
Chalk, Spanish,	2790
———, Coarse Briançon,	2727
———, British,	2784
Gypsum, opaque,	2168
————, semi-transparent,	2306
————, fine ditto,	2274
————, rhomboidal,	2311
————, ditto 10 faces,	2312
————, cuneiform crystallised,	2306
Glass, green,	2642
——, white,	2892
——, bottle,	2733
——, Leith crystal,	3189
——, fluid,	3329
Granite, red Egyptian,	2654
Hone, white razor,	2876
Lapis nephriticus,	2894
—— Lazuli,	3054
—— Hæmatites,	4360
—— Calaminaris,	5000
—— Judaicus,	2500
—— Manati,	2270
Limestone,	3179
————, white fluor,	3156
————, green	3182
Marble, green, Campanian,	2742
———, red,	2724
———, white Cassara,	2717
———, white Parian,	2838
———, Pyrenean,	2726
———, black, Biscayan,	2695
———, Brocatelle,	2650
———, Castilian,	2700
———, Valencian,	2710
———, white Grenadan,	2705
Marble, Siennien,	2678
———, Roman violet,	2755
———, African,	2708
———, violet Italian,	2858
———, Norwegian,	2728
———, Siberian,	2718
———, green Egyptian,	2668
———, Swisserland,	2714
———, French,	2649
Obsidian stone,	2348
Peat, hard,	1329
Phosphorus,	1714
Porcelaine, Seves,	2146
————, Limoges,	2341
————, China,	2385
Porphyry, red,	2765
————, green,	2676
————, red, from Dauphiny,	2793
————, red, from Cordoue,	2754
————, green, from ditto,	2798
Pyrites, coppery,	4954
———, feruginous cubic,	3900
———, ditto round,	4101
———, ditto, of St. Domingo,	3440
Serpentine, opake, green Italian,	2430
————, ditto, veined black and olive,	2594
————, ditto red and black,	2627
————, semi-transpa. grained,	2586
————, ditto fibrous,	3000
————, ditto from Dauphiny,	2669
Slate, common,	2672
——, new,	2854
——, black stone,	2186
——, flesh polished,	2766
Stalactite, transparent,	2324
————, opake,	2478
Stone, pumice,	915
——, prismatic basaltes,	2722
——, touch,	2415
——, Siberian blue,	2945
——, oriental ditto,	2771
——, common,	2520
——, Bristol,	2510
——, Burford,	2049
——, Portland,	2496
——, rag,	2470
——, rotten,	1981
——, hard paving,	2460
——, rock of Chatillon,	2122
——, clicard, from Brachet,	2357
——, ditto from Ouchain,	2274
——, Notre-Dame,	2378
——, St. Maur,	2034
——, St. Cloud,	2201
Sulphur, native,	2033
———, molten,	1991
Talc, of Muscovy,	2792
——, black crayon,	2089
——, ditto German,	2246
——, yellow,	2655

Talc, black,	2900
——, white,	2704

Liquors, Oils, &c.

Acid, sulphuric,	1841
——, ditto highly concentrated,	2125
——, nitric,	1271
——, ditto, highly concentrated,	1580
——, muriatic,	1194
——, red acetous,	1025
——, white acetous,	1014
——, distilled ditto,	1010
——, fluoric,	1500
——, acetic,	1063
——, phosphoric,	1558
——, formic,	994
Alcohol, commercial,	837
———, highly rectified,	829
———, mixed with water	
15-16ths alcohol	853
14-16ths ditto	867
13-16ths ditto	882
12-16ths ditto	895
11-16ths ditto	908
10-16ths ditto	920
9-16ths ditto	932
8-16ths ditto	943
7-16ths ditto	952
6-16ths ditto	960
5-16ths ditto	967
4-16ths ditto	973
3-16ths ditto	979
2-16ths ditto	985
1-16th ditto	992
Ammoniac, liquid,	897
Beer, pale,	1023
——, brown,	1034
Cyder,	1018
Ether, sulphuric,	739
——, nitric,	909
——, muriatic,	730
——, acetic,	866
Milk, woman's,	1020
——, cow's,	1032
——, asses',	1036
——, ewe's,	1041
——, goat's	1035
——, mare's,	1034
——, cow's clarified,	1019
Oil, essential, of turpentine,	870
—, essential, of lavender,	894
—, ditto, of cloves,	1036
—, ditto, of cinnamon,	1044
—, of olives,	915
—, of sweet almonds,	917
—, of filberts,	916
—, linseed,	940
—, of walnuts,	923
—, of whale,	923
—, of hempseed,	926
Oil of poppies,	924
—, rapeseed,	919
Spirit of wine. See Alcohol.	
Turpentine, liquid,	991
Urine, human,	1011
Water, rain,	1000
———, distilled,	1000
———, sea, (average)	1026
———, of Dead-sea,	1240
Wine, Burgundy,	992
——, Bourdeaux,	994
——, Madeira,	1038
——, Port,	997
——, Canary,	1033

Resins, Gums, and Animal Substances, &c.

Aloes, socotrine,	1380
——, hepatic,	1359
Asafœtida,	1328
Bees-wax, yellow,	965
———, white,	969
Bone of an ox,	1656
Butter,	942
Calculus humanus,	1700
————, ditto,	1240
————, ditto,	1434
Camphor,	989
Copal, opake,	1149
——, Madagascar,	1060
——, Chinese,	1063
Crassamentum, human blood,	1126
Dragon's blood,	1205
Elemi,	1018
Fat, beef,	923
—, hog's,	937
—, mutton,	924
—, veal,	934
Galbanum,	1212
Gamboge,	1222
Gum, ammoniac,	1207
——, Arabic,	1452
——, Euphorbia,	1124
——, seraphic,	1201
——, tragacanth,	1316
——, bdellium,	1372
——, scammony of Smyrna,	1274
——, ditto of Aleppo,	1235
Gun-powder, shaken,	932
—————, in a loose heap,	836
—————, solid,	1745
Honey,	1450
Indigo,	769
Ivory,	1826
Juice of liquorice,	1723
—— of Acacia,	1515
Labdanum,	1186
Lard,	948
Mastic,	1074

Myrrh,	1360
Opium,	1336
Scammony. See Gum.	
Serum of human blood,	1030
Spermaceti,	943
Storax,	1110
Tallow,	942
Terra Japonica,	1398
Tragacanth. See Gum.	
Wax. See Bees'-wax.	
——, shoemaker's,	897

Woods.

Alder,	800
Apple-tree,	793
Ash, the trunk,	845
Bay-tree,	822
Beech,	852
Box, French,	912
——, Dutch,	1328
——, Brazilian red,	1031
Campechy-wood,	913
Cedar, wild,	596
——, Palestine,	613
——, Indian,	1315
——, American,	561
Citron,	726
Coco-wood,	1040
Cherry-tree,	715
Cork,	240
Cypress, Spanish,	644
Ebony, American,	1331
Ebony, Indian,	1209
Elder-tree,	695
Elm, trunk of,	671
Filbert-tree,	600
Fir, male,	550
——, female,	498
Hazel,	600
Jasmin Spanish,	770
Juniper-tree,	556
Lemon-tree,	703
Lignum-vitæ,	1333
Linden-tree,	604
Logwood. See Campechy.	
Mastick-tree,	849
Mahogany,	1063
Maple,	750
Medlar,	944
Mulberry, Spanish,	897
Oak, heart of, 60 years old,	1170
Olive-tree,	927
Orange-tree,	705
Pear-tree,	661
Pomegranate-tree,	1354
Poplar,	383
——, white, Spanish,	529
Plum-tree,	785
Quince-tree,	705
Sassafras,	482
Vine,	1327
Walnut,	671
Willow,	585
Yew, Dutch,	788
——, Spanish,	807

Weight and Specific Gravities of different Gases.

Fahrenheit's Thermom. 55° Barometer 30 inches.

	Spec. Grav.	Wt. Cub. Foot.
Atmospheric air,	1·2	525·0 grs.
Hydrogen gas,	0·1	43·75
Oxygen gas,	1·435	627·812
Azotic gas,	1.182	517·125
Nitrous gas,	1·4544	636·333
Ammoniac gas,	·7311	319·832
Sulphureous acid gas,	2·7611	1207·978

In this table the weights and specific gravities of the principal gases are given, as they correspond to a state of the barometer and thermometer which may be chosen for a medium. The specific gravity of any one gas to that of another will not conform to exactly the same ratio under different degrees of heat and other pressures of the atmosphere.

CHAPTER III.

Equilibrium, Stability, and Oscillations of Floating Bodies.

411. Among the different bodies which float on the surface of a fluid, the most important are ships and other vessels employed on seas, rivers, and canals, in commerce. It is of much consequence to determine the stability of such vessels, and the positions they assume when they float freely on the water. The principles on which the stability of the different positions depend flow naturally from what has been already done, and may be here illustrated and explained. Two or three definitions are premised.

Defs. 1. *The plane of floatation* is the horizontal surface of the fluid in which the vessel floats.

2. *The line of support* is the vertical line passing through the centre of gravity of the part of the solid or vessel immersed.

3. *The metacentre* is the point of intersection of the axis passing through the centre of gravity (round which the body revolves through an indefinitely small angle) and the line of support.

412. That a body floating upon a quiescent fluid may be in equilibrio, something more is necessary than that the weight of the body should be equal to that of the fluid displaced: for this condition only ensures an equality of the upward push of the fluid and the weight of the body; and these two vertical forces cannot destroy each other unless they are directly opposed: they must therefore pass through the same point; that is to say, *the right line which joins the centres of gravity of the floating body and of the fluid displaced must be vertical.* Without this condition the two vertical forces, though equal, cannot annihilate each other's effects; and, of consequence, the body will have about its centre of gravity a rotatory motion, as if that point were fixed.

Indeed, positions may be assumed in which the circumstances just recited concur, and yet the solid will take some other position in which it will permanently float. If a cylinder, for example, having its specific gravity to that of the fluid on which it floats as 3 to 4, and its axis to the diameter of its base as 2 to 1, be placed on the fluid with its axis vertical, it will sink to a depth equal to a diameter and a half of the base; and while its axis is preserved in a vertical position by external force, the centres of gravity of the whole solid and of the immersed part will remain in the same vertical line: but when the external force that sustained it is removed, it will decline from its upright position, and will permanently float with its axis horizontal. If the axis be supposed to be half the diameter of the base, and be placed vertically, the solid will sink to the depth of three-eighths of its diameter; and in that position it will float permanently. If the axis be made to incline to the vertical line, the solid will change its position until it settles permanently with the axis perpendicular to the horizon.

DEF. Whether a solid float permanently, or overset when placed on the surface of a fluid, so that the centre of gravity of the solid and that of the part immersed shall be in the same vertical line, it is said to be in a position of equilibrium; and of this equilibrium there are three species, viz. the equilibrium of *stability*, in which the solid floats permanently in a given position; the equilibrium of *instability*, in which the solid, though the two centres of gravity already mentioned are in the same vertical line, spontaneously oversets, unless supported by external force; and the equilibrium of *indifference*, or the insensible equilibrium, in which the solid rests on the fluid indifferent to motion, without tendency to right itself when inclined, or to incline itself further.

These states of a floating body are also those in which the stability may be said to be *positive, negative,* and *nothing*, respectively.

413. From the properties investigated in the chapter on the centre of gravity, it follows that the right line which joins the centres of gravity of a body and of a segment of that body formed by any plane whatever, must pass through the centre of gravity of the other segment: in the case which we propose to treat here, the plane cutting the body is the upper surface of the fluid, or the plane of floatation; and we conclude that, in order that a body may be in equilibrium upon a fluid which is specifically heavier than itself, the right line which passes through the centres of gravity of the body and either of the segments formed by the plane of floatation must be perpendicular to that plane. Thus the general problem of the determination of the positions of equilibrium with regard to a homogeneous body in a fluid is reduced to the following.

PROP. *To cut by a plane a body of given figure in such manner that the capacity of either of the segments shall be to* *that of the whole body in a given ratio; and that the right line which passes through the centre of gravity of the body and those of its segments shall be perpendicular to the intersecting plane.*

The ratio of the segments into which the body must be cut, or of either of them to the whole body, will depend upon the ratio of the specific gravities of the body and the fluid: and the positions of equilibrium of any body will be given by the roots deduced from the equations of equilibrium, and the equation defining the nature of the body. We cannot here enter much into the detail: but shall merely present a few obvious examples.

414. It is manifest, then, that right prisms and cylinders of any base whatever, if homogeneous, will have two positions of equilibrium; viz. when the generating axis is placed vertically, and when it is placed horizontally. The equilibrium obtains also, with regard to solids of revolution, and symmetrical bodies relative to any line, when that line is placed vertically. Hence, a floating sphere will continue at rest in any position; because it is, in every position, symmetrical with respect to the line of support passing through its centre of gravity.

Moreover, prismatic or cylindric bodies, and in general those which are symmetrical with regard to a plane (as are all which need be considered in the theory), have their positions of equilibrium, when that plane, relatively to which they are symmetrical, is vertical: in which case we have only to ascertain the positions of equilibrium for an area or vertical section of the body, perpendicular to that with respect to which the whole is symmetrical.

415. Let us propose an instance for more particular examination, in the calculus of which it will appear in what manner different positions of equilibrium of a body will be furnished by the same equation.

Suppose that the body is a triangular prism, which floats with its ends vertical, and its edges horizontal.

Here we must consider two cases: 1st. That where the bases of the prism have an angle F immersed in the fluid, and the other two E, H, out of it (fig. 10. pl. XVI.). 2d. The reverse of this, where the two angles E, H, are immersed (fig. 11.). The two cases may be treated at the same time; for the method in each is the same.

Let XX′ be the plane of floatation. To determine the line MN, in which this plane should intersect the triangle, we must find FM, and FN. Call the first of these x, the second y: and taking P in the middle of EG we may represent the data thus: $FE=a$, $FH=b$, $FP=k$, angle $PFE=m$, $HFP=n$; and let the ratio of the specific gravities of the solid and fluid $=r$ to 1.

Then, that the weight of the solid may be equal to that of the fluid displaced, we must have,

In the first case (fig. 10.) .. r . FEH = FMN ;

In the second case (fig. 11.) .. r . FEH = HEMN.

Now the triangles FMN, FEH, having the angle F common, have their areas proportional to the rectangles of the sides about that angle: that is,

FEH : FMN :: FH × FE : FM × FN :: $ab : xy$;

whence FEH − FMN, or HNME : FEH :: $ab - xy : ab$;

consequently $xy = rab$, and $xy = ab\ (1 - r) \ldots (1)$.

These two equations include the condition of the equality of the vertical pressures, upwards and downwards.

Let us next find equations to satisfy the second condition of the centres of gravity residing in the same vertical line. Now if PR be taken = $\frac{1}{3}$PF, R (art. 114.) will be the centre of gravity of the triangle FE ; in like manner, if Q be the middle point of MN, and QG = $\frac{1}{3}$QF, G will be the centre of gravity of the triangle FMN : the right line RG, or its parallel PQ, will therefore be vertical in the case of the equilibrium ; and this condition will evidently be expressed by the equation PM = PN, in both figures. From the point P demit upon the sides FE, FH, the perpendiculars PA, PD : then will PA $= k \sin m$, PD $= k \sin n$, FA $= k \cos m$, FD $= k \cos n$. Therefore ND $= k \cos n - y$, and AM $= k \cos m - x$. But PM = PN gives $\text{AP}^2 + \text{AM}^2 = \text{PD}^2 + \text{DN}^2$: that is, $k^2 \sin^2 m = (k \cos m - x)^2 = k^2 \sin^2 n + (k \cos n - y)^2$; from which, by reduction, we find

$$y^2 - 2ky \cos n = x^2 - 2kx \cos m.$$

If for y in this equation we substitute separately each of its values deduced from the two equations No. 1, there will result two equations containing only x and known quantities, and which will express the conditions of equilibrium for the respective cases of figs. 10. and 11. They are as follows:

$$\left.\begin{array}{l} x^4 - 2kx^3 \cos m + 2abkrx \cdot \cos n - r^2a^2b^2 \ldots\ldots\ldots\ldots = 0 \\ x^4 - 2kx^3 \cos m + 2\ (1 - r)\ abkx \,.\, \cos n - (1 - r)^2 a^2b^2 = 0 \end{array}\right\} \ldots \text{(II.)}$$

These equations being of the fourth degree, and having the last term negative, have at least *two* real roots (Lacroix's Alg. No. 219.): but the roots may be all four real, and then the disposition of the signs indicate that three of the roots are positive, and the fourth negative. This latter root is obviously useless in the present case, because the solicitation of gravity being constantly downwards, the right-line FM can only be placed on one side with regard to the point F. There must, consequently, be either *one* or *three* positions of equilibrium, which will be determined by the positive roots of the equations marked (II.) The corresponding values of y will be furnished by the equations (I.) We must always, however, have $x < a$, and $y < b$.

416. The preceding reasonings may now be applied to the isosceles triangle, as the most likely to occur in practice: and to prevent the calculus from being very complicated, let us merely consider the case where one angle only of the triangle is immersed in the fluid: the other case being easily deduced by transforming in the results r into $1 - r$, as may be shewn by comparing together either the equations No. II. or those of No. I. Here, therefore, we have $m = n$, and $a = b$, from which there arises

$$xy = a^2 r, \text{ and } x^4 - 2kx^3 \cos m + 2a^2 rkx \cos m - r^2 a^4 = 0.$$

The factors of the second degree of this latter equation being $x^2 - a^2 r = 0$, and $x^2 - 2kx \cos m + a^2 r = 0$, we thence conclude, taking only the positive roots, that,

$$x = a\sqrt{r}, \text{ and } x = k \cos m \pm \sqrt{(k^2 \cos^2 m - a^2 r)}$$

$$\text{whence, } y = x, \text{ and } y = k \cos m \mp \sqrt{(k^2 \cos^2 m - a^2 r)}.$$

The first root indicates that there will be only one position of equilibrium when EH is horizontal: and the same applies to the case of fig. 11. The other positions are given by the other two roots; but they must be such as to correspond with the conditions of the remarkable limits: for we ought to have $x < a$, and every root real; and from this we find

$$r > \frac{2ak \cos m - a^2}{a}, \text{ and } r < \frac{k^2 \cos^2 m}{a^2}$$

In like manner we shall find the limits for the second case, by changing r into $1 - r$; whence there results

$$r < \frac{2a - 2k \cos m}{a}, \text{ and } r > \frac{a^2 - k^2 \cos^2 m}{a^2}$$

If we suppose the triangle equilateral we shall have $k = a\sqrt{\frac{3}{4}}$, and $\cos m = \sqrt{\frac{3}{4}}$; therefore $\cos m = \frac{3}{4}a$: and the preceding limits will become—

First case, $r > \frac{1}{2}$, and $r < \frac{9}{16}$: second case, $r < \frac{1}{2}$, and $r > \frac{7}{16}$.

417. Prop. *When a floating body is in equilibrio in any fluid, and an extraneous cause as an impulsion move the body, it is proposed to determine whether this perturbation will permit the body to return to its first position, or constrain it, on the contrary, to move still further from that of equilibrium.*

As the investigation of this problem in its utmost extent would lead into very tedious and complex discussions, we shall premise three hypotheses, in order that we may obtain comparatively simple results.

1. We suppose the floating body to be symmetrical with respect to a vertical plane passing through its centre of gravity when the whole is at rest; and of such a kind that we need

only resolve the problem for the area of a plane section of the body.

2. We conceive the derangement to be indefinitely small. Thus, in fig. 12. pl. XVI. where *ab* is the plane of floatation, G the centre of gravity of the floating body DFE, and O the centre of gravity of AFB, or of the fluid displaced when the line AB coincides with the plane of floatation; the inclination of the body will then be the angle GOV formed by GO and the vertical line OV passing through the centre of *a*F*b*; and we put this evanescent angle, or the arc which measures it, = A.

3. We disregard the vertical motion of the centre of gravity of the body as indefinitely small; and suppose that the new position is consistent with the equality of the weights of the body and of the fluid displaced. Thus will the portion AC*a*, which is moved out of the fluid, be equal to the part BC*b*, which has entered it in consequence of the motion.

Granting these suppositions, the equal areas AC*a* and BC*b* may be regarded as isosceles triangles, since we may suppose AC = *a*C and CB = C*b*: hence it will follow that *a*C = C*b*; that is, the intersection C of the two lines of floatation is in the middle of AB.

The upward pressure of the fluid on *a*F*b* is equal to the weight of a lamina of that fluid of an equal magnitude; and this vertical force acts at the centre of gravity of *a*F*b*. So that, as it is necessary to know this centre, we shall proceed to assign its position. In order to this, it may be observed, that *a*F*b* = AFB + CB*b* − AC*a*; and if we conceive the laminæ of the fluid equal to these areas, it is evident that the moment of the upward pressure of the fluid, taken with relation to any vertical whatever, is equal to the sum of the moments of the fluid laminæ of which the volumes are AFB and CB*b*, minus the moment of the lamina AC*a*. Let us then estimate these moments with respect to the vertical G*i*. So shall we have

I. The moment of the weight of the fluid laminæ AFB is W · GV; W representing the weight of the body, or that of the fluid displaced.

II. Representing by *w* [CB*b*] the weight of a volume of fluid equal to CB*b*, we have for the moment of this weight its product by the distance *qi*: the point *q* being the foot of the perpendicular demitted upon C*b* from the centre of gravity of CB*b*, that is, taking C*q* = $\frac{2}{3}$C*b*. Thus the moment of CB*b* is *w* [CB*b*] × *qi*.

III. Lastly, the moment of AC*a* is, in like manner, *w* [AC*a*] × *pi*; making *p*C = $\frac{2}{3}$*a*C. Where, as this latter force tends to produce a motion in the contrary direction to the former, it must be taken negatively; and since, moreover, it ought to be subtracted from the two others, it becomes positive, and the moment M of the pressure of the fluid upon aFb is (because $\mathrm{CB}b = \mathrm{AC}a$) expressed thus; $\mathrm{M} = (\mathrm{W} \times \mathrm{GV}) + w\,[\mathrm{CB}b] \times pq$.

Now, since the fluid is supposed homogeneous, the areas AFB and $\mathrm{BC}b$ are proportional to the weights of the quantities of fluid to which they are respectively equal; that is to say, if we put the surface $\mathrm{AFB} = \mathrm{S}$, since $\mathrm{CB} = \mathrm{C}b$, and moreover $\mathrm{B}e = \mathrm{CB} \times \sin \mathrm{A} = \mathrm{A} \cdot \mathrm{CB}$, we have the triangle $\mathrm{CB}b = \frac{1}{2}\mathrm{A} \cdot \mathrm{CB} \cdot \mathrm{CB}$: therefore,

$$\mathrm{S} : \mathrm{W} :: \tfrac{1}{2}\mathrm{A}\,(\mathrm{BC})^2 : w\,[\mathrm{CB}b] = \frac{\mathrm{W}}{2\mathrm{S}}\,\mathrm{A}\,(\mathrm{BC})^2.$$

But $pq = \frac{2}{3}ab$, and $\mathrm{GV} = \mathrm{A} \cdot \mathrm{GO}$. Putting therefore $\mathrm{GO} = a$, and $\mathrm{AB} = b = 2\mathrm{BC}$, there results

$$\mathrm{M} = \mathrm{A} \cdot \mathrm{W}\left(a + \frac{b^3}{12\mathrm{s}}\right) \ldots\ldots\ldots \text{(i.)}$$

418. Since we consider the body as if it were retained by a fixed axe passing through G, the preceding value of M will measure the pressure of the fluid. So that, if we would compare together these pressures upon different bodies, we must consider them in reference to the same angle of inclination A, and the relative stabilities will be measured by

$$\mathrm{W}\left(\frac{b^3}{12\mathrm{s}} \pm a\right) \ldots\ldots\ldots \text{(ii.)}$$

Cor. If the solid be of an irregular form, the stability will be as the sum of all the $\mathrm{W}.\left(\frac{b^3}{12\mathrm{s}} \pm a\right)$.

By *stability* we would here be understood to mean the resistance which a body opposes to its change of position; that is, the effort made by the fluid to retain the body in its position of equilibrium, or to carry it further from it. In fig. 12. we have placed the centre of gravity of the body *below* that of the fluid displaced, whence it follows that the body is not homogeneous: and this is what happens most frequently (ships of war being excepted), because it is customary to dispose, in the inferior parts about F, some substances specifically heavier than the materials of the floating body. If the centre of gravity of the body were, on the contrary, higher than that of the fluid displaced, it would be necessary to take a as negative: on this account we have put in the formulæ (ii.) the double sign ± before the quantity a.

419. The value ii. may (in conformity with art. 412.) be either positive, nothing, or negative. 1st. When the centre of gravity of the body is lower than that of the fluid displaced, the stability will be positive so long as a retains the upper sign;

and it will be positive in the contrary case, when O is below G, whenever $a < \frac{b^3}{12s}$. In these cases the pressure of the fluid tends obviously to carry back the body to its first position. This tendency is, besides, the stronger, as the value (ii) is the greater. We see also the advantage of having the floating body narrower towards the bottom than towards the plane of floatation; and the corresponding advantage of making the upper parts of such materials as have the least specific gravity.

2dly. In the two other cases the centre of gravity of the body is *higher* than that of the fluid displaced: if we have $a > \frac{b^3}{12s}$, the stability is *negative*, and the aggregate pressure tends to move the body in the opposite direction, or so as to increase the angle A; for then the centre of gravity of $a\mathrm{F}b$ being on the other side of Gi must cause the body to proceed further from its first position of equilibrium. If $a = \frac{b^3}{12s}$, the stability is *nothing* or *indifferent;* and the total pressure does not exert any such effort calculated to turn the body, because the centre of gravity of $a\mathrm{F}b$ is then upon Gi, and consequently in the same vertical line as G.

420. If the quantity (ii) be divided by W, the weight of the fluid displaced, the quotient will express the distance from the vertical Gi to the direction of the resultant of the upward pressure of the fluid: that distance being taken $= \mathrm{G}n$, we have

$$\mathrm{G}n = \frac{\mathrm{A}\cdot b^3}{12s} \pm \mathrm{GV} = \mathrm{A}\cdot\left(\frac{b^3}{12s} \pm a\right) \ldots\ldots \text{(iii)}.$$

If b is positive, the vertical Gi falls to the left of the centre of gravity of the fluid displaced; in this case Gn must be taken to the right of Gi: the same thing obtains in the contrary case, provided that we have $\mathrm{GV} < \frac{\mathrm{A}\cdot b^3}{12s}$, the vertical ng intersecting the right line GO in a point g more elevated than G. But if, in the latter case, $\mathrm{GV} = \frac{\mathrm{A}\cdot b^3}{12s}$, we have then $\mathrm{G}n = 0$, and g falls upon Gi. Finally, if we have $\mathrm{GV} > \frac{\mathrm{A}\cdot b^3}{12s}$, and G constantly higher than O, the line Gn must be taken to the left of Gi, and the point g will fall below G. To find the distance Gg it may be observed that $\mathrm{G}g\cdot\sin\mathrm{A} = \mathrm{G}n$, or, because an evanescent arc may be substituted for its sine, $\mathrm{A}\cdot\mathrm{G}g = \mathrm{G}n$;

$$\text{wherefore} \ldots \mathrm{G}g = \frac{b^3}{12s} \pm a \quad \ldots\ldots \text{(iv.)}$$

The point g corresponds with our definition of *metacentre* (art. 411.), we may therefore announce the result of this investigation thus: *the stability of a body or vessel is positive, nothing, or negative, according as the metacentre is more elevated, the same, or lower than the centre of gravity of the body.*

421. Prop. *If the floating body be a homogeneous rectangular parallelopiped, whose altitude is perpendicular to the surface of the fluid, its stability will be proportional to the difference between the sixth part of the square of the breadth of the base and the product of the square of the altitude, into the difference between the number expressing the specific gravity of the solid and its square, that of the fluid being unity.*

Let h be the height of the solid, b the breadth of the base, and s the specific gravity of the solid; then will the cube of the line coinciding with the plane of floatation $= b^3$ as before, the height of the part immersed $= sh$, and s the space proportional to the part immersed $= sbh$; GO the distance between the centres of gravity of the whole body and of the part immersed $= \frac{1}{2}h - \frac{1}{2}sh = \frac{h-sh}{2}$. Consequently the stability which varies as $\frac{b^3}{12s} \pm$ GO, varies as $\frac{b^3}{12sbh} - \frac{h-sh}{2} = \frac{b^2}{6} - [h^2 \cdot (s - s^2)]$.

As an example to illustrate this proposition, suppose the height of the parallelopiped equal to the breadth of the base, or $h = b$, and let it be required to ascertain the specific gravity of the solid, when it will float in the equilibrium of indifference. Here $\frac{b^2}{6} = h^2 \cdot (s - s^2)$, or, since $h = b$, we have $s^2 - s = -\frac{1}{6}$; whence $s = \frac{1}{2} \pm \sqrt{(\frac{1}{4} - \frac{1}{6})} = \frac{1}{2} \pm \frac{1}{6}\sqrt{3} = {\cdot}78868$ or ${\cdot}21132$ nearly, either of which may be taken for the specific gravity of the solid, that of the fluid being unity. If the fluid were rain-water, the bodies would have nearly the same specific gravity as apple-tree and cork.

422. Prop. *If the floating body be an homogeneous cylinder whose axis is vertical, its stability will be as the quotient of the square of the radius of the base divided by four times the height of the part immersed, diminished by the distance between the centres of gravity of the whole solid and the part immersed.*

Let DEFG (fig. 2. pl. XVII.) be the section of the cylinder coinciding with the plane of floatation, AB any line in that section, or the breadth of any variable vertical section of the cylinder; CG $= r$, CV $= x$; then AB $= 2\sqrt{(r^2 - x^2)}$, and AB^3 $= 8(r^2 - x^2)^{\frac{3}{2}}$. But $(r^2 - x^2)^{\frac{3}{2}} = (r^2 - x^2)^{\frac{1}{2}} \times r^2 - (r^2 - x^2)^{\frac{1}{2}} \times x^2$: Also $r^2(r^2 - x^2)^{\frac{1}{2}} = r^2 \times \frac{1}{4}$ of the circle whose radius is r, when

x increases from 0 to r; and $x^2(r^2-x^2)^{\frac{1}{2}} = rx^2 - \frac{x^4}{2r} - \frac{x^6}{8r^3}$, &c. and the quantity generated $= \frac{rx^3}{3} - \frac{x^5}{10r} - \frac{x^7}{56r^3}$, &c. which, when x increases from 0 to r, becomes $r^4 \times (\frac{1}{3} - \frac{1}{10} - \frac{1}{56}$, &c.) $= r^4 \times \frac{1}{4}$ of a circle whose diameter is unity (vide *Anal. per Equationes*, pa. 74.) $= r^2 \times \frac{1}{16}$ of a circle whose radius is r. Consequently, the whole quantity generated by $(r^2-x^2)^{\frac{3}{2}}$ will be $r^2 \times (\frac{1}{4} - \frac{1}{16})$ of a circle whose radius is $r = r^2 \times \frac{3}{16}$ of that circle $= r^2 \times \frac{3}{16}\pi r^2 = \frac{3}{16}\pi r^4$. Hence the sum of all the AB^3 in one semicircle $= 8r^2 \times \frac{3}{16}\pi r^2, = \frac{3}{2}\pi r^4$, and the sum of all the AB^3 in both semicircles $= 3\pi r^4$; while the volume of the part immersed is equal to its depth (d) × circle DEFG $= \pi dr^2$. Therefore $\int \frac{\text{AB}^3}{12s} = \frac{3\pi r^4}{12\pi dr^2} = \frac{r^2}{4d}$; and the general expression for the stability (ii) is proportional to $\frac{\text{CG}^2}{4d} - \text{GO}$, as in the proposition.

COR. If s be the specific gravity of the solid, that of the fluid being unity, and h the height of the cylinder, then will the depth of the part immersed $= sh$, and $\text{GO} = \frac{h-sh}{2}$: so that the stability of the cylinder will be proportional to $\frac{r^2}{4sh} - \frac{h-sh}{2}$, and will vanish or become indifferent when $\frac{r^2}{4sh} = \frac{h-sh}{2}$, or when $s^2 - s = -\frac{r^2}{2h^2}$, or $s = \frac{1}{2} \pm \sqrt{\left(\frac{1}{4} - \frac{r^2}{2h^2}\right)}$: where the expression under the radical vanishes if $r = h\sqrt{\frac{1}{2}}$, or $2r^2 = h^2$. So that, *if the specific gravity of the cylinder* [illegible] *half that of the fluid, and the radius of the base be to the height of the cylinder as* 1 *to* $\sqrt{2}$, *the cylinder will float in a state of insensible equilibrium.*

COR. 2. *If* s $= \frac{3}{4}$, *then in the case of insensible equilibrium* r *will be to* h *as* $\sqrt{3}$ *to* $2\sqrt{2}$. And if $s = \frac{1}{4}$ *we shall have the same ratio.*

423. PROP. *Considering the fluid as non-resisting, the floating body will oscillate incessantly about a horizontal axis passing through its centre of gravity, according to laws analogous to those of bodies in a vacuum: it is proposed to examine the particular nature of these oscillations, regarding them as indefinitely small.*

Supposing that the vertical line passing through the centre of gravity of the body, in its equilibrated position, has been inclined by the quantity f, then, denoting by A the inclination of the body at the end of the time t, and by α the arc described by the point placed at the unit of distance from G (fig. 12. pl. XVI.), we shall have $A = f - \alpha$. Now the angular accelerating force, as has been seen in the Dynamics, is the quotient of the sum of the momenta of the moving forces divided by the momentum of inertia: the dividend, being the total vertical pressure of the fluid, is given by the value art. 418. i.; the divisor is Mk^2, M being the mass of the body, and k^2 the quotient of the momentum of inertia (with respect to an axe passing through the centre of gravity) divided by the mass. Hence, as the weight W of the body is equivalent to gM (art. 108.) we have, for the angular accelerating force,

$$\frac{\ddot{\alpha}}{\dot{t}^2} = \frac{g}{k^2} \cdot A \left(a + \frac{b^2}{12s}\right) = \frac{g}{k^2}\left(a + \frac{b^3}{12s}\right) \cdot (f - \alpha).$$

Or, if we put l for $\left(a + \frac{b^3}{12s}\right) \div k^2$, there will arise $\frac{\ddot{\alpha}}{\dot{t}^2} = \frac{g}{l}(f - \alpha)$. Multiplying this by $\dot{\alpha}$, we obtain for the fluent $\left(\frac{\dot{\alpha}}{\dot{t}}\right)^2 = \frac{g}{l}(2f\alpha - \alpha^2) + C$, a constant quantity. Here $\frac{\dot{\alpha}}{\dot{t}}$ is the angular velocity; so that at the same time that $\frac{\dot{\alpha}}{\dot{t}} = 0, \alpha = 0$, and $C = 0$, which gives

$$\dot{t} = \sqrt{\frac{l}{g}} \cdot \frac{\dot{\alpha}}{\sqrt{(2f\alpha - \alpha^2)}}.$$

Taking the fluents again, we have

$$t = \sqrt{\frac{l}{g}} \cdot \text{arc}\left(\cos = \frac{f-\alpha}{f}\right) + C'.$$

To correct this we must consider that when $t = 0$, $\alpha = 0$, and $C' = 0$; whence we find

$$\frac{f-\alpha}{f} = \cos\left(t\sqrt{\frac{g}{l}}\right), \text{ or } \alpha = f\left(1 \cos\left(t\sqrt{\frac{g}{l}}\right)\right).$$

The variables comprised in this equation are the arcs α and $t\sqrt{\frac{g}{l}}$: where, if we attribute successively to

$t\sqrt{\frac{g}{l}}$ the values . . 0, $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$, 2π, &c.

we find . . . 0, f, $2f$, f, 0, &c. for

the corresponding values of α; which shews that the moving body, after describing the descending arc f, will describe an ascending arc f in the same time, then return and rise the contrary way, and so on. It is obvious also, that the time of making a complete oscillation is given by taking $\pi = t\sqrt{\frac{g}{l}}$, since that hypothesis gives $\alpha = 2f$, as it ought to do. Hence there results

$$t = \pi\sqrt{\frac{l}{g}}.$$

This expression for the time does not comprise the quantity f; whence it follows, that the vibrations are isochronal: and, by comparing the above with the value of t for the simple pendulum vibrating in indefinitely small arcs (art. 269.), it will appear that they exactly agree: consequently, the chief circumstances traced there apply immediately to the present case. Now if we restore the value of l we shall have

$$l = \frac{12k^2s}{12as + b^3};$$

an expression which shews the length of the synchronal simple pendulum.

And, for the time of oscillation,

$$t = \pi k\sqrt{\frac{12s}{g(12as + b^3)}}.$$

424. By way of applying these principles, take the instance of a prismatic body, such that the transverse section of the immerged part in the position of equilibrium is an isosceles triangle, of which the vertices of the equal angles are in the plane of floatation: the upper part of the section may have almost any variety of form. Let b represent the semi-base of the isosceles triangle, which is the transverse section of the part immersed, h the height of this triangle, and d the distance of the upper surface of the fluid from the centre of gravity of the body: the distance of that surface from the centre of gravity of the triangle immersed will be $\frac{1}{3}h$: consequently the distance between the centres of gravity of the body and of the fluid displaced will be $a = d - \frac{1}{3}h$; the area immersed will be $s = bh$, and the value (418. ii.) of the stability will be

$$\frac{w}{3h}(3dh - h^2 + 2b^2)$$

which may be either positive, nothing, or negative, as in the various cases before specified. The length of the synchronal pendulum will be $= \frac{3hk^2}{3dh - h^2 + 2b^2}$.

We may here remark, that the oscillations of a floating body in *finite* angles are not analogous to those of a cycloidal pendulum: for the force of stability varies in a proportion very different from that of the distance from quiescence, unless the arcs of vibration be of evanescent magnitude.

SCHOLIUM.

425. The determination of the stability and the time of oscillation of floating bodies, a sketch of which is given in this chapter, is by no means a matter of mere curiosity; but is, when considered more at large, of considerable practical importance. Those students who wish to pursue this subject may read with much advantage the admirable treatise by Leonard Euler, entitled *Théorie Complette de la Construction et de la Manœuvre des Vaisseaux*, or the English translation by colonel Henry Watson; and an ingenious paper in the *Philosophical Transactions* for 1796, by Mr. Atwood. In ships of war and merchandise the calculations are very complex and operose, and not always so accurate as is desirable; but in river and canal boats the regularity and simplicity of the form of the vessel itself, together with the compact disposition and homogeneal quality of the burden, render the computations far more easy.

In that valuable miscellany *Tilloch's Philosophical Magazine*, there is a paper on this subject by Mr. John English; from which we shall make an extract, shewing the application of the principles to river and canal boats.

" Vessels of this kind," says Mr. E. " are generally of the same transverse section throughout their whole length, except a small part in prow and stern, formed by segments of circles or other simple curves; therefore a length may easily be assigned such, that any of the transverse sections being multiplied thereby, the product will be equal to the whole solidity of the vessel. The form of the section ABCD is for the most part either rectangular, as in fig. 13. pl. XVI. trapezoidal, as in fig. 1. pl. XVII. or mixtilineal, as in fig. 14. pl. XVI., in all which MM represents the line of floatation when upright, and EF that when inclined at any angle MXE; also G represents the centre of gravity of the whole vessel, and R that of the part immersed.

" If the vessel be loaded quite up to the line AB, and the specific gravity of the boat and burden be the same, then the point G is simply the centre of gravity of the section ABCD; but if not, the centres of gravity of the boat and burden must be found separately, and reduced to one by the common method, namely, by dividing the sum of the momenta by the sum of weights, or areas, which in this case are as the weights. The point R is always the centre of gravity of the section MMCD, which, if consisting of different figures, must also be found by dividing the sum of the momenta by the sum of the weights as common. These two points being found, the next thing necessary is to determine the area of the two equal triangles MXE, MXF, their centres of gravity *o*, *o*, and the perpendicular projected distance *nn* of these points on the water line EF. This being done, through R and parallel to EF draw RT = a fourth proportional to the whole area MMCD, either triangle MXE or MXF, and the distance *nn*; through T, and at right angles to RT or EF, draw TS meeting the vertical axis of the vessel in S the metacentre; also through the points G, B, and parallel to ST, draw NGW and BV; moreover through S, and parallel to EF, draw WSV, meeting the two former in V and W; then SW is as the stability of the vessel, which will be positive, nothing, or negative, according as the point S is above, coincident with, or below, the point G. If now we suppose W to represent the weight of the whole vessel and burden (which will be equal to the section MMCD multiplied by the length of the vessel), and P, to represent the required weight applied at the gunwale B to sustain the vessel at the given angle of inclination; we shall always have this proportion: as VS : SW : : W : P : which proportion is general, whether SW be positive or negative; it must only in the latter case be supposed to act upward, to prevent an overturn.

" In the rectangular vessel, of given weight and dimensions, the whole process is so evident, that any further explanation would be unnecessary. In the trapezoidal vessel, after having found the points G and R, let AD, BC, be produced until they meet in K. Then, since the two sections MMCD, EFDC are equal, the two triangles MMK, EFK, are also equal; and therefore the rectangle EK × KF = KM × KM = KM^2; and since the angle of inclination is supposed to be known, the angles at E and F are given. Consequently, if a mean proportional be found between the sines of the angles at E and F, we shall have the following proportions:

" As the mean proportional thus found; sin ∠ E : : KM : KF, and as the said mean proportional : sin ∠ F : : KM : KE; therefore ME, MF become known; from whence the area of either triangle MXE or MXF, the distance *nn*, and all the other requisites, may be found.

" In the mixtilineal section, let AB = 9 feet = 108 inches, the whole depth = 6 feet = 72 inches, and the altitude of MM the line of floatation 4 feet or 48 inches; also let the two curvilinear parts be circular quadrants of two feet, or 24 inches radius each. Then the area of the two quadrants = 904·7808 square inches, and the distance of their centres of gravity from the bottom = 13·8177 inches very nearly; also the area of the included rectangle *abie* = 1440 square inches, and the altitude of its centre of gravity 12 inches; in like manner, the area of the rectangle AB*cd* will be found = 5184 square inches, and the altitude of its centre of gravity 48 inches: therefore we shall have

Momentum of the two quad.	= 904·7808	× 13·817	= 12501·98966016
Momentum of the rectan. *abic*	= 1440·	× 12	= 17280·
Momentum of the rectan. ABc*d*	= 5184·	× 48	= 248832·
	7528·7808		278613·98966016

" Now the sum of the momenta, divided by the sum of the areas, will give $\frac{278613 \cdot 98966016}{7528 \cdot 7808} \doteq 37 \cdot 006$ inches, the altitude of G, the centre of gravity of the section ABCD above the bottom. In like manner, the altitude of R, the centre of gravity of the section MMCD, will be found to be equal $\frac{123093 \cdot 98966016}{4936 \cdot 7808} = 24 \cdot 934$ inches; and consequently their difference, or the value of GR = 12·072 inches, will be found.

" Suppose the vessel to heel 15°, and we shall have the following proportion; namely, As radius : tangent of 15° : : MX = 54 inches : 14·469 inches = ME or MF; and consequently the area of either triangle MXE or MXF = 390·663 square inches. Therefore, as 4936·7808 : 390·663 : : 72 = *nn* = $\frac{2}{3}$ AB : 5·6975 inches = RT; and, again, as radius : sine of 15° : : 12·072 = GR : 3·1245 inches = RN; consequently RT − RN = 5·6975 − 3·1245 = 2·573 inches = SW, the stability required.

" Moreover, as the sine of 15° : radius : : 5·6975 = RT : 22·013 = RS, to which, if we add 24·934, the altitude of the point R, we shall have 46·947 for the height of the metacentre, which taken from 72, the whole altitude, there remains 25·053; from which, and the half width = 54 inches, the distance BS is found = 59·529 inches very nearly, and the angle SBV = 80° 06′ 42″; from whence SV = 58·645 inches.

" Again: Let us suppose the mean length of the vessel to be 40 feet, or 480 inches, and we shall have the weight of the whole vessel equal to the area of the section MMCD = 4936·7808 multiplied by 480 = 2369654·784 cubic inches of water, which weighs exactly 85708 pounds avoirdupoise, allowing the cubic foot to weigh 62·5 pounds.

" And, finally, as SV : SW (*i. e.*) as 58·645 : 2·573 : : 85708 : 3760 + the weight on the gunwale which will sustain the vessel at the given inclination. Therefore a vessel of the above dimensions, and weighing 38 tons, 5cwts. 28lbs. will require a weight of 1 ton, 13 cwts. 64 lbs. to make her incline 15°.

" In this example, the deflecting power has been supposed to act perpendicularly on the gunwale at B; but if the vessel is navigated by sails, the centre velique[*] must be found; with which, and the angle of deflection, the projected distance thereof on the line SV may be obtained; and then the power, calculated as above, necessary to be applied at the projected point, will be that part of the wind's force which causes the vessel to heel, and conversely, if the weight and dimensions of the vessel, the area and altitude of the sails, the direction and velocity of the wind, be given, the angle of direction may be found."

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[*] The *centre velique* (a term first used by M. *Bouguer*) is the centre of gravity of the *equivalent* sail; that is, of the sail whose position and magnitude are such as cause it to be acted upon by the wind when the vessel is sailing, in a manner equivalent to the action of the wind upon all the sails together which the vessel actually carries. See also Euler on Vessels, Book III. Chap. ii. § 10. 11., and Bossut Hydrodynamique, part i. ch. 12, 13, 14.

and the altitude of its centre of gravity 48 inches: th
shall have

Momentum of the two quad. $= 904 \cdot 7808 \times 13 \cdot 817 = 12501 \cdot 9896$
Momentum of the rectan. *abio* $= 1440 \cdot \times 12 = 17280 \cdot$
Momentum of the rectan. ABC*d* $= 5184 \cdot \times 48 = 2488$
$7528 \cdot 7808$ 27

“ Now the sum of the momenta, div
areas, will give $\frac{278613 \cdot 98966016}{7528 \cdot 7808} = 37$
of G, the centre of gravity of th
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point R, constant quantity is found by experiment to be about
centre, the inch when the fluid is water; ·036 of an inch for vin
mains tar. *per deliquium*; and ·024 of an inch for sweet
the quicksilver is put into the tube, the contrary to all
the takes place; for that fluid stands lower within the tube tha
ir surface in the vessel, and the *lower* as the tube is smaller.

Among the various methods of determining the diameter of a uni
capillary tube, the following seems the best and most accurate.
into the tube a certain quantity of mercury whose weight in troy g
is w, and measure carefully the length l of the tube which it occu
then is the diameter $d = \cdot 0192523 \sqrt{\frac{w}{l}}$. For, the specific gravit
mercury being 13568, a cubic inch weighs 3435·16 grains; h
$1 : d^2 l \times \cdot 785398 :: 3435 \cdot 16 : w$. Whence, by multiplying means
extremes, &c. we find $d = \sqrt{\frac{w}{3435 \cdot 16 \times \cdot 785398 \times l}} = \cdot 0192523 \sqrt{\frac{w}{l}}$
If l be the whole length of the tube in inches, and w the difference in grains between its weight when empty and when full of mercury, same theorem will obviously ascertain the diameter.

CHAPTER IV.

On the Phenomena of Attraction in Capillary Tubes.

426. The appellation *capillary*, in a general sense, is given to any thing on account of its extreme fineness, because it in that respect resembles hair. In physics, *capillary tubes* are small pipes of glass whose canals are extremely narrow. The internal diameter of these tubes may vary from $\frac{1}{20}$ to $\frac{1}{50}$ of an inch: indeed Dr. Hook affirms that he drew tubes in the flame of a lamp much smaller, and resembling a spider's thread.

If several capillary tubes of different diameters, and open at both ends, be immersed a little way into water, the fluid will be seen to stand higher in the tubes than the surface of the water without, and this is in a greater degree as the tube is smaller; the height of the surface of the fluid in the tube above that of the fluid in the reservoir, being nearly in the reciprocal ratio of the diameter of the tube: so that the diameter of the tube * multiplied into the altitude of the fluid in it (above that in the reservoir) is nearly a constant quantity for the same fluid, This constant quantity is found by experiment to be about ·048 of an inch when the fluid is water; ·036 of an inch for vinegar or *ol. tar. per deliquium*; and ·024 of an inch for sweet oil. When quicksilver is put into the tube, the contrary to all this takes place; for that fluid stands lower within the tube than its surface in the vessel, and the *lower* as the tube is smaller.

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* Among the various methods of determining the diameter of a uniform capillary tube, the following seems the best and most accurate. Pour into the tube a certain quantity of mercury whose weight in troy grains is w, and measure carefully the length l of the tube which it occupies; then is the diameter $d = 0192523\sqrt{\frac{w}{l}}$. For, the specific gravity of mercury being 13568, a cubic inch weighs 3435·16 grains; hence $1 : d^2l + \cdot785398 :: 3435\cdot16 : w$. Whence, by multiplying means and extremes, &c. we find $d = \sqrt{\frac{w}{3435\cdot16 \times \cdot785398 \times l}} = \cdot0192523\sqrt{\frac{w}{l}}$. If l be the whole length of the tube in inches, and w the difference in troy grains between its weight when empty and when full of mercury, the same theorem will obviously ascertain the diameter.

Another phenomenon of these tubes is, that such of them as would naturally only discharge water by drops, when electrified yield a continued and accelerated stream; and the acceleration is proportional to the minuteness of the tube. Nay, the effect of electricity is so considerable that it produces a continued stream from a very small tube, out of which the water would not drop at all previous to the excitation by electricity. But we shall not attempt here any explanation of this phenomenon of the continued stream: our present object being merely to state the most approved method of accounting for the ascent and suspension of fluids in these tubes, according to the principle of attraction or of adhesion.

In accounting for these phenomena of capillary tubes we must deviate from the method we have generally followed hitherto, of demonstrating a series of connected propositions: for the mode of elucidation we adopt has, at most, only probability on its side; and we would not willingly delude the student with an appearance of strict demonstration, when that kind of proof is incompatible with our present knowledge of the subject.

427. It will be necessary first to premise that the attraction between the particles of glass and water is greater than the cohesive attraction between the particles of water itself. For if this were not the case, the least quantity or drop of water applied to the underside of a glass tube placed horizontally would not adhere to it, but fall down immediately in the direction of gravity; whereas this does not happen till the bulk and weight of the drop be so far increased as to exceed the attractive power of the glass, and then it falls off.

Since, then, we find such a strong attractive power exerted at the surface of glass, it will be natural to conceive that such power must act sensibly on the surface of a fluid that is not viscid (water, for instance) contained within the small cavity of a glass tube, and that it will be proportionally stronger as the internal diameter of the tube is the smaller: for that the efficacy of the power to hold up the fluid in the tube will follow the inverse proportion of the diameter will be highly probable, if we consider that only such particles as are in contact with the fluid, and those immediately above the surface, can affect it.

428. Hence most philosophers assert that *the suspension of the fluid in capillary tubes is owing to the attraction of the narrow ring of glass contiguous to the upper surface of the fluid.* The reasoning adduced is of this kind: Every ring of glass below the surface attracts the water above it as much downwards as it attracts the water below it upwards, and con-

sequently can contribute nothing towards the support of the column: and the action of the lowest ring upon all the fluid of the tube within its surface of attraction, must either concur with the force of gravity to bring the fluid downwards, or acting upon it at right angles can have no effect in suspending it within the tube. The fluid, therefore, can only be supported by the ring of glass contiguous to its upper surface, which, attracting upwards, opposes the action of gravitation by which the fluid is solicited downwards. And the same kind of reasoning may be applied to the fluid raised between parallel planes of glass.

429. The preceding reasoning being admitted, it will follow, in conformity with the experiments, that *in capillary tubes the heights to which the fluid rises by virtue of the attraction are inversely as the internal diameters.* For the fluid being suspended by the action of the annulus of glass contiguous to the upper surface, and the distance to which the attraction of glass upon any one fluid reaches being unvaried, tho force which sustains the fluid will be as the number of attracting particles, that is, as the circumference, or as the diameter of the upper ring, or of the tube. Let Q, q, then, represent the quantities of fluid to be raised in two tubes of different bores; D, d, the respective internal diameters; and H, h, the heights to which fluids rise in the tubes: then, because Q, q, represent two cylinders of the fluid, we have $Q : q :: D^2H : d^2h$; and, from the nature of this attraction, which is as the diameters of the tubes, $D : d :: Q : q$; consequently $D^2H : d^2h :: D : d$, or $DH : dh :: 1 : 1$, or, finally, $D : d :: h : H$.

Dr. *Jurin*, who first offered this solution of the phenomenon, says, the effect is the same *in vacuo* as in the air: but in his time the air-pumps would not exhaust sufficiently to determine this point: the pumps which are now made may perhaps shew that the water will not be supported after a *very great* degree of exhaustion.

Mr. *Martin* says, the power by which the fluid is raised will keep it in the tube for any time without exhaling or evaporation; as he tried by hanging several capillary tubes thus charged with their fluids for months together in the summer sun, whose heat did not appear to diminish the fluids in the least sensible degree.

430. Another curious circumstance ascribed to the same cause is the following: *Between two glass plates, meeting on one side, and kept open at a small distance on the other, water will rise unequally; and its upper surface will form a hyperbolic curve, in which the altitudes of the several points above the surface of the fluid in the reservoir will be to one another*

reciprocally as their perpendicular distances from the line in which the plates meet. This was first observed by Dr. *Brook Taylor*, and is thus accounted for on the same hypothesis as before. Let AE (fig. 3. pl. XVII.) be the horizontal surface of the fluid in the reservoir, AF the line in which the plates meet, HGIKL the curve formed by the surface of the suspended fluid; GB, IC, KD, LE, perpendiculars to AE, shewing the heights of the respective points, G, I, K, L, above the horizontal surface AE, and AB, AC, AD, AE, perpendiculars to AF, measuring the distances of the same points from the line in which the planes meet: these heights and distances are reciprocally proportional. For let the lines GB, IC, KD, LE, represent pillars of fluid of an equal evanescent breadth: those portions of the glass plates which by their attraction support these pillars, being of equal breadth, will sustain equal quantities of fluid; that is, the pillars will be equal. But the pillars may be considered as parallelopipeds, which are equal when their bases and altitudes are reciprocally proportional. And the bases, being of equal breadth, are as their lengths; that is, as the intervals between the plates: and since these intervals continually increase as the distance from the line AF increases, these intervals at the points B, C, D, E, are as their distances AB, AC, AD, AE, from the line AF. Since, then, the heights of the pillars are reciprocally as the intervals, the heights GB, IC, &c. are reciprocally as the distances AB, AC, &c. and HGIKL is a hyperbola whose asymptotes are AF and AE.

431. If two long glass planes are first smeared over with oil and then set together at their ends, and inclined to each other under a very small angle, and a drop of the oil so placed between them as to touch both planes, it will immediately begin to move towards the touching ends, or angles of the planes; and that motion will be continued with an accelerated velocity, by reason of the increasing action of the planes, on account of the decreasing distance between them, and the larger portion of touching surface on each side of the drop.

432. If glass be applied to any other fluid whose particles attract each other more strongly than glass attracts them, all the phenomena of such a fluid in capillary tubes, and between glass planes, will be just the reverse of those which we have stated as taking place with regard to water. Now quicksilver is such a fluid, and therefore it will stand lower within a capillary tube than without (art. 426.); the surface will be convex, and not concave, as in water; and between the planes it will move the contrary way. But if a basin or dish be made of copper or brass, and polished well within, and then tinned all over, mercury put into such a vessel will every-where unite

with the tin, and may perhaps be said to *wet* it, as water does glass; and the mercury put into this mercurial basin will, if clean, be attracted and rise all around the sides, and have the same phenomena with water put into a wet glass, transparency alone excepted.

433. If water rise in any capillary tube T to a certain height, and another vessel be put into the water, having the upper end capillary and of the same diameter as the tube T, but the lower part of any greater size; then, if the air be drawn out of this vessel by suction until the water enters into the capillary part, it will stand at the same altitude as in the tube T after the suction ceases and the air is admitted into the capillary part. In this case the cylindrical part of the irregular vessel whose diameter and height equal those of the capillary tube seems to be supported by the same power as the water in the latter, that is, by the attraction of the glass annulus contiguous to the upper surface of the fluid: and the other part surrounding this capillary cylinder is supported by the pressure of the air upon the surface of the water in the reservoir; which is proved hence, that if the whole be placed under the receiver of an air-pump, and the air exhausted from the surface of the vessel, the water will not be supported as before.

434. The opinion that the suspension of the fluid in capillary tubes is occasioned by the attraction of the glass annulus contiguous to the upper surface has been pretty generally acquiesced in: but the ingenious Dr. *Hamilton* is of opinion, on the contrary (see his Essays), that the fluid is supported by the *lower* annulus contiguous to the bottom of the tube; this he imagines will first draw up a plate of water immediately under it, and then a succession of plates, till the weight of the whole is equivalent to the attraction of that annulus. A similar opinion is embraced by Dr. *Matthew Young*. But Dr. *Parkinson* and Mr. *Vince* object to Dr. Hamilton's solution. Mr. Vince says, "If this were the case, the quantity supported, and consequently the altitude of the fluid, would depend upon the orifice at the bottom; whereas experiments show that the altitude at which the fluid is supported depends upon the diameter of the tube at the upper surface of the fluid, without any regard to the form of the tube below it. Again, if in a capillary tube water will stand at the altitude of an inch above the surface of the fluid in the vessel, and you depress the tube till there be only an inch of it above the surface, the water will then not rise to the top of the tube, and if you depress the tube still lower, the water will not rise to the top. Thus there will always be an annular surface of the tube *above* the fluid, which is a strong argument in favour of the fluid being supported by the attraction of such a

surface. If the fluid were raised by the attraction of the annulus at the bottom, when the length of the tube above the surface was less than an inch, the fluid ought to run over, and thus a perpetual motion would be formed." *Hydrostatics*, p. 68.

435. Still, though the attraction of the glass, and perhaps of the annulus immediately above the upper surface of the fluid in the tube, be the chief cause of the ascent of any fluids, yet it must be allowed that the nature and quality of each particular fluid is to be regarded in most of the phenomena, which are not proportioned to the attracting power of the glass solely, but to that conjointly with the various disposition of fluids to yield to it: nor is the density of the fluid of primary consideration; it being manifest from experiment, that some lighter fluids will rise to a less height than others which are much heavier. Nor, again, does it depend on the viscidity or tenacity of parts; for hard white varnish (very thick and viscid) and spirit of wine ascend nearly to the same height. From experiments it likewise appears that heat and cold are not concerned in this ascent of fluids, very hot water standing at the same height as cold. And a solution of any salt in water makes but very little difference in the heights.

Mr. *Martin* made and repeated frequently a series of experiments on several fluids, to ascertain the altitudes at which they would be sustained in a tube whose internal diameter was about $\frac{1}{30}$, or more accurately $\frac{1}{25}$ of an inch: the results are given below:

	Alt.		Alt.
Common spring water	1·2 *inch.*	Red wine . . .	0·75 *inch.*
Spirit of urine. .	1·1	White wine . .	·75
Tincture of galls .	1·1	Ale	·75
Recent urine . .	1·1	Ol. sulp. per campanum	·65
Spirit of salt . .	·9	Oil of vitriol . .	·65
Ol. tar. per deliq. .	·9	Sweet oil . . .	·6
Vinegar	·95	Oil of turpentine .	·55
Small-beer . . .	·9	Geneva . . .	·55
Strong spirit of nitre	·85	Rum	·5
Spirit of hartshorn	·85	Brandy	·5
Cream	·8	White hard varnish	·5
Skimmed milk .	·8	Spirit of wine .	·45
Aqua fortis . . .	·75	Tincture of Mars	·45

M. *Clairaut*, in his excellent work on the figure of the earth, says, "The truth is, that when we compare the elevation of the same fluid in two different tubes, the attraction of each surface is the result of all the particular attractions exerted by the different moleculæ of the glass upon all those of the liquid which are at distances small enough to be subjected to the effect of these attractions."

The Abbe *Hauy*, in his *Traité Elémentaire de Physique*, adopts the same hypothesis, and ingeniously applies it to the elucidation of various phenomena: see my translation of that work, vol. i. p. 170.

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The celebrated *Laplace* has published several distinct essays on capillary action. A syllabus of one of these is given in a note to the translation of Haüy's Philosophy just cited. In another published in the *Journal de Physique*, Dec. 1806, he adopts a method which is very simple and general, and resembles in some measure the mode of demonstration employed in Dr. M. Young's Analysis, although it is certainly much more unexceptionable. He begins with shewing that the weight of the fluid supported in any prismatic and vertical tube, is equal to the length of the interior circumference of the prism, multiplied by a certain constant quantity; and that this quantity for the same fluid is proportional to the excess of twice the attractive force of the solid above that of the fluid. His reasoning is exhibited in a condensed form, in the remainder of this scholium. The attraction of a capillary tube, which tends to elevate the column of fluid contained in it, consists of two equal parts, one of which is derived from the action of the portion above the fluid on the upper part of the column, the other from that of the end of the tube on the imaginary continuation of the base of the column, a little below the tube; and these forces are opposed by the attraction of that part of the fluid which forms an imaginary continuation of the tube, tending to draw the column downwards, in the same manner as each of the other forces draws it upward.: so that the weight of the fluid elevated, must be proportional to the excess of twice the density of the solid above that of the fluid. Mr. Laplace compares this proposition with the experiments of Gellert on rectangular and triangular tubes, and finds that they agree as well as could be expected from the degree of accuracy with which these experiments appear to have been performed. Secondly, if the vessel into which the tube is immersed contain fluids of different kinds, the weight sustained in consequence of the capillary action will be the same as if the vessel contained only that fluid in which the lower end of the tube is situated; the only effective action being exerted on this fluid. Thirdly, if the tube be wholly immersed, and its superior part be in one, and its inferior in another, of two fluids contained in the vessel, the weight of the portion of the lower fluid raised above its natural level, considered as situated in the upper, will be equal to the difference of the weights of the two fluids which would be separately supported in the same tube. Fourthly, where the tube is inclined to the horizon, the relative weight of the fluid elevated will be the same as when it is vertical, and the actual quantity raised will be inversely as the sine of the angle formed by the tube with the horizon. Fifthly, the space between two parallelopipeds, the one being placed within the other, may be considered as a single tube. Sixthly, if the parallelopipeds be of different substances, the weight of the fluid raised may be determined from the length of the line of contact of each with the surface, considered as acting independently of the other in raising its appropriate weight. Seventhly, the angle formed by the termination of the fluid is investigated with the assistance of the author's former demonstrations; his reasoning may again be thus briefly expressed, as in the fourth paper. Supposing the fluid to be elevated in a very narrow space of a given breadth, the half of this breadth being the radius, the secant of the angle of contact will become equal to the radius of curvature of the surface, which is always inversely as the height of the elevated column; hence, the cosine of the angle of contact will be directly as the height; that is, as the difference between the density of the fluid and twice that of the solid, the whole density of the fluid being represented by the radius. Lastly, for discs of the same diameter raised from a given fluid, the squares of the forces required to separate them are proportional to the magnitude of the attraction; and if different fluids be thus compared, the squares of the forces, divided by their specific gravities, will shew their comparative attractions. It must however be observed in all cases, that where the attractive power of the solid is greater than that of the fluid, it must always be supposed to be covered by a thin stratum of the fluid, which becomes the immediate agent in all the effects of the capillary force.

Dr. *Thomas Young*, also, in an ingenious paper on the cohesion of fluids, published in the Philosophical Transactions for 1805, has developed many analogous results. See also his Natural Philosophy, vol. ii. p. 649. After all, however, the theory of capillary action, is without an *experimentum crucis*, or a fact which will compel those who know it, to adopt one mode of explication only.

HYDRODYNAMICS.

BOOK IV.

Introductory Remarks.

436. *Hydrodynamics* is that part of mechanical science which relates to the motion of non-elastic fluids, and the forces with which they act upon bodies.

This branch of mechanics is the most difficult, and the least advanced: whatever we know of it is almost entirely due to the researches of the moderns; for the only work on the mechanism of fluids which has reached us from the ancients is the piece of *Archimedes* in two books, *De Insidentibus humido,* in which the inquiries respect solely the sinking and floating of bodies in fluids, their relative gravities, levities, situations, and positions, while in equilibrio. We find, it is true, some hints and rules upon the motion of fluids in a treatise attributed to *Sextus Julius Frontinus,* inspector of public fountains at Rome, under the emperors Nerva, Cocceius, and Trajan, entitled *De Aquæ ductibus urbis Romæ commentarius;* but they are not of sufficient importance to deserve much attention by a student of this science. *Benedict Castelli* was the first who opened the way to a true measure of the flux of waters, in his treatise *Della mesura dell' Acque Currenti;* which measure he found to depend upon the area of the section, and the velocity of the water, conjointly. The most valuable and important discoveries and theorems in this department of science are given in Sir *Isaac Newton's Principia,* book II.; the *Hydrodynamique* of *Daniel Bernoulli*; the *Traite des Fluides* by M. *D'Alembert;* the *Examen maritimo Theorico Practico* of *D. George Juan;* the *Hydrodynamique* of M. *Bossut*; *Principes d'Hydraulique* by M. *Buat;* and the *Handbuch der Mechanik und der Hydraulik* by Mr. *Eytelwein.* To these may be added an ingenious paper on the Motion and Resistance of Fluids by Mr. *Vince,* in the Philosophical Transactions for 1795; those by the late Dr. *Matthew Young* in the Irish Transactions; and Dr. *Thomas Young's* Investigations relative to the Motion of Water in Pipes,

437. Could we know with certainty the mass, the figure, and the number of particles of a fluid in motion, the laws of its motion might be determined by the resolution of this problem, viz. to find the motion of a proposed system of small free bodies acting one upon the other in obedience to some given exterior force. We are, however, very far from being in possession of the data requisite for the solution of this problem: and even if we were in possession of them, it is doubtful whether we should be much further advanced; as it might be extremely difficult to deduce any convenient practical results from the intricate and complex expressions which might stand at the foot of the investigation. The wisest philosophers have had their doubts with regard to every abstract theory concerning the motion of fluids; and the greatest geometers and analysts affirm that those methods, which have directed them to such curious and useful conclusions in the mechanics of solid bodies, do not furnish any conclusions with respect to fluids but such as are too general and uncertain for the greater number of particular cases. On these accounts a detailed exhibition of the theory alone would scarcely be of any utility: we shall, therefore, enter but little into the theoretical part of Hydrodynamics; but present merely a few propositions that are least dubious in their nature, and endeavour to supply the deficiency by stating the results of some of the most ingenious, careful, and satisfactory experiments with which we are acquainted.

The motion of the water in pumps, depending in part upon the principles of Pneumatics, will be treated of in our fifth book.

CHAPTER I.

On the Discharge of Fluids, through Apertures in the Bottom and Sides of Vessels, and on Spouting Fluids.

438. **Prop.** *If a fluid run through any tube, which is kept continually full, and the velocity of the fluid in every part of one and the same section be the same, the velocities in different sections will be inversely as the areas of the sections.*

For as the tube is always equally full, the same quantity of fluid will run through every section in the same time: but the quantity passing through any section S with the velocity V in any given time manifestly varies as S and V conjointly, or as $S \cdot V$; and, in like manner, the quantity passing through any other section s with velocity v must vary as $s \cdot v$ in a given time: consequently we must have $S \cdot V = s \cdot v$, and $S : s :: v : V$.

It is supposed in this proposition that the changes in the diameters of the tube are *continual*, and no-where abrupt so as to break the law of continuity in the sides of the tube: for if there be any angles, or considerable sinuosities in the tube, they will produce eddies in the motion of the fluid, and the proposition will not obtain.

439. **Prop.** *If a fluid flowing through a very small orifice in the bottom of a vessel be kept constantly at the same height in the vessel, by being supplied as fast above as it runs out below, the velocity of the effluent fluid will be equal to that which a heavy body would acquire in falling freely through the height of the fluid above the orifice.*

Let MNOP (fig. 4. pl. XVII.) represent a vessel filled with a fluid up to the level GH; MP the bottom in which is the aperture CD (*very* small compared with MP); CIKD the column of the fluid standing directly above the aperture, and CABD the lowest plate of the fluid immediately contiguous to the aperture. Also let v denote the velocity which a heavy body would acquire in falling freely through BD the height of the plate, and V the velocity acquired by the same plate during its descent through the same space until it is discharged by the pressure of the column CIKD. If we suppose the lowest plate of fluid ACBD to fall as a heavy body through the height BD, its moving force will be its own weight. Again, suppose it to be accelerated by its own weight, together with the pressure of the ambient fluid, about the column CIKD, that is, (art. 386.) by the weight of the column CIKD, through the same space, that is, while it is accelerated from quiescence until it is actually discharged: then (by what has been shewn in Dynamics, art. 228.), the velocity in the former case will be to that in the latter as the moving forces and the times in which they act directly, and the quantities of matter moved inversely. But the moving forces are to each other as the heights BD and KD; the times in which they act are inversely as the velocities, the space through which the body is accelerated being given; and the quantities of matter moved are equal: therefore, $v : V :: \frac{BD}{v} : \frac{KD}{V}$, whence $v^2 : V^2 :: BD : KD$, or $v : V :: \sqrt{BD} : \sqrt{KD}$. Now v is the velocity which a heavy body would actually acquire in falling through the space BD; consequently V the velocity of the effluent fluid is that which a heavy body would acquire in falling through KD, the whole altitude of the fluid above the orifice.

Cor. 1. In the same manner it may be shewn, that if a pipe be inserted horizontally in the vessel MNOP (fig. 5.), the plate of fluid ACBD will be discharged with the same velocity as before (if its centre of pressure be of the same depth) whatever be the thickness of the plate; this velocity not depending upon a continual acceleration through the length of the tube, otherwise the effluent fluid could not attain its full velocity until a column had been discharged whose base is equal to the orifice and height equal to the length of the tube: whereas we find by experience that this full velocity can be attained by the thinnest plate which can be let escape from the aperture.

Cor. 2. The velocities and quantities discharged at different depths are as the square roots of the depths.

Cor. 3. The quantity run out in any time is equal to a cylinder, or prism, whose base is the area of the orifice, and its altitude the space described in that time by the velocity acquired by falling through the height of the fluid.

So that if h denote the height of the fluid,
a the area of the aperture,
g, 32⅙ feet, or 386 inches,
and t the time of efflux,

we shall have for the quantity discharged $Q = at\sqrt{2gh}$;
Or, when a and h are in feet . . . $Q = 8{\cdot}0208\,at\sqrt{h}$, feet;
When a and h are in inches . . . $Q = 27{\cdot}7387\,at\sqrt{h}$, inch.
If the orifice is a circle whose diameter is d, then $\cdot785398\,d^2$ must be substituted for a;
And, when d and h are in feet . . $Q = 6{\cdot}29952\,d^2t\sqrt{h}$, feet;
When d and h are in inches . . . $Q = 21{\cdot}78592\,d^2t\sqrt{h}$, inc.

And from either of these it will be easy to find either a, t, or h, when the other three quantities are given.

COR. 4. The force with which the effluent water impinges against any quiescent body is proportional to the altitude of the fluid above the orifice.

For the force is as velocity × quantity of matter; but the quantity discharged in a given time is as the velocity: therefore the force is as the square of the velocity, that is, by the demonstration of the proposition, as the height of the fluid.

COR. 5. The water spouts out with the same velocity whether it be downwards, or upwards, or sideways; because the pressure of fluids is the same in all directions at the same depth.

COR. 6. Hence, if the adjutage be turned upwards, the jet will ascend to the height of the surface of the water in the vessel. And this is confirmed by experience, from which it appears that jets really ascend *nearly* to the height of the reservoir; the small quantity abated arising from the friction against the sides, the resistance occasioned by the oblique motion of the fluid in the bended pipe, and the resistance of the air.

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440. What is said in this proposition and corollaries of the velocity of the effluent water, is true only of the middle filament of particles which issue through the centre of the aperture, which are supposed in theory to experience no retardation, and which, in fact, suffer no other retardation than what arises from the resistance of the air, and their mutual adhesion and attrition against each other. But those which issue near the edges of the aperture undergo a greater attrition, and therefore suffer a greater retardation. Hence it follows that the mean velocity of the whole column of effluent fluid will be considerably less than according to theory.

Sir Isaac Newton, who examined every subject that came before him with peculiar accuracy, first discovered a contraction in the vein of effluent water; and found, that at the distance of about a diameter of the orifice the section of the vein contracted nearly in the subduplicate ratio of 2 to 1. Hence he concluded that the velocity of the water, after its exit from the aperture, was increased in this proportion, the same quantity passing in the same time through a narrower space. From the quantity of water discharged in a given time through that narrow section, he found that its velocity there was that which a heavy body would acquire in falling through the height of the water above the orifice; and since the velocity there was greater than immediately in the orifice in the subduplicate ratio of 2 to 1, he concluded that the velocity of the effluent water in the orifice was equal to that which a heavy body would acquire in falling through *half* the altitude. But all this is true only of the *mean* velocity; for there is no cause which can actually accelerate the water after its exit from the orifice, whatever causes may contribute to its retardation. The manner in which the mean velocity of the water is increased after its discharge, though the actual velocity of the several particles continues unvaried, might be easily explained: but it need not be dwelt upon here. A circumstance of considerable importance in the escape of fluids through a horizontal orifice seems to have been entirely neglected by most writers; we allude to a *whirling* motion: the fluid will revolve about the aperture, and at some distance from it, and escape with a revolving motion; the fluid rushes from all sides in spiral streams to supply the continual waste. This circumstance will be explained more distinctly in a subsequent proposition: but, excepting that, the remainder of this chapter will be treated in conformity with the common hypothesis; which, when corrected by the experiments in the ensuing chapter, will furnish tolerably correct results.

441. Prop. *When a vessel is left gradually to discharge itself by an orifice in the bottom, if the area of the section parallel to the bottom be every-where the same, the velocity of the surface of the fluid, and consequently the velocity of the efflux will be uniformly retarded.*

For (art. 438.) the velocity of the descending surface is to the velocity at the orifice, as the area of the orifice to the area of the surface, which is a constant ratio; consequently, the velocity of the descending surface varies as the velocity at the orifice, or as $\sqrt{h}$, by cor. 2. of the last article; that is, the velocity of the descending surface varies as the square root of the space which it has to describe: so that this exactly corresponds with the case of a body projected perpendicularly upwards (art. 246.), where the velocity is as the square root of the space to be described: whence, as the retarding force is constant in the instance referred to, it must also be constant in the case before us, and the retardation uniform.

Comparing this with the instance just pointed out, we deduce these obvious corollaries:

COR. 1. The quantities of water in a prismatic vessel discharged through an aperture in the bottom decrease in equal times as the series of odd numbers, 1, 3, 5, 7, 9, &c. taken in an inverted order.

COR. 2. Hence, if it were proposed to construct a *clepsydra*, or water clock, by means of a prismatic or cylindric vessel, having an orifice at the bottom, let the altitude DB (fig. 9. pl. XVII.) which will be completely exhausted in a given time, say 12 hours, be determined experimentally. Call that height 144, and set off from the bottom towards the top, portions increasing as the odd numbers, 1, 3, 5, 7, &c. up to 23; they will mark the spaces through which the upper surface of the fluid will successively descend in each hour of the exhaustion.

COR. 3. The quantity of water contained in an upright prismatic vessel is *half* that which would be discharged in the time of the entire gradual evacuation of the vessel, if the water be kept always at the same altitude.

442. PROP. *To determine the time of emptying a vessel of water by an orifice in the bottom of it, or in the side contiguous to the bottom, the height of the orifice being very small compared with the altitude of the fluid.*

Let a = the area of the aperture;
h = the whole height of the fluid above the aperture;
x = the vertical space descended by the upper surface in any time t;
A = the area of the upper surface;
$g = 32\frac{1}{6}$ as before, the measure of the force of gravity.

Then will the velocity of the effluent fluid at any time be represented, not by $\sqrt{2gh}$ as in art. 439. but by $\sqrt{[2g(h-x)]}$. This velocity will vary continually, because x increases, and the difference $h-x$ diminishes continually, but it may be regarded as constant during the indefinitely small time $\dot{t}$: so that in the time $\dot{t}$ there will escape through the orifice a prism of the fluid which has that orifice a for its base, and $\sqrt{[2g(h-x)]}$ for its altitude. Thus the quantity of fluid discharged during the instant $\dot{t}$ is $= a\dot{t}\sqrt{[2g(h-x)]}$. But during the same time the upper surface has descended through the space $\dot{x}$, and the vessel has lost a prism or cylinder of the fluid whose height is $\dot{x}$ and base A, that is, a prism whose capacity is $A\dot{x}$. Hence we have $A\dot{x} = a\dot{t}\sqrt{[2g(h-x)]}$;

and .... $\dot{t} = \frac{A\dot{x}}{a\sqrt{[2g(h-x)]}}$ ...... (I.)

As the area A will be given in functions of x, by the form of the vessel, the second member of this equation may be considered as containing only the variable quantity x; and it will be very easy in most cases, by simply finding the fluents, to discover the successive depressions and discharges of the fluid in any vessel of known form.

443. By way of application, take the following examples:

I. *Let the vessel be an upright prism or cylinder.* Here the area A will be constant, because every horizontal section of the prism will be equal to its base. Hence we have

$$t = \frac{A}{a\sqrt{2g}} \cdot \int \frac{\dot{x}}{\sqrt{(h-x)}} = -\frac{2A}{a\sqrt{2g}} \sqrt{(h-x)} + C.$$

Now when the time t is nothing, the depression of the upper surface A of the fluid is nothing also: thus we have at the same time $x=0$, and $t=0$; this condition determines the constant quantity $C = \frac{2A}{a\sqrt{2g}} \sqrt{h}$; and gives for the time of depressing the upper surface through the space x,

$$t = \frac{2A}{a\sqrt{2g}} [\sqrt{h} - \sqrt{(h-x)}] \text{........(II.)}$$

To find the time of completely emptying the vessel, we have only to make $x=h$, in which case the preceding expression will become....... $t = \frac{A}{a}\sqrt{\frac{2h}{g}}$.

Cor. The time just found is double that in which an equal quantity would be discharged, if the vessel were kept constantly full. For, in art. 439. cor. 3. we have $Q = at\sqrt{(2gh)}$, where, if $Q = Ah$, we have $t = \frac{Ah}{a\sqrt{2gh}} = \frac{A}{a}\sqrt{\frac{h}{2g}} = \frac{A}{2a}\sqrt{\frac{2h}{g}}$, which is half the preceding value of t.

Hence the result of this example corresponds with cor. 2. art. 441. as it manifestly ought to do.

II. *Let the vessel be any solid of rotation, of which the axis is vertical.* Here A will be the area of a circle which has for its radius the ordinate y of the generating curve: that is, if $\pi = 3{\cdot}141593$, we shall have $A = \pi y^2$. Introducing this value into the equation marked I. we have

$$t = \frac{\pi}{a\sqrt{2g}} \cdot \int \frac{\dot{x}y^2}{\sqrt{(h-x)}} \text{......(III.)}$$

In any particular examples it will be necessary to put for y its value deduced in terms of x from the equation of the generating curve, and to find the fluents, which will be corrected by making at the same time $t=0$, and $x=0$.

III. *Let the solid of rotation be a paraboloid with its vertex downwards.* If p be the parameter of the generating parabola, the equation of the curve will be $y^2 = px$, when the origin of the curve is supposed at V (fig. 2. pl. XIII.), the point where we conceive the aperture to be; or if we transfer the origin to A, the equation will be $y^2 = p(h-x)$: this substituted for y^2 in equation II. gives

$$t = -\frac{2\pi p}{3a\sqrt{2g}} \cdot (h-x)^{\frac{3}{2}} + \text{c}.$$

Determining the constant quantity as above directed, we obtain for the correct fluent

$$t = \frac{2\pi p}{3a\sqrt{2g}} \cdot \left(h^{\frac{3}{2}} - (h-x)^{\frac{3}{2}}\right) \ldots\ldots \text{(IV.)}$$

If we make $h = x$, we have $t = \frac{2\pi p}{3a}\sqrt{\frac{h^3}{2g}} = \frac{2\pi y^2}{3a}\sqrt{\frac{h}{2g}}$ $= \frac{s}{3a}\surd\frac{2h}{g}$, where s is the area of the upper surface of the fluid at the beginning of the exhaustion, and h the whole height.

The above may suffice as a specimen of the method, on the supposition that the velocity of efflux is that due to the *whole* height of the fluid. A very ingenious and complete paper on this subject, according to the hypothesis that the velocity is that due to *half* the height of the fluid above the aperture, is given by Dr. *Hutton* in his Mathematical Miscellany.

444. PROP. *If the water of a reservoir which flows through a horizontal aperture be influenced by any foreign motion, it will form a hollow whirl above the orifice itself.*

Let DQ (fig. 6. pl. XVII.) represent a horizontal plane near the orifice EF, through which the fluid of the reservoir MN flows. A fluid particle, D, situated in this plane, has a motion DB inclined to the axis AB; which may be decomposed into two DC, CB. Let us suppose the plane DQ to descend parallel to itself along the axis with the motion, and then enquire into the motion of the particle D on the plane DQ; a motion which impresses upon all the particles situated in the plane DQ a centripetal force towards the centre C. Let any other horizontal motion whatever, not coincident in direction with DC, be impressed upon the same particles: under the joint influence of the two moving forces the particles will describe round the centre C areas proportional to the times, and by the equilibrium of these motions they may assume a horizontal circular rotation.

Imagine that, during this horizontal circulation, the particle D in its approach towards the center C, as in a spiral, shall describe circular orbits, of which the diameters are successively diminished: put the rotatory velocity of the particle D $= v$; its distance CD from the centre $= r$; the time of one revolution $= t$:

then, since the areas must be as the times, we shall have nearly $v \propto \frac{1}{r}$, $t \propto r^2$; and the centrifugal force of the particle D will $\propto \frac{1}{r^3}$. Indeed, when the particles which revolve at the surface of the funnel at MN are observed, it is seen that the effect which really takes place in nature is conformable to $t \propto r^2$ very nearly. Since, therefore, the centrifugal force in approaching the centre C increases as $\frac{1}{r^3}$, it will become sufficient to form an equilibrium against the upper pressure SD, which produces the centripetal force DC: a cavity KRTHPV will therefore be formed, round which the whirling fluid will support itself by the centrifugal force of its rotation.

Let DQPR represent a circular fluid zone, the particles of which revolve about the cavity RP according to the law here indicated. Let the gravity of a fluid particle be $= w$, CR $= a$, RD $= b$, DX $= z$, XZ $= \dot{z}$, and the velocity of the particle D $= v$. If the centrifugal force of the particle D were equal to its gravity, its velocity (art. 282. cor. 3.) would be equal to that of a body falling by gravity alone through the space $\frac{a+b}{2}$. And as a heavy body falls in one second through the space $16\frac{1}{12}$ feet $= \frac{1}{2}g$, the velocity of the particle D will be represented by $\sqrt{\left(2g \cdot \frac{a+b}{2}\right)} = \sqrt{[g(a+b)]}$. The centrifugal force in the circle is as v^2; therefore the centrifugal force of D will actually be $= \frac{-v^2w}{g(a+b)}$. And since the centrifugal force $\propto \frac{1}{r^3}$; taking $\frac{1}{(a+b)^3} : \frac{1}{(a+b-z)^3} :: \frac{-v^2w}{g(a+b)}$: a fourth term, we shall have the centrifugal force of the element of DX in $\dot{x} = \frac{v^2w(a+b)^2\dot{z}}{g(a+b-z)^3}$; and that of the filament DX itself $= \frac{v^2w(a+b)^2}{2g(a+b-z)^2} + c$. When $z = 0$, the fluent $= 0$, whence $c = \frac{-v^2w}{2g}$: and taking $z = b$, the centrifugal force of the filament DR will be $= \frac{bv^2w}{2a^2g} \cdot (2a+b)$. Now the quantity bw is the gravity of the filament DR: hence the gravity of this filament is to its centrifugal force as $v^2(2a+b)$, to $2a^2g$.

445. When the fluid zone DRPQ is nearer the orifice RF, the pressure SD increases; whence the centrifugal force of the zone must also be increased, in consequence of the radius RC of

the cavity being diminished; hence we may determine the nature of the curve which forms the vertical section of the cavity KRT. For greater simplicity, let it be supposed that the sides of the vessel MD, NQ, have the same shape as those of the cavity itself; so that $DR = b$ may be constant. Let $AC = x$, and $CR = y$. In the preceding formula substitute y for a; and, since the gravity of the filament DR is to that of the filament SD as b to x, we shall have, by composition of ratios, the centrifugal force of the filament DR to the pressure SD, as $bv^2(2y+b)$ to $2gxy^2$. In order that there may be an equilibrium, this last must be a ratio of equality, that is, $bv^2(2y+b) = 2gxy^2$; whence we deduce $xy^2 - \frac{bv^2}{g}y - \frac{b^2v^2}{2g} = 0$, for the equation of the curve KRT. This is the 64th species in Sir Isaac Newton's Enumeration of Lines of the Third Order: its convexity is turned toward the axis; and it has two asymptotes, one of which is the axis AB, and the other is in MN, supposing the two points MN to be infinitely distant.

If the assumed positions in this and the preceding articles do not absolutely coincide with nature, they approximate to its effects very nearly. It is not only possible, but there does actually exist in nature a whirling stream, of which the cavity turns its convex part to the axis, and in which $t \propto \frac{1}{r^2}$ very nearly as is shewn by numerous experiments.

If any body, which floats at the surface of the liquor after the funnel has been formed, be of sufficient size to cover the whole cavity, it will destroy the funnel in the upper part, and sometimes also in the lower. For the body itself can only turn round its centre according to the law $v \propto r$; it therefore destroys by friction the law $v \propto \frac{1}{r}$ in the parts of the fluid contiguous to it, and consequently destroys the upper part of the funnel, if not the funnel itself.

446. PROP. *If a notch or sluice in form of a rectangle be cut, in the vertical side of a vessel full of water, or any other fluid, the quantity of water flowing through it will be* $\frac{2}{3}$ *of the quantity which would flow through an equal orifice placed horizontally at the whole depth, in the same time; the vessel being constantly kept full.*

Let ABCD (fig. 7. pl. XVII.) be the vertical side of the reservoir, EFGH the rectangular notch in it, and let ILil be a parallelogram of evanescent breadth. Then the velocity with which the water escapes at GH is to the velocity with which it escapes through ILil as $\sqrt{\text{EG}}$ to $\sqrt{\text{EI}}$ (art. 439. cor. 2.); and by the same corollary the quantities discharged in a given time through an evanescent parallelogram at the depth EG, and the evanescent parallelogram ILil, are in the same ratio; that is, as GH to IK, the ordinates in a parabola EKH, whose axis is EG, and greatest ordinate GH. Therefore, the sum of all the quantities discharged through all the parallelograms ILil, of which the rectangle EFGH is composed, is to the sum of all the quantities discharged through as many equal parallelograms ILil at the depth EG as the sum of all the elements IKki of the parabola, to the sum of all the corresponding elements ILli of the rectangle; that is, as the area of the parabola EKHG to that of the rectangle EFGH: or, the quantity running through the notch EFGH is to the quantity running through an equal horizontal area placed at GH, as EKHG to EGHF; that is, as 2 to 3, by the nature of the parabola.

Cor. 1. The mean velocity of the fluid in the notch is equal to two-thirds of that at GH.

Cor. 2. The quantity flowing through the rectangle ILHG is to the quantity discharged through an equal rectangle placed horizontally at GH, as the parabolic zone IKHG is to the rectangle ILHG. As is evident from the demonstration of the proposition.

447. Prop. *To determine the relation between the time and the quantity of water, or other non-elastic fluid, discharged from a vessel through any vertical orifice; the velocity of the effluent fluid at any point being (as heretofore) supposed that due to the altitude of the upper surface of the fluid in the vessel above that point.*

Let XYVZ (fig. 8. pl. XVII.) be the vertical side of the vessel, and AM*st*M′ a vertical orifice therein, of which the contour is a plane curve; SB a vertical line passing through A, the highest point of the orifice; and let MM′, *mm*′, be two horizontal lines indefinitely near the one to the other. Then, the upper surface of the fluid being supposed to pass through S, put SA$=h'$, SB$=h$, AP$=x$, MM′$=y$, the velocity $32\frac{1}{6}$f. which gravity communicates at the end of a second $=g$, and the time $=t$: so shall AB$=h-h'$, and P*p*$=\dot{x}$.

I. We propose first to ascertain what will be the discharge (q) at this orifice during a determinate time T, *supposing the vessel is all that time kept full up to the level of* S.

Now the velocity of the fluid discharged through the elementary trapezoid MM′*m*′*m*, whose surface is $y\dot{x}$, may be considered as that due to the height SP$=h'+x$: consequently, if for a we substitute its value $y\dot{x}$, and for h its value in the first formula in art. 439. cor. 3. we shall have for the quantity discharged through that trapezoid in the time t

$$ty\dot{x}\sqrt{[2g(h'+x)]}, \text{ or } t \times y\dot{x}\sqrt{(h'+x)} \times \sqrt{2g}.$$

Therefore the quantity discharged during the time t through the portion of the orifice AMM′ is equal to

$$t(\textstyle\int y\dot{x}\sqrt{(h'+\dot{x})} + \text{c})\sqrt{2g}.$$

Thus taking the fluent contained between $x=0$, and $x=$ AB $=h-h'$, we have for the quantity of efflux sought

$$q=t\sqrt{2g}(\textstyle\int y\dot{x}\sqrt{(h'+x)}+\text{c}) \;.\;.\;.\;.\;.\;.\; \text{(i.)}$$

Here it may be observed, that as the nature of the line which bounds the orifice is considered as known, we may always substitute for y a function of x, and in finding the fluent there will in fact be no other variable quantity than x; and of course the constant quantity c will be determined by considering that when $x=0$, the quantity discharged is also $=0$. And when c is known, the whole quantity discharged is readily found by introducing into the value of the discharge through AMM′, for x its value $h-h'$.

448. Let s be such a height as if the fluid issued from all points of the orifice with the velocity due to that height, the total discharge through the orifice would be the same as has place naturally, conformably to equation i.; then will the velocity of the issuing fluid be $=\sqrt{2gs}$, and the discharge through the elementary trapezoid M′M$m'm=y\dot{x}$, in the time t, will be represented by $t\sqrt{2gs}\int y\dot{x}$: and since this ought to be equal to the value of q in equation i. we shall by making that equality, and reducing, find

$$s=\frac{(\int y\dot{x}\sqrt{(h'+x)}+\text{c})^2}{(\int y\dot{x}+\text{c}')^2} \;.\;.\;.\;.\;.\; \text{(ii.)}$$

The fluent of the denominator must manifestly be taken between the same values of x as that of the numerator; and the constant quantity c′ is determined by considering that when $x=0$ the portion AMM′$=0$ also.

The quantity s, which we have just shewn how to determine, is generally called *the mean height* of the fluid above the orifice: the rule for finding this height is obviously nothing more than to *divide the value of* q *found equa.* i. *art.* 447. *by the product of* t$\sqrt{2g}$ *into the surface of the orifice, and to square the quotient.*

449. II. Let us enquire, secondly, into the relation between the time and the quantity discharged, *supposing the vessel empties itself*; that is, let us ascertain, according to this

hypothesis, the quantity of fluid expelled during a given time.

This determination will evidently furnish, at the same time, the depression of the upper surface of the fluid in the vessel, of which the interior form is known; because the portion of the capacity of the vessel which is exhausted is equal to the volume of fluid expelled at the orifice. Let us, in order to solve this branch of our general proposition, suppose that at the commencement of the motion the surface of the fluid is at S, and that at the end of the given time t it has descended to K: making SK$=z$, we shall have to obtain an equation between z, t, g, x, h, and h'. This may be accomplished thus: we conceive, at first, that the vessel is kept constantly full to the height K during the time t, and thence find, from what has just been done, for the discharge in that time the value $t\sqrt{2g}\left[\int y\dot{x}\sqrt{(h'+x-z)}+\text{c}\right]$. Since z is here supposed constant, it will be very easy to find the fluent of this expression; and it may be corrected as the former fluent, by the condition that when $x=0$ the discharge is likewise nothing; the whole fluent being found by taking $x=h-h'$.

The value of $t\sqrt{2g}\left[\int y\dot{x}\sqrt{(h'+x-z)}+\text{c}\right]$, found in the manner just explained, comprises only z and constant quantities: we next suppose that the vessel empties during the element of time $\dot{t}$ and that the surface Qq is depressed to Rr, through a distance K$k=\dot{z}$; which, if we put the section Qq of the vessel $=$ S, gives for the quantity discharged S$\dot{z}$. Now, on this hypothesis, the descent through Kk may be imagined to take place with a uniform velocity, and the velocity at each point of the orifice to continue the same as when the fluid had its surface at K: therefore, since when the velocities are equal the quantities discharged vary as the times, we have this analogy,

$$t\sqrt{2g}\left[\int y\dot{x}\sqrt{(h'+x-z)}+\text{c}\right] : \text{s}\dot{z} :: t : \dot{t},$$

$$\text{whence } \dot{t}=\frac{\text{s}\dot{z}}{\left[\int y\dot{x}\sqrt{(h'+x-z)}+\text{c}\right]\sqrt{2g}}$$

$$\text{and the time } t=\frac{1}{\sqrt{2g}}\int\frac{\text{s}\dot{z}}{\int y\dot{x}\sqrt{(h'+x-z)}+\text{c}}+\text{c}' \ldots\ldots \text{(iii.)}$$

When the form of the vessel is known, S is given in functions of z, and of constant qualities; on the other hand, when the integration indicated by the denominator is effected, it contains likewise only the quantity z and invariable quantities: so that the complete value of t may be found by the integration of an expression which contains only the variable quantity z. The constant magnitude c′ will be determined from the consideration that when $z = 0$, $t = 0$.

Now it is known that fluents, such as those in equations (i.) and (iii.) of this proposition, which comprise, under the sign of integration, only one variable quantity and its fluxion, may always be referred to the rules for the quadrature of curves. In effect, let x be an expression composed of the variable quantity x and a constant quantity, the fluxion $\mathrm{x}\dot{x}$ will be the element of the surface of a plane, terminated by a curve of which x will be the ordinate, x the corresponding abscissa, and $\int \mathrm{x}\dot{x}$ the surface itself. Thus, granting the equation to be $y' = \mathrm{x}$, and squaring the curve, which is designed by this equation, between a certain value of x and another certain value of x, determined by the conditions of the case in hand, the resulting surface will be equivalent to the value of $\int \mathrm{x}\dot{x}$ taken within suitable limits.

450. Let us now proceed to give a few examples of the application of the preceding formula to particular cases.

Ex. I. *Suppose the vessel constantly kept full, and the orifice a rectangle whose sides are horizontal and vertical respectively.*

Let the horizontal sides of the rectangle be each equal to b: putting this value instead of y in the equation (i.) of art. 447. and taking the fluents, we have

$q = tb\sqrt{2g} \times \frac{2}{3}\left[(h'+x)^{\frac{3}{2}} + \mathrm{c}\right]$. The constant quantity, determined according to the method previously explained, will be $\mathrm{c} = -h'^{\frac{3}{2}}$. Substituting this value of c for it, in the equation just given, it becomes $q = \frac{2}{3}bt\sqrt{2g}\left[(h'+x)^{\frac{3}{2}} - h'^{\frac{3}{2}}\right]$. Then making, conformably to the directions in art. 447. $x = h - h'$, we obtain for the total discharge through the orifice

$$\mathrm{Q} = \frac{2}{3}bt\sqrt{2g}\,(h^{\frac{3}{2}} - h'^{\frac{3}{2}}) \; \ldots\ldots\ldots \; \text{(iv.)}$$

If the theorem above be compared with the known rule for finding the area of a parabolic zone, it will thence appear that this value of Q furnishes the same result as cor. 2. art. 446. A corroboration, if any were needed, of the theoretic truth of both.

The *mean height* of the fluid, determined by the rule in art. 44. will be found

$$s = \frac{4(h^{\frac{3}{2}} - h'^{\frac{3}{2}})^2}{9(h-h')^2}.$$

451. Ex. II. *Suppose the orifice a triangle whose vertex is*

uppermost, and the base horizontal, the vessel being constantly kept full.

Let the altitude of the triangle be to the base as m to 1, then is $y = mx$, and the equation art. 447. (i.) will become

$$q = t\sqrt{2g}\left[\int mx\dot{x}.\sqrt{(h'+x)} + \text{c}.\right]$$

Making $h'+x = z$, we have $\dot{x} = \dot{z}$, and $x = z - h'$: whence arises $\int x\dot{x}\sqrt{(h'+x)} = \int(z-h')z^{\frac{1}{2}}\dot{z} = \int(z^{\frac{3}{2}} - h'z^{\frac{1}{2}})\dot{z} = \frac{2}{5}z^{\frac{5}{2}} - \frac{2}{3}h'z^{\frac{3}{2}} + \text{c} = \frac{2}{5}h' + x)^{\frac{5}{2}} - \frac{2h'}{3}(h'+x)\frac{3}{2} + \text{c}$. The fluent vanishing when $x = 0$, we have $\text{c} = -\frac{2}{5}h'^{\frac{5}{2}} + \frac{2}{3}h'^{\frac{5}{2}} = \frac{4}{15}h'^{\frac{5}{2}}$, and the value of q becomes... $q = mt\sqrt{2g}[\frac{2}{5}(h'+x)^{\frac{5}{2}} - \frac{2}{3}h'(h'+x)^{\frac{3}{2}} + \frac{4}{15}h'^{\frac{5}{2}}]$.

When $x = h - h'$, we have for the total discharge at the aperture

$$\text{Q} = \tfrac{2}{15}mt\sqrt{2g}\,(3h^{\frac{5}{2}} + 2h'^{\frac{5}{2}} - 5h'h^{\frac{3}{2}}) \ldots\ldots (\text{v.})$$

The area of the triangular orifice is $\frac{1}{2}m(h-h')^2$, which is the value of $\int y\dot{x}$, taken with reference to the present instance.

$$\text{Thus } t\sqrt{2g}\cdot\int y\dot{x} = \tfrac{1}{2}mt\sqrt{2g}\,(h-h')^2$$

whence, by the method pointed out in art. 448. we find the *mean height* of the fluid by the equation

$$s = \frac{16\,(3h^{\frac{5}{2}} + 2h^{\frac{5}{2}} - 5h'h^{\frac{3}{2}})^{\frac{3}{3}}}{225\,(h-h')^4} \ldots\ldots\ldots (\text{vi.})$$

452. Ex. III. *Let the orifice be a triangle, as in the preceding example, but having its vertex downwards, and its base horizontal.*

In this case $y = m(h - h' - x)$, and, by a calculus little more difficult than the above, we shall find

$$\text{Q} = \tfrac{2}{15}m\sqrt{2g}\,(2h^{\frac{5}{2}} + 3h'^{\frac{5}{2}} - 5hh'^{\frac{3}{2}}) \ldots\ldots (\text{vii.})$$

And the mean height of the fluid as below,

$$s = \frac{16\,(2h^{\frac{5}{2}} + 3h'^{\frac{5}{2}} - 5hh'^{\frac{3}{2}})^2}{225\,(h-h')^4} \ldots\ldots (\text{viii.})$$

453. Ex. IV. *Suppose the orifice a circle, and the vessel kept constantly full.*

The general theorem (art. 447. i.) may be most readily applied to the present example, by an approximation which will be sufficiently accurate for practice, thus: Put d = the diameter of the circle, then from the property of that figure $\frac{1}{2}y = \sqrt{(dx - x^2)}$, whence the general equation just referred to becomes

$$q = 2t\sqrt{2g}\int(dx - xx)^{\frac{1}{2}}(h'+x)^{\frac{1}{2}}\dot{x} + \text{C}).$$

Making $\int (dx-xx)^{\frac{1}{2}}(h'+x)^{\frac{1}{2}}\dot{x}=\int z\dot{x}$, the integration is reduced to the finding the area of a curve whose equation is $z = (dx-xx)^{\frac{1}{2}}(h'+x)^{\frac{1}{2}}$, its origin being at the point where $x+0$. We may approximate to the quadrature of this curve by the method of equidistant ordinates, adopting the first general proposition in Sect. II. Part iv. *Hutton's Mensuration.* To this end, suppose x divided into four equal parts; then shall we find the five ordinates corresponding to the points of division, by substituting in the preceding equation for z the values 0, $\frac{1}{4}x$ $\frac{1}{2}x$, $\frac{3}{4}x$, and x: these values found, we add to the sum of the first and fifth four times the sum of the second and fourth, and twice the third; one third of this latter sum, multiplied into the common distance between the ordinates, will be the approximate value of the surface $\int z\dot{x}$; which, multiplied into $2t\sqrt{2g}$, produces at length

$$q=\left\{\begin{array}{l} 2\left(\frac{1}{4}dx-\frac{1}{16}xx\right)^{\frac{1}{2}}\left(h'+\frac{1}{4}x\right)^{\frac{1}{2}} \\ +\left(\frac{1}{2}dx-\frac{1}{4}xx\right)^{\frac{1}{2}}\left(h'+\frac{1}{2}x\right)^{\frac{1}{2}} \\ +2\left(\frac{3}{4}dx-\frac{9}{16}xx\right)^{\frac{1}{2}}\left(h'+\frac{3}{4}x\right)^{\frac{1}{2}} \\ +\frac{1}{2}\left(dx-xx\right)^{\frac{1}{2}}\left(h'+x\right)^{\frac{1}{2}} \end{array}\right\}\frac{1}{3}tx\sqrt{2g}.$$

If in this equation we make $x=d=h-h'$, and perform the requisite reductions, we shall have for the total discharge at the orifice

$$Q=td^2\sqrt{2g}\left(\frac{\sqrt{(h+3h')}+\sqrt{(h'+3h)}}{4\sqrt{3}}+\frac{1}{6}\,\frac{\sqrt{(h+h')}}{\sqrt{2}}\right)\ldots\ldots(\text{ix.})$$

This theorem may be reduced to computation in any particular case with tolerable facility. Had the number of equidistant ordinates been much increased, the additional accuracy of the approximation would not have compensated for the additional labour which would attend the resulting formula.

454. A rigorous integration would require the aid of infinite series. If r be the radius of the orifice; n the quotient of the distance of its centre from the upper surface of the fluid, divided by r; and $\pi = 3{\cdot}141593$; we should then have

$$Q=\pi tr^2\sqrt{2grn}\times\left(1-\frac{1}{32n^2}-\frac{5}{1024n^4}-\&c.\right)\ldots\ldots(\text{x.})$$

$$\text{and } s=rn\left(1-\frac{1}{16n^2}-\frac{9}{1024n^4}-\&c.\right)\ldots\ldots(\text{xi.})$$

In both these series the first three terms are all which will be needed in practice.

454. Ex. V. *To determine in what time* t, *the upper surface of the fluid, shall be depressed through a vertical distance* z, *the vessel being supposed prismatic, and the orifice rectangular.*

The section s of the vessel, and the breadth y of the orifice, are in this instance constant quantities. According to the method explained in art. 449, we first determine the quantity of fluid which would escape through the orifice during the time t, on the supposition that the vessel were kept full to the height $h'-z$, above the upper part of the orifice; for which purpose we must find the fluent of the expression $y\int \dot{x}\sqrt{(h'+x-z)}$, supposing z constant. This gives for the discharge sought $\frac{2}{3}y(h'+x-z)^{\frac{3}{2}}\times c$. But this quantity being nothing when $x=0$, we have $c=-\frac{2}{3}y(h-z)^{\frac{3}{2}}$, and the correct fluent is $\frac{2}{3}y\left[(h'+x-z)^{\frac{3}{2}}-(h'-z)^{\frac{3}{2}}\right]$. Extending this to the whole of the orifice by making $x=h-h'$, we shall have $\frac{2}{3}y\left[(h-z)^{\frac{3}{2}}-(h-z)^{\frac{3}{2}}\right]$; and if this value be substituted for $\int y\dot{x}\sqrt{(h'+x-z)}+c$, in the equa. iii. 449. it will become

$$t=\frac{3s}{2y\sqrt{2g}}\cdot\int\frac{\dot{z}}{(h-z)^{\frac{3}{2}}-(h'-z)^{\frac{3}{2}}}\;\ldots\ldots\;\text{(xii.)}$$

If we put a for the altitude of the orifice, b for its breadth, and δ for the distance of the upper surface of the fluid from the bottom of the orifice, we shall have $\delta=h-z$; $\dot{z}=-\dot{\delta}$; $h=h'+a$; $h'=\delta-a$; and the preceding equation will be transformed to this:

$$t=\frac{-3s}{2b\sqrt{2g}}\cdot\int\frac{\dot{\delta}}{\delta^{\frac{3}{2}}-(\delta-a)^{\frac{3}{2}}}\;\ldots\ldots\;\text{(xiii.)}$$

The quantity which is found under the sign of integration in this equation is susceptible of being made rational; but the equation would be very complicated. The most easy method of obtaining a result is to square by approximation the curve, which for each value of δ has an ordinate equal to $\frac{1}{\delta^{\frac{3}{2}}-(\delta-a)^{\frac{3}{2}}}$, in the same manner as was adopted in the last article. Observing now that when $\delta=h$, $t=0$, the surface may be estimated between the limits $\delta=h$, and $\delta=a$; and as in this case we should only have the terms to transcribe from the preceding article, there can of course be no occasion to copy here the final equation.

If the altitude of the orifice a be equal to the height h of the vessel, the time of exhaustion to any variable depth z, reckoned from the bottom, would be equal to $t=\frac{3s}{b\sqrt{2g}}\times\frac{h\sqrt{z}-z\sqrt{h}}{hz}$: when $z=0$, this expression is infinite; that is, the time of complete exhaustion is infinite.

The student who wishes to pursue further this part of Hydrodynamics may consult *l'Hydrodynamique de M. Bossut*, and the *Select Exercises* at the end of *Dr. Hutton's Conics*, or in the 3d vol. of his 8vo. Tracts.

455. PROP. *If upon the altitude of the fluid in a vessel as a diameter, we describe a semicircle, the horizontal space described by the fluid spouting from a vertical orifice at any point in the diameter will be as the ordinate of the circle drawn from that point, the horizontal space being measured on the plane of the bottom of the vessel.*

When the aperture is vertical, and indefinitely small (as supposed here), the fluid will spout out horizontally with the velocity due to the altitude of the fluid above the orifice (art. 439.); and this velocity, combined with the perpendicular velocity arising from the action of gravity, will cause every particle, and consequently the whole jet, to describe the curve of a parabola. Now the velocity with which the fluid is expelled from any hole, as G (fig. 9. pl. XVII.), is such as, if uniformly preserved, would carry a particle through a space equal to 2BG in the time of the fall through BG: but, after quitting the orifice, it describes the parabolic curve, and arrives at the horizontal plane CF in the same time as a body would fall freely through GD; so that, to find the distance DE, since the times are as the roots of the spaces, we have this analogy $\sqrt{GB} : \sqrt{GD} :: 2BG : DE = \frac{2BG\sqrt{GD}}{\sqrt{GB}} = 2\sqrt{(BG \cdot GD)} = 2GH$, by the nature of the circle. And the same will hold with respect to any other point in BD.

COR. 1. If apertures be made at equal distances from the top and bottom of the vessel (kept full of the fluid), the horizontal distances to which the water will spout from these apertures will be equal. For when $Dg = BG$, we shall have $2\sqrt{(Bg \cdot gD)} = 2\sqrt{(BG \cdot GD)}$, and consequently DE the same in both cases.

COR. 2. When the orifice is at the point bisecting the altitude of the fluid in the vessel, the fluid will spout to the greatest distance on the horizontal plane; and that distance, if measured on the plane of the bottom of the vessel, will be equal to the depth of fluid in it. For IK, the ordinate from the centre I, is the greatest which can be drawn in the semicircle, and DF, which is $= 2IK$, is then $= 2BI = BD$.

COR. 3. If a right line be drawn from B to F, since $BD = DF$, that line BF will make an angle of 45° with BD. And all the jets which are formed by adjutages at G, I, F, &c. in the vertical BD, will have for a common tangent the right line BF, or its prolongation.

COR. 4. Since the distance to which the fluid spouts depends upon the height of its surface AB above the orifice, and not in any degree upon the depth of the fluid below the orifice, it will follow that whether the fluid in any vessel reach down to any horizontal plane CF, or whether the bottom of the vessel stand at some higher point P, the distances DE, DF, to which the fluid will spout from the apertures G, I, will be the same; and the maximum distance will be when the fluid is expelled from an orifice half way between the planes AB and CF. If the bottom of the vessel be higher than the point I, then the nearer to that point the orifice is placed the greater will be the distance to which the fluid will spout on CF.

Cor. 5. Since the middle filament of particles issuing from an orifice in the side of a vessel is discharged with the full velocity due to the entire altitude of the fluid above the orifice (art. 440.), experiments made on the distance or height to which fluids spout will be found to agree very well with theory; but it by no means follows that *all* the filaments should be expelled with the same velocity: consequently, the quantity of the fluid discharged in a given time may be less than that which would be discharged if all the filaments were expelled with the velocity due to the entire altitude; because this quantity depends on the mean velocity of all the filaments. Hence, therefore, we cannot infer (as several authors have done) from these experiments, compared with those which relate to the height or distance to which the fluid spouts, that the velocity of the water in the orifice is less than that which is due to the entire altitude; and that it is accelerated immediately after it gets out of it; because the distance to which the fluid spouts depends upon the central filament only; but the quantity discharged on the mean velocity of the whole.

SCHOLIA.

456. I. If the fluid issue from the orifice not horizontally, but in any oblique direction, it will describe the curve of a parabola very nearly; consequently the theorems given in arts. 252. 253. for the motion of projectiles in a non-resisting medium, may be applied to the determination of the various analogous particulars respecting the motion of fluids expelled from springs or reservoirs. The letter I, which denotes the impetus in those theorems, will here represent the height of the fluid in the reservoir above the orifice, H the greatest height to which the fluid spouts above any plane passing through the aperture, and the other letters the range, the velocity, &c. As an example of the use of these formulæ, suppose that a short pipe elevated in any direction from an aperture in a conduit, throws the water in a parabolic curve to the distance or range R on an horizontal plane passing through the orifice, and that the greatest

height of the spouting fluid above that plane is H, we may thence *determine the height* I *of the reservoir which supplies the conduit, without actually measuring the angle of elevation* E *of the pipe.* For from art. 253, we have $R = 2I \sin 2E$, or $I = \frac{R}{2\sin 2E}$, and $I = \frac{H}{\sin^2 E}$. Hence, $\frac{R}{2\sin 2E} = \frac{H}{\sin^2 E}$, and, by reduction, we find $\frac{R}{2H} = \frac{\sin 2E}{\sin^2 E} = \frac{2\cos E}{\sin E} = 2 \cot E$. Thus the angle E will become known; and then either the equation $I = \frac{H}{\sin^2 E}$, or $I = \frac{R}{2\sin 2E}$, will determine the height of the reservoir, as required.

456. II. The foregoing investigations relate to the issuing of a fluid from an orifice in a vessel at *rest*; but it may not be amiss to point out the mode of conducting the enquiry when the fluid escapes from an orifice in a vessel in *motion*. To assume one of the simplest cases: let a vessel supposed full, at first, be raised by means of a weight W acting upon the vessel through the intervention of a cord passing over two fixed pulleys, so that the weight hangs freely and draws up the vessel; and let it be proposed *to investigate the circumstances of the discharge through an aperture, of area* a, *in the bottom of the vessel, during its motion upwards.* Suppose G to be the centre of gravity of the total mass of the vessel and water which it contains, M denoting that mass. Suppose, also, that if the two masses, W and M, were left to the free action of gravity, they would descend, in an instant, through the small equal spaces Ww, Gx, (easily conceived without a diagram); but that, because of the mutual action of the two masses W and M through the intervention of the cord which unites them, W descends only through Wk, and M ascends through the equal space Gy. Then, the quantity of motion lost by W being equal to the quantity gained by M in the same direction, if g be put for the force of gravity, and f for the simple accelerating force measured by Wk or Gy, we shall have $W(g - f) = M(g + f)$, whence we obtain $f = g\frac{W-M}{W+M}$; and consequently $xy = xG + Gy = g + g\frac{W-M}{W+M} = \frac{2gW}{W+M}$, the expression of the force which carries upwards each particle of the mass M: so that, if there should be impressed an equal and contrary motion upon all the particles of the system the whole would be in equilibrio. But in this latter case, by reason of the force $\frac{2gW}{W+M}$ which would act vertically downwards on each particle of the fluid, every point of the bottom or sides of the vessel would experience a pressure,

which is to the pressure which each such point sustains when the fluid is subjected to the action of gravity singly, as $\frac{2g\mathrm{W}}{\mathrm{W}+\mathrm{M}}$ to g, or as $2\mathrm{W}$ to $\mathrm{W}+\mathrm{M}$. Hence, putting h for the height of water in the vessel regarded as immoveable, the height due to the velocity at the orifice, in the hypothesis of the problem, will be $h\frac{2\mathrm{W}}{\mathrm{W}+\mathrm{M}}$.

Therefore, to express the elementary quantity of liquid $\dot{\mathrm{Q}}$ which will issue during the instant of time $\dot{t}$, we have only to substitute in the first theorem in cor. 3, art. 439. $h\frac{2\mathrm{W}}{\mathrm{W}+\mathrm{M}}$ for h, $\dot{t}$ for t, and $\dot{\mathrm{Q}}$ for Q, it will thus become adapted to the present case, and be $\dot{\mathrm{Q}} = 2a\dot{t}\sqrt{\frac{gh\mathrm{W}}{\mathrm{W}+\mathrm{M}}}$.

The vessel being imagined to empty itself by the aperture a, without receiving fresh supplies of the liquid, call the variable height of the liquid x, the horizontal section of the vessel at the surface of the liquid X, (this being a function of x given by the form of the vessel), then we shall have $\dot{\mathrm{Q}} = -\mathrm{X}\dot{x}$, and $\mathrm{M} = \mathrm{A} + \int \mathrm{X}\dot{x}$, A being a constant quantity. Therefore $-\mathrm{X}\dot{x} = 2a\dot{t}\sqrt{\frac{g\mathrm{W}x}{\mathrm{W}+\mathrm{A}+\int\mathrm{X}\dot{x}}}$: and $\dot{t} = \frac{-\mathrm{X}\dot{x}\sqrt{(\mathrm{W}+\mathrm{A}+\int\mathrm{X}\dot{x})}}{2a\sqrt{(g\mathrm{W}x)}}$.

If the vessel be a solid of revolution and the altitude the axis, and the figure be required so that in equal times the surface of the liquid in the ascending vessel shall be depressed by equal quantities; then must $\dot{t} = -n\dot{x}$, n being a given coefficient: afterwards, supposing y the ordinate, and of consequence $\mathrm{X} = \pi y^2$, π being $= 3{\cdot}141593$, freeing the equation of radicals, putting the expression $\int\mathrm{X}\dot{x}$ or $\int\pi y^2\dot{x}$, on one side of the equation, the process of differentiation will give an equation of this form $y^7\dot{x} + \mathrm{B}y\dot{x} + \mathrm{C}x\dot{y} = 0$, B and C being given constant quantities. Whence we obtain the "separate equation" $\frac{\dot{x}}{x} = -\frac{\mathrm{C}\dot{y}}{y^7+\mathrm{B}y}$,

Recurring to the equation $f = g\frac{\mathrm{W}-\mathrm{M}}{\mathrm{W}+\mathrm{M}}$, it is obvious that when $\mathrm{W} = \mathrm{M}$, $f = 0$, and $\frac{2\mathrm{W}}{\mathrm{W}+\mathrm{M}} = 1$. Of course, the vessel is at rest, at least during an instant, when the discharge of the quantity $2a\dot{t}\sqrt{\tfrac{1}{2}gh}$, will destroy the equilibrium, and the vessel will ascend.

If $W = 0$, then $\frac{2W}{W+M} = 0$, the pressure of the fluid on the bottom will vanish, and none will issue through the orifice.

When M exceeds W, the vessel will *descend;* and, to determine the discharge, f must be regarded as negative. In this case, however, the height due to the velocity is $h\,\frac{2W}{W+M}$.

For problems analogous to this, the reader may consult Bossut's Hydrodynamique, part ii. cap. 8, and the controversial papers between Mr. Dawson and Mr. Wildbore, in Hutton's Mathematical Miscellany.

Another interesting application of the theory of hydrodynamics, is to the motion of water in rivers; for which see *Playfair's Outlines of Natural Philosophy,* vol. i. p. 191—197; *Du Buat, Principes d'Hydraulique;* and *Bossut,* Hydrodynamique, tom. ii. p. 254—288.

CHAPTER II.

Account of Experiments made by different Philosophers on the Discharge of Water through Apertures and Tubes; and the practical Deductions from those Experiments.

457. THE greater part of the propositions and theorems in the preceding chapter are founded upon the hypothesis, that the *whole* of the fluid particles issuing from a horizontal orifice in a vessel is expelled with the velocity due to the height of the fluid in the reservoir above that orifice; and the whole of the particles discharged through an evanescent horizontal element of a vertical aperture, with one and the same velocity. But this hypothesis, as we have hinted more than once, is not altogether consistent with fact. For, when water issues from an orifice, the particles will flow from all sides towards the orifice, with an accelerated motion, and in all directions. If the orifice be horizontal, that filament of particles which answers to its centre will, as Dr. M. Young observes, descend in a vertical line, and suffer no other resistance than that of the friction caused by the excess of its velocity above that of the collateral filaments, or by the retardation which arises from the attraction subsisting between them. The other filaments are soon compelled to turn from their vertical course, and to approach the orifice in spiral curves, (art. 444.); and, when they arrive at it, their directions become more or less horizontal, according as they pass nearer to, or further from, the edge of the orifice. The motion, therefore, may be decomposed into two directions; the one horizontal, which is destroyed by the equal and contrary resistance of the filaments which are diametrically opposite; the other vertical, in proportion to which the quantity of water discharged is to be estimated. Hence it appears that the vertical velocity of the filaments decreases from the centre of the orifice to the circumference; and that the total discharge is less than if the filaments had issued vertically, as that filament does which corresponds to the centre of the orifice. It also follows that the

filaments which are nearer to the centre, moving faster than those which are nearer to the edges, the vein of the fluid, after it has issued from the aperture (if that be circular) will form a conic frustrum whose greater base is the aperture; that is to say, its diameter will diminish to a certain distance, because the exterior filaments are gradually drawn, in consequence of their mutual attraction, by the interior filaments whose velocity is greater, whence follows a diminution in the diameter of the vein.

This diminution in the section of the vein is often called the *contraction* of the vein; and the vein itself, from the orifice to the greatest diminution, is called the *vena contracta*, the *contracted vein.*

The contraction of the stream is found to take place not only when water is discharged from horizontal apertures, but when the discharge is from vertical apertures, or apertures inclined to the horizon in any manner whatever: in these latter cases, however, the form of the contracted vein is by no means so regular as in the discharge from horizontal orifices; the stream often assumes a very curious form, having for a small distance from the aperture the appearance of a plaited band.

When the orifice is horizontal and circular, the length of the contracted vein is very nearly equal to the semi-diameter of the orifice: and *Polenus* makes the ratio of the diameters of the contraction and of the aperture to be as 5¼ to 6½; *Bernoulli,* 5 to 7; *Chev. Du Buat,* 6 to 9; *Bossut,* 41 to 50; *Michelotti,* 4 to 5; *Venturi,* nearly 4 to 5. And the latter author has shown, by many experiments, that the contraction of the stream takes place at a greater distance under strong charges than in those which have but little elevation. The ratio of the surface of the section of the fluid vein at the place of greatest contraction, to the area of the orifice, does not much depend on the figure of the orifice; but this ratio is subject to variations, to which we should have regard when the side or the bottom of the vessel is more or less thick, or when an additional tube is adapted to the orifice. The situation of the orifice with respect to the sides of the vessel has likewise a certain influence on the contraction; but the differences thus occasioned may be generally neglected in practice. We may infer from what has now been said, that to obtain formula which will furnish results applicable to practice, we should, in the different cases, substitute for the actual area of the orifice the area of the smallest section of the contracted vein. This reduced area should be considered as the true orifice through which the discharge is made, and its vertical distance from the upper surface of the fluid as the height due to the velocity of the fluid issuing through this little orifice.

458. From the preceding remarks the necessity of giving

extension to the theory of Hydrodynamics, by combining with it the results of experiments, must be pretty obvious: we shall in this chapter, therefore, give a concise view of the most useful conclusions deduced from the experiments of different philosophers, commencing with those of M. *Bossut*, which appear to have been very numerous, judicious, and exact.

We propose to determine from the experiments, first, what is *the ratio between the area of the contraction of the vein and that of the orifice*; secondly, *the quantity of discharge through thin plates*; thirdly, *the discharge through additional tubes, their length being small compared with the depth of the water in the reservoir.*

In M. Bossut's first experiments, the apertures for the efflux of the water were all pierced perpendicularly in plates about ½ a line thick.

M. *Bossut* has given in his *Hydrodynamique* (tome II. pa. 46.) the following table relative to the discharge through orifices pierced in thin plates: the measure is the Paris foot royal, which is to the English foot as 1535 to 1440, or 1066 to 1000: the fourth column, which expresses the relation between the results of the experiments and those of the theory, is from M. *Prony*.

Constant altit. of the water in the reserv. above the apert. in Paris ft.	Theoret. discharges in 1^m, through a circular apert. of tin, diameter expressed in cubic inches.	Real discharges in the same time through the same orifice, expressed also in cubic inc.	Ratio of actual to theor. discharges, the latter being denoted by unity.
1	4381	2722	0·62133
2	6196	3846	0·62073
3	7589	4710	0·62064
4	8763	5436	0·62034
5	9797	6075	0·62010
6	10732	6654	0·62000
7	11592	7183	0·61965
8	12392	7672	0·61911
9	13144	8135	0·61892
10	13855	8574	0·61883
11	14530	8990	0·61873
12	15180	9384	0·61819
13	15797	9764	0·61810
14	16393	10130	0·61795
15	16968	10472	0·61716

It appears from this table that the actual discharges, as well as those resulting from the theory, are sensibly proportional to the square roots of the depths of fluid in the reservoir. Thus, for example, if we take the depths 4 and 9, whose square roots are as 2 to 3; the corresponding actual discharges taken from the third column, are 5436 and 8135: and these numbers are very nearly in the ratio of 2 to 3, the ratio being 2 to 2·9931.

If the numbers in the last column are multiplied together, and the 15th root of the last product taken, we shall have ·61932 for the true mean of the effective discharges, compared with the theoretic discharge 1; and the arithmetical mean between the numbers in the last column standing against the heights 7 and 8, is ·61938: the mean ratio between the actual and theoretic discharges, then, is not widely distant from that of ·62 to 1: whence it follows, from the remarks in art. 457, that ·62 is the number by which we must multiply the real area of the orifice to obtain the area of the smallest section of the contracted vein.

459. Another set of experiments made by M. *Bossut*, with *different* apertures, are the following, in which the water was kept constantly at the altitude of 11 feet, 8 inches, 10 lines, from the centre of each aperture.

Exp.		No. of cubic inches furnished in 1 min.
1.	With an horizontal circular aperture, 6 lines diameter	2311
2.	With a circular horizontal aperture, 1 inch diameter	9281
3.	With a circular horizontal aperture, 2 inches diameter	37203
4.	With a rectangular horizontal aperture, 1 inch by 3 lines	2933
5.	With a square horizontal aperture, the side 1 inch	11817
6.	By a square horizontal aperture, the sides 2 inches	47361
	Constant height 9 feet.	
7.	Lateral circular aperture, 6 lines diameter . .	2018
8.	Lateral circular aperture, 1 inch diameter . .	8135
	Constant height 4 feet.	
9.	Lateral circular aperture, 6 lines diameter . .	1353
10.	Lateral circular aperture, 1 inch diameter . .	5436
	Constant height 7 lines.	
11.	By a lateral and circular orifice, 1 inch diameter	628

From the preceding experiments we may make the following deductions:

1. 'The quantities of fluid discharged in equal times from different sized apertures, the altitude of the fluids being the same, are nearly to each other as the areas of the apertures.'

Thus in the second and third experiments the areas of the apertures are as one to four, and the water discharged 9281 cubic inches to 37203 is nearly in the same ratio.

2. 'The quantities of water discharged, in equal times, by the same aperture, with different altitudes of the reservoir, are nearly as the square roots of the corresponding altitude of the water in the reservoir above the centre of the aperture.' Comparing together the eighth and tenth experiments, in which the respective altitudes of the reservoir were of 9 and 4 feet, of which the square roots are 3 and 2, we find the water discharged by the first was 8135 cubic inches, the second 5436 cubic inches nearly in the proportion of 3 to 2, as before observed.

3. 'That, in general, the quantities of water discharged in the same time, by different apertures and under unequal altitudes of the reservoirs, are to each other in a compound ratio of the areas of the apertures and the square roots of the altitudes.'

4. 'That, on account of the friction, the smallest apertures discharge less water than those that are larger and of a similar figure, the water in the respective reservoirs being at the same height.'

5. 'That of several apertures whose areas are equal, that which has the smallest circumference will discharge more water than the others, the water in the reservoirs being at the same altitude,' and this because there is less friction. Hence circular apertures are most advantageous, as they have less rubbing surface under the same area.

Hence, then, to make the formulæ in the theory, for instance, those in cor. 3. art. 439. furnish such results as would agree with experiments, we must deduce the aperture *a* in those theorems in the ratio of ·62 to 1; or multiply the quantities resulting from the theorems as they now stand by the decimal ·62; or, lastly, if great accuracy be required, take, instead of the constant multiplier ·62, the number standing against the height of fluid in the reservoir above the orifice, in the last column of the table in the preceding article: thus, if the altitude of the fluid be 10 feet, the multiplier will be ·61883.

460. If the water, instead of flowing through an aperture pierced in a thin substance, pass through the end of a vertical tube of the same diameter as the aperture, there is a much greater discharge of water, because the contracted stream is greater in the first instance than in the second. In the following experiments the constant height of the water in the reservoir above the upper aperture of the tube was 11 feet 8 inches 10 lines, the diameter of the tube 1 inch.

Different lengths of the tube expressed in lines.			Number of cubic inches of water discharged in 1 minute.
	Lines		
Exp. 1	48	The stream filling the tube.	12274
2	24		12188
3	18		12168
4	18	The water not filling the tube.	9282

It appears, on comparing the first three experiments, that the longer the vertical tube is, the greater is the discharge of the water, because the contraction of the stream is less; it is, however, always somewhat contracted, even when it appears to fill the tube.

By comparing the quantities of water discharged in the third and fourth experiments, we find the two discharges 12168, 9282, are to each other nearly in the proportion of 13 to 10; but we have seen that the water discharged through a thin aperture without any contraction in the stream, would be to the same aperture with a contracted stream as 1 to ·62, or as 16 to 10. From hence we may conclude, that, the altitude in the reservoir and the apertures being the same, the discharge through a thin aperture without any contraction in the stream, the discharge through an additional tube, and the discharge through a similar aperture with a contracted stream, are to each other nearly as the numbers 16, 13, 10: these proportions are sufficiently exact for practice. Hence it is plain that an additional tube only destroys in part the contraction of the stream, which contraction is greatest when the water passes through a thin aperture from a large reservoir.

If the additional tube, instead of being vertical, or placed at the bottom of the reservoir, were horizontal, or placed in the side, it would furnish the same quantity of water, provided it was of the same length, and that the exterior aperture was at the same distance from the surface of the water in the reservoir.

If the additional tube, instead of being cylindrical, were conical, having its largest base nearest the reservoir, it would discharge a greater quantity of water. The most advantageous form that can be given, in order to obtain the greatest quantity of water in a given time by a given aperture, is that which the stream assumes in coming out of the aperture; i. e. the tube must be of the form of a truncated cone, whose largest base should be of the same diameter as the aperture; the area of the small base should be to that of the larger base as 10 to 16; and the distance from one base to the other should be the semidiameter of the largest base; and the efflux of water will be as abundant as it would be through a thin aperture equal to the smallest base, and where the stream was not contracted. This form may be applied where it is necessary to obtain a certain quantity of water from a river, an aqueduct, &c. by a canal or lateral tube.

On comparing the efflux of water through additional tubes of different diameters, and with different altitudes of the water in the reservoirs, the following results were obtained; the additional tubes were two inches long, and were vertical and placed at the bottom of the reservoir.

Constant altitude of the water above the tubes.		Diameter of the tubes expressed in lines.		Number of cub. in. in 1 min.
Exp. 1	3 feet 10 inches	6	Water filling the tube	1689
2		10		4703
3		6	The water not following the sides	1293
4		10		3598
5	2 feet	6	The water filling the tube	1222
6		10		3402
7		6	The water not filling the tube	935
8		10		2603

It results from these experiments, 1. 'That the discharges by different additional tubes, with the same altitude of the reservoir, are nearly in proportion to the area of the apertures, or to the squares of the diameters. 2. That the discharges of water by additional tubes of the same diameter with different altitudes of water in the reservoir, are nearly proportional to the square root of the altitude of the reservoir. 3. That in general the discharges of water in the same time, through different additional tubes, with different altitudes of water in the same reservoir, are to each other nearly as the product of the square of the diameters of the tubes by the square root of the altitude of the reservoirs.' So that additional tubes, transmitting water, ollow (amongst themselves) the same laws as through the thin orifice. The following table was formed from the foregoing experiments.

Constant altitude of the water in the reservoir above the aperture, expressed in feet.	Water discharged in one minute through a hole 1 inch diameter, the stream not contracted, in cubic inches.	Water discharged in one minute through an additional tube of 1 inch diameter, 2 inches long, in cubic inches.	Water discharged during 1 minute, through a hole 1 inch diameter, with a contracted stream, in cubic inches.	Ratio between the actual and the theoretic discharges, the latter being 1.
1	4381	3539	2722	0·81781
2	6169	5002	3846	0·80729
3	7589	6126	4710	0·80724
4	8763	7070	5436	0·80681
5	9797	7900	6075	0·80638
6	10732	8654	6654	0·80638
7	11592	9340	7183	0·80573
8	12392	9975	7672	0·80496
9	13144	10579	8135	0·80485
10	13855	11151	8574	0·80483
11	14530	11693	8990	0·80477
12	15180	12205	9384	0·80403
13	15797	12699	9764	0·80390
14	16393	13197	10130	0·80382
15	16968	13620	10472	0·80270

The mean of the numbers in the last column of this table is somewhat less than ·81; we may, however, in most cases, take ·81 as a very good approximation to the truth: using it as a constant co-efficient in the formula for the value of Q given in cor. 3. art. 439. when we wish to know the discharge through a cylindric tube of the dimensions specified at the head of column the third. Thus we shall have $Q = \cdot 81\, at \sqrt{(2gh)}$; the dimensions being all in feet, or all in inches.

461. We now pass to M. *Bossut's* experiments on the exhaustion of vessels (which have no extraneous supply) by little orifices.

The experiments upon the time of complete exhaustion of vessels which empty freely are not easy to make, at least in a conclusive manner: for, besides that in some cases the complete exhaustion would, according to the theory, require an unlimited time, it is found that, when the surface of the water arrives within a small distance, as two or three inches, of a horizontal orifice, it forms above that orifice a conical or rather conoidal funnel, which diminishes the effect, and makes the conclusion of the discharge uncertain. It is best, therefore, not to make experiments upon the time of total discharge, but upon the time in which the upper surface is depressed through a certain vertical distance x, the greater the better, provided the upper surface has not sunk so low as to permit the formation of the funnel just spoken of.

It was shewn (art. 443. equa. II.) that when the primitive height of the water in a prismatic vessel was $= h$, the constant section of the vessel $=$ A, the time t employed by the fluid to descend through the space x was expressed by this equation:

$$t = \frac{2\text{A}}{a\sqrt{2g}}\left[\sqrt{h} - \sqrt{(h-x)}\right].$$

Now when the orifice a is supposed pierced in a thin plate we must substitute the contracted orifice ·62 a, and the formula will become

$$t = \frac{2}{\cdot 62 \times 8\cdot 0208}\,\frac{\text{A}}{a}\left[\sqrt{h} - \sqrt{(h-x)}\right] = \cdot 40218\,\frac{\text{A}}{a}\left[\sqrt{h} - \sqrt{(h-x)}\right].$$

When the aperture is a circle, and its diameter d, we have $\cdot 785398\, d^2 = a$: this value of a substituted for it in the preceding equation gives

$$t = \cdot 51208\,\frac{\text{A}}{d^2}\left[\sqrt{h} - \sqrt{(h-x)}\right],$$

the dimensions being given in English statute feet; or if the dimensions are in terms of the Paris foot royal, then

$$t = \cdot 52852\,\frac{\text{A}}{d^2}\left[\sqrt{h} - \sqrt{(h-x)}\right].$$

This latter theorem may be applied to the experiments made by M. *Bossut*, in order to compare the results which it gives with those which are furnished by the experiments: the whole is comprised in the following table.

The constant section A of the vessel is 9 square feet. The primitive height h of the water is 11·667 Paris feet.	Diameter of the horizontal circular orifice, or value of d.	Depression of the water, or value of x.	Time of the depression of the water.	Time of the depression, calculated by the last theorem.
		Feet.	Seconds.	Seconds.
	·033333	4	445·5	443·04
		9	1224·5	1221·2
	·166666	4	112·	110·76
		9	306·	305·25

Hence we see that the difference between the results of the computation and those of the experiments are extremely small: much smaller indeed than might be expected, considering the many circumstances which may contribute to vary the times given by observation. So that we may regard the formulæ given in this article as sufficiently correct for practice; at least within the limit suggested by the formation of the conoidal funnel. What has been here said applies principally to horizontal apertures; but it may be applied without fear of material error to small laterial orifices, when the fluid in the reservoir stands higher than the upper side of the orifice, and taking for the height *h* the distance of the centre of gravity of the orifice from the upper surface of the fluid.

For an account of Bossut's experimental researches on the motion of water in rectangular canals, see Hydrodynam. tom. ii. p. 211—253.

Experimental Enquiries of Venturi.

462. The experiments and researches of M. *J. B. Venturi*, professor of natural philosophy at Modena, are neither so extensive nor so important as those of M. Bossut: but as he has noticed two or three curious circumstances relative to the motion of fluids, which seem to have escaped the observation of preceding philosophers, we shall present the reader with a concise account of the result of his enquiries.

I. In any fluid those parts which are in motion carry along with them the lateral parts which are at rest.

To shew the truth of this proposition M. *Venturi* introduced a horizontal current of water into a vessel filled with the same fluid at rest. This stream entering the vessel with a certain velocity, passes through a portion of the fluid, and is then received in an inclined channel, the bottom of which gradually rises until it passes over the border or rim of the vessel itself. The effect is found to be, not only that the stream itself passes out of the vessel through the channel, but carries along with it the fluid contained in the vessel; so that after a short time no more of the fluid remains than was originally below the aperture at which the stream enters. This fact is adopted as a principle or primitive phenomenon by the author, under the denomination of the *lateral communication* of motion in fluids, and to this he refers many important hydraulic facts. He does not undertake to give an explanation of this principle, but shews that the mutual attraction of the particles of water is far from being a sufficient cause to account for it.

II. If that part of an additional cylindric tube which is nearest the side of the reservoir be contracted, according to the form of the contracted vein of fluid which issues through a hole of the same diameter in a thin plate, the expenditure will be the same as if the tube were not contracted at all; and the velocity of the stream within this tube will be greater than that of a jet through a thin plate in the ratio of 121 to 100.

III. The pressure of the atmosphere increases the expence of water through a simple cylindrical tube, when compared with that which issues through a hole in a thin plate, whatever may be the direction of the tube.

IV. In descending cylindrical tubes, the upper ends of which possess the form of the contracted vein, the discharge is such as corresponds with the height of the fluid above the inferior extremity of the tube.

V. In an additional conical tube the pressure of the atmosphere increases the expenditure in the proportion of the exterior section of the tube to the section of the contracted vein, whatever may be the position of the tube, provided its internal figure be adapted throughout to the lateral communication of motion.

VI. In cylindrical pipes the expenditure is less than through conical pipes, which diverge from the place of the contracted vein, and have the same exterior diameter.

For, in the space between the inverted contracted vein and the sides of the cylinder, eddies, or circular whirls, are produced, as in a basin which receives water by a channel; and these retard the efflux of the stream, and produce a corresponding failure in the effect.

VII. By means of proper adjutages applied to a given cylindric tube placed horizontally, it is possible to increase the expenditure of water through that tube in the proportion of 24 to 10, the charge or height of the reservoir remaining the same.

For this purpose, the inner extremity of the tube AD (fig. 11. pl. XVII.) must be fitted at AB with a conical piece of the form of the contracted vein; this increases the expenditure as 12·1 to 10. Every other form will afford less. If the diameter at A be too great the contraction must be made beyond B, and the section of the vein will be smaller than the section of the tube. Secondly, at the other extremity of the pipe BC apply a truncated conical tube CD, of which let the length be nearly nine times the diameter C, and its external diameter D must be 1·8C. This additional piece will increase the expenditure as 24 to 12·1. So that the quantity of effluent water will be increased by the two adjutages in the ratio of 24 to 10. All this is on the supposition that the pipe BC has no elbows or sinuosities.

VIII. The expenditures out of a straight tube, a curved tube in a quadrantal arc, and an elbowed tube having the angle 90°, (each being posited horizontally) are *cæt. par.* nearly as 70, 50, and 45.

IX. The internal roughness of a pipe diminishes the expenditure, though the friction of the water against these asperities does not form any considerable part of the cause. A right-lined tube may have its internal surface highly polished throughout its whole length; it may every-where have a diameter greater than the orifice to which it is applied; but, notwithstanding the expenditure will be greatly diminished if the pipe should have enlarged parts, or swellings: for, by reason of these sudden changes in the interior dimensions of the pipe, much of the motion will be consumed in eddies. This, as M. *Venturi* remarks, is a very interesting circumstance, to which, perhaps, sufficient attention has not been paid in the construction of hydraulic machines. It is not enough that elbows and contractions are avoided; for it may happen, by an intermediate enlargement, that the whole advantage may be lost, which may have been procured by the ingenious dispositions of the other parts of the machine.

The above comprises what to us appeared most important in M. *Venturi's* researches, relative immediately to the subject of hydrodynamics. Those, however, who are desirous of seeing a more detailed account of this ingenious author's experiments, may consult Mr. Nicholson's translation of his work "*On the Lateral Communication of Motion in Fluids*," sold by Taylor, Holborn.

Practical Conclusions of Mr. Eytelwein.

463. Mr. *Eytelwein* published, at Berlin, in 1801, a treatise entitled *Handbuch der Mechanik und der Hydraulik*; from the second part of which, relative to hydrodynamics, we shall extract a few particulars.

I. In the chapter on the motion of water flowing out of reservoirs, and on the contraction of the stream, this philosopher makes the area of a section at the distance of about half its diameter from the orifice about $\frac{16}{25}$ of that of the aperture: hence the diameter is reduced to $\frac{4}{5}$. The quantity of water discharged is very nearly, but not quite, sufficient to fill this section with the velocity due to the height: for, finding more accurately the quantity discharged, the orifice must be supposed diminished to ·619, or nearly $\frac{5}{8}$. Hence the square root of the height may be multiplied by 5 instead of 8 (an approximate root of $64\frac{1}{3}$) for the mean velocity in a simple orifice.

II. If we apply the shortest pipe that will cause the stream to adhere every-where to its sides, which will require its length to be twice its diameter; the discharge will be about $\frac{13}{16}$ of the full quantity, and the velocity may be found by taking $6\frac{1}{2}$ for a multiplier.

III. The greatest diminution is produced by inserting a pipe so as to project within the reservoir, probably because of the greater interference of the motions of the particles approaching its orifice in all directions: in this case the discharge is reduced nearly to a half.

IV. A conical tube approaching to the figure of the contraction of the stream procured a discharge of ·92, and, when its edges were rounded off, a discharge of ·98, calculating on its least section.

V. Mr. Eytelwein is of opinion that the assertion of Venturi (art. 462. vii.) is too strong, and observes, that where the pipe is already very long, scarcely any effect is produced by the addition of such a tube. He proceeds to describe a number of experiments made with different pipes, where the standard of comparison is the time of filling a given vessel out of a large reservoir, which was not kept always full, as it was difficult to avoid agitation in replenishing it; and this circumstance was perfectly indifferent to the results of the experiments. They confirm the assertion that a compound conical pipe may increase the discharge to twice and a half as much as through a simple orifice, or to more than half as much more as would fill the whole section with the velocity due to the height: but where a considerable length of pipe intervenes the additional orifice appears to have little or no effect.

VI. The first chapter concludes with a general table of the coefficients for finding the mean velocity of the water discharged by the pressure of a given head under different circumstances.

1. For the whole velocity due to the height, the coefficient by which its square root is to be multiplied is 8·0208.

2. For an orifice of the form of the contracted stream, 7·8.

3. For wide openings, of which the bottom is on a level with that of the reservoir; for sluices with walls in a line with the orifice; for bridges with pointed piers, 7·7.

4. For narrow openings, of which the bottom is on a level with that of the reservoir; for smaller openings in a sluice with side walls; for abrupt projections and square piers of bridges, 6·9.

5. For short pipes, from two to four times as long as their diameter, 6·6.

6. For openings in sluices without side walls, 5·1.

7. For orifices in a thin plate, 5.

VII. In the chapter on the discharge by rectangular orifices in the side of a reservoir, extended to the surface, this author makes use of the same principles as we have adopted in art. 446. and shews that the quantity of water discharged may be found by taking two-thirds of the velocity due to the mean height, and allowing for the contraction according to the form of the opening.

On comparing the factors here deduced by Eytelwein from his researches, with those in our account of Bossut's experiments, it will be found that there is no very great disagreement between them: though Bossut's manifestly claim the preference in point of accuracy. A further account of Eytelwein's labours (by Dr. *T. Young)* may be seen in Nicholson's Journal, Nos. 9, 10. N.S.

Researches of Dr. M. Young and Mr. Vince.

464. Dr. *Matthew Young*, the late bishop of Clonfort, paid particular attention to the various circumstances connected with the discharge of fluids from orifices of different kinds; but he appears to have been most successful in his enquiries into the cause of the increased velocity of efflux through additional tubes. This chapter will, therefore, be terminated by an extract from his paper in vol. VII. of the *Transactions of the Royal Irish Academy*, which contains some judicious remarks relative to his own experiments, and applicable at the same time to some experiments made by Mr. *Professor Vince.*

"When a tube *mnrs* (fig. 10. pl. XVII.) is inserted into the vessel ABCD, it is found that the velocity is increased nearly in the subduplicate ratio of the length of the pipe; and that it approaches nearer to that subduplicate ratio according as the length of the pipe is increased. To account for this increase of velocity has appeared a matter of some difficulty, since the water cannot issue at *rs* with a greater velocity than it enters at *mn*; and it does not appear how the velocity at *mn* can be increased by inserting a tube beneath it. In order to explain the cause of this effect, we are to consider that the whole force with which the plate *mn* is pressed down is the weight of a column of water equal to *emnf*, together with the weight of a column of air of the same base, reaching to the top of the atmosphere; and the whole force with which it is pressed up is the weight of an equal column of air, diminished by the weight of a column of water equal to *mnrs*; therefore the actual force with which the plate *mn* is pressed down is the weight of a column of water equal to *efrs*: the velocity therefore with which the plate *mn* will issue through the orifice *mn* will be the same as through the orifice *rs* in the vessel A*bc*D; that is, equal to the velocity which a heavy body would acquire in falling through the altitude *tr*: and all the plates of water in the tube *mnrs* will descend with the same velocity; for they cannot descend faster, because otherwise there would be a vacuum left in the tube, which is prevented by the upward pressure of the atmosphere. And the velocity of the effluent water will be the same, whatever be the pressure of the atmosphere, provided the weight of a column of air of the same base with *rs*, and whose height is equal to that of the atmosphere, be either greater than or equal to the weight of the pillar of water *mnrs*. This might be proved experimentally by a vessel of water with a pipe inserted in the bottom placed under an exhausted receiver. But as the operation of exhaustion is obstructed more by the evaporation of water than of mercury, it will be better to use mercury in these experiments. Now if D be the defect of the gauge from the standard altitude, it will measure the pressure of the air on the surface of the mercury in the vessel: let A be the altitude of the mercury in the vessel: above the upper orifice of the pipe, and P the length of the pipe; then the whole force pressing downwards the plate of mercury, which is immediately in the upper orifice of the pipe, will be $=D+A$; and the whole force pressing the same plate upwards will be $D-P$; and the difference between these forces will be the absolute force pressing the same plate of mercury downwards: while D is greater than P, this absolute force will consequently be equal to $A+P$; when $D=P$, $D-P$ vanishes, and the force pressing the plate downwards is $=D+A=P+A$; hence therefore no variation in the time of the efflux will be perceived, while the altitude of the mercury in the gauge is equal to or less than the difference between the length of the pipe and the standard altitude. When D is less than P, the force upwards is also nothing; and therefore, as before, the whole force pressing the plate downwards is $=D+A$; and A being given, it decreases according as D decreases; and when D vanishes, that is, when the receiver is absolutely exhausted, the force becomes equal to A, and the time of the efflux will be the same as if the pipe had not been inserted in the bottom of the vessel. To try the truth of these things by experiment, I inserted a tube 7·8 inches long in a cylindrical vessel, and, closing the orifice of the pipe, I filled the vessel with mercury to the height of 6 inches; then placing the apparatus under the receiver of an air-pump, when the barometer was at 30 inches, and the gauge at 28·5, the time of the efflux was 26 seconds; when the experiment was repeated precisely in the same manner, but in the open air, the time of the efflux was only 19 seconds. Now as the gauge stood at 28·5, the defect D was $30-28·5=1·5$, and the pressure on the plate of mercury was

$=6+1\frac{1}{2}=7\frac{1}{2}$; in the open air the pressure was $=6+7{\cdot}8=13{\cdot}8$: therefore the ratio of the velocity of efflux in both cases, which is the same with the reciprocal ratio of the times, was $\sqrt{7\frac{1}{2}}$ to $\sqrt{13{\cdot}8}$, or as 2·73 to 3·7; but 2·73 is to 3·7 as 19 to 26, very nearly.

"No difference was observed in the times of the efflux, when in the open air and exhausted receiver, unless the gauge stood higher than 22½ inches; that is, unless the height of the mercury in the gauge was greater than the difference between the length of the pipe and the standard altitude. In another experiment, when the gauge stood at 27·9, the height of the barometer was 29·9; the defect therefore was $=2$, and the pressure $=8$. But $\sqrt{8}=2{\cdot}828$, and $\sqrt{13{\cdot}8}=3{\cdot}7$; but $2{\cdot}828 : 3{\cdot}7 :: 19 : 24$, and by experiment the time of the efflux appeared to be 23 seconds. When the efflux is made in vacuo, it is obvious to observe that the pipe is not filled during the efflux, as it is while the discharge is made in the open air.

465. "Since the column of water in the pipe *mnrs* adds to the pressure which the plate *mn* sustains, by diminishing the upward pressure of the air through the pipe, it appears that it produces this increase of pressure in the plate *mn* alone, without producing any lateral pressure in the water which is on a level with *mn*; for it is manifest, that if an aperture were made in *m*B or *n*C, the velocity of the water issuing through it would not be affected by the insertion of the pipe; and consequently that the plate *mn*, which is immediately in the orifice of the pipe, is the only one, on the same level, whose tendency downwards is increased by the insertion of the pipe. Hence the particles of water at the edge of the aperture having their perpendicular pressure increased by the weight of the column *mnrs*, without any increase of their lateral pressure, they will issue through the orifice *mn* more perpendicularly; the sides also of the tube will obstruct the converging motion of the particles, and consequently, on both these accounts, the quantity of water discharged through a pipe thus inserted will exceed that discharged through a simple orifice, in a greater ratio than the sub-duplicate of the height of the water. And according as the length of pipe increases, the ratio of the quantity of water actually discharged by experiment, to that which should be discharged according to theory, will increase; because the ratio of the perpendicular to the horizontal pressure increases, in the ratio of the sum of the depth of the vessel, and length of the pipe, to the depth of the vessel. It follows therefore, that experiments made in this manner will approach nearer to coincidence with theory than when made with a simple orifice; except either when the tube is so long as that the friction of the fluid against the sides of the tube

shall produce a sensible effect, or when it is so short as not to be sufficient to give the particles a vertical direction. All which agrees very well with the experiments made by the ingenious Mr. Vince, of which he has given us an account in the Phil. Trans. for the year 1795. Thus he tells us, that having inserted a tube, a quarter of an inch in length, into a cylindrical vessel 12 inches deep, he found that the velocity did not sensibly differ from that through the orifice; the cause of which he discovered to be this, that the stream did not fill the pipe, but that the fluid was contracted, as when it flowed through the simple orifice. When the pipe was half an inch long, inserted into a vessel of the same depth as before, the velocity of the water from the pipe and from the orifice, which ought by theory to have been as $\sqrt{12{\cdot}5}$ to $\sqrt{12}$, or 49 to 48, was by experiment found to be nearly in the proportion of 4 to 3. Now if the ratio of 49 to 48 be increased in the ratio of 7 to 6 (because this is the ratio of the diminution of the velocity on account of the contraction of the vein, and this contraction either nearly or entirely vanishes in a pipe), we shall have the ratio of 3·57 to 3. When the pipe was an inch long, the velocity from the pipe and from the orifice, which, according to theory, ought to have been as $\sqrt{13}$ to $\sqrt{12}$, or as 26 to 25, appeared by experiment very nearly in the ratio of 4 to 3; now if the ratio of 26 to 25 be increased in the ratio of 7 to 6, we shall have the ratio of 3·64 to 3. When he made use of longer pipes, the velocity of the effluent water by experiment approached nearer to that which ought to have been discharged according to theory; so that in long pipes the difference between theory and experiment, he says, was not greater than what might be expected from the friction of pipes, and other causes which may be supposed to retard the velocity. When he inserted a pipe of the same diameter with the aperture, which terminated in a truncated cone fixed in the orifice (see the Philosophical Transactions for the year 1795), he expected that the quantity of water discharged in a given time would have been diminished, because the water, issuing through the orifice *mn*, would have room to form the *vena contracta* in the enlarging cone; but he found that the same quantity of water was discharged as if the pipe had continued throughout of the same diameter with the orifice. The reason of this is manifest from what has been said, for the pressure of the air will not suffer the truncated cone to remain partly empty, as it would be if the *vena contracta* were formed; it will therefore continue full, and consequently, the water will pass through it in the same manner as if the water in the cone, surrounding the pipe *mabn*, were congealed.

466. " Mr. Vince likewise inserted into the bottom of the

vessel a perpendicular pipe, in form of a truncated cone, the narrower part being fixed in the orifice, by which he found the efflux to be increased more than if he had inserted a cylindrical pipe of the same length, whose diameter was equal to that of the narrowest part of the conical pipe. This effect may be explained on the same principle by which we account for the augmentation of the diameter of a vertical vein of water through a simple orifice, when the velocity of the efflux is considerable. For when a perpendicular pipe is inserted, the velocity of the discharge being considerably increased, the resistance from the air will be so likewise; and thus the diameter of the vein has a tendency to enlarge itself: now, in the widening cone, the pipe admits of this augmentation, at the same time that it increases the velocity; but the cylindrical pipe, though it equally increases the velocity, yet it does not permit the vein to enlarge itself, and, by thus confining it, the efflux is obstructed, and the quantity discharged in a given time is diminished. Accordingly, under the receiver of an air-pump, even in a moderate degree of exhaustion, there is no difference perceived between the velocities, with which a fluid is discharged through a conical or cylindrical pipe."

For more on this subject consult Mr. Vince's paper in *Phil. Trans.* vol. lxxxv. or *New Abridgement*, vol. xvii. p. 466.

CHAPTER III.

On the Effect of Water upon the Motion of Water-wheels.

467. Water-mills are distinguished into four kinds: *Breast-mills, undershot-mills, overshot-mills*, and *mills with horizontal wheels;* of which, however, the latter kind is by far the least common, being very disadvantageous and deficient in point of utility. In a breast-mill the water falls down upon the wheel at right-angles to the float-boards or buckets placed all round the wheel to receive it: if float-boards are used, the water acts only by its impulse; but if buckets, it acts also by the weight of water in the buckets in the under quadrant of the wheel, which is considerable. In the undershot wheel float-boards only are used, and the wheel is turned merely by the force of the current running under it, and striking upon the boards. In the overshot-wheel the water is poured over the top, and thus acts principally, though not altogether, by its weight; for the fall upon the upper part cannot be very considerable, lest it should dash the water out of the buckets. Hence it is evident that an undershot-wheel must require a much larger supply of water than any other; the breast-mill the next, unless the fall is very great; and an overshot-mill the least.

It was long believed that the float-boards of an undershot-wheel ought to be so proportioned that when one of them was in a vertical position, or at the middle of its immersion, the next board should be just entering the water; but it is now well known that the more float-boards such a wheel has, the greater and more uniform will be its effect. According to the experiments made by M. *Bossut* on this subject, a wheel furnished with 48 float-boards produced a much greater effect than one furnished with 24; and the latter a greater effect than one with 12; their immersion in the water being equal. And the same thing will appear from some of the experiments of Mr. *Smeaton*, described in the fourth chapter of this book.

It is, also, well established, that in fluids, which may be regarded as of indefinite extent, when a surface is struck directly by a current, the force of the impulsion *is equal to the weight of a column of that fluid, whose base is the surface impelled, and altitude the height due to the velocity of the current.* When the stream is comprized within a canal whose section is exactly occupied by the surface struck, the altitude of the column must be *double* the height due to the velocity with which the impulsion is made. Hence the absolute measure of the force with which a given fluid, moving with a given velocity, strikes a given surface, may be easily ascertained. Take, for example, water moving with a velocity of 1 foot per second, and striking a surface of a foot square. The base of the equivalent column is 1^2; its altitude, $\frac{v^2}{2g} = \frac{1}{64\frac{1}{4}}$; the weight of a cubic foot $62\frac{1}{2}$lbs. avoirdupois. Consequently the weight of the column is $1^2 \times \frac{1}{64\frac{1}{4}} \times 62\frac{1}{2} = \frac{62\frac{1}{2}}{64\frac{1}{4}} = \frac{250}{257}$, or nearly 1lb. avoirdupois, in the case of an indefinite fluid. When the fluid is confined to a narrow channel, the measure is $\frac{1}{2}$ a pound. Hence, it appears that if the several measures be in feet, the force of the direct impulsion in pounds, will be indicated either by the product, or by the half product, of the surface struck into the square of the velocity, according as the fluid is indefinite or definite.

468. The varying force which communicates motion to water-wheels, and the resistances occasioned by friction, tenacity, and various other causes, render the application of the theory to practice in these cases extremely difficult. From this reason probably it happens that the art of constructing machines to be moved by the force of water has been almost wholly practical; the best improvements having generally been deduced from constant observation and reiterated experiments. Since the theory of mechanics is confessedly inadequate to the complete investigation of every circumstance in the motion of hydraulic machines, it has been thence supposed that the same laws of motion would not extend to all branches of mechanics; but that different principles were to be accommodated to the motion and mutual action of different kinds of bodies. If this were truly the case, the science of mechanics would fall far short of that superior excellence and extent which it is generally allowed to possess. But it is highly probable that there is no kind of motion, whether of the most simple or complicated nature, but what may be referred to the same principles as were assumed at the commencement of this work (art. 21.): and if we are not enabled to investigate the effects from the data in every case which can arise, the deficiency must not be imputed

to the first principles of mechanics, but to the want of methods of applying our mathematical knowledge to the peculiar circumstances under discussion; or to our inability to reduce to computation the various causes of resistance and obstruction to the moving forces. For these reasons, however, we shall not dwell long upon the theory of the motion of water-wheels; but shall merely exhibit a few propositions, the consideration of which may enable the student to appreciate more exactly the value and importance of the experimental researches of Smeaton, and some other practical writers.

469. Prop. *If a stream of water impinge on the float-boards of an undershot-wheel, and escape from it the very instant after it has made its impact, the quantity of water which actually impinges against the wheel will be to the whole quantity which passes by it in a given time, as the difference between the velocities of the water and of the wheel to the absolute velocity of the water.*

Let WH (fig. 2. pl. XVIII.) be the wheel, BA the stream of water, and let the float-board F first receive the impact from the water at F, and quit it at C; also let DF be to FC as the absolute velocity of the water to the velocity of the float-board. Then, when F arrives at C, the particle at D will have passed at F; and, taking DE = FC, all the water in the space DE will pass by the wheel without impinging against it: for it cannot impinge on the float F, because that float emerges from the water at C; neither can it impinge on the subsequent float, because it has already passed it. Therefore the whole quantity of water which passes by the wheel in a given time is to that which actually impinges against it, as DF to EF.

Cor. 1. Hence we may correct the mistake of Mr. *Waring*, in his New Doctrine of Mills, who lays it down as a fundamental principle, that while the stream is invariable, whatever be the velocity of the wheel, the same number of particles or quantity of the fluid must strike it somewhere or other in a given time. See the 3d vol. of the Transactions of the American Society, page 144.

Cor. 2. *The force of the impinging water is as the square of the difference between the velocities of the wheel and the water.*

For the force is as the relative velocity into the quantity of impinging matter, and the latter is manifestly as the relative velocity; therefore the force will be as the square of the relative velocity.

470. Prop. *If* W *be a weight fastened to a line which is wound round the horizontal axis of an under-shot water-wheel,* A *the altitude of a column of water equivalent to the force of the impact of the water on the wheel, when the wheel is quiescent,*

V *the velocity with which the water impinges on the float-boards,* v *the velocity of the circumference of the wheel,* R *the radius of the wheel, and* r *the radius of the axle; then will the velocity of the wheel be* $v = V - V\sqrt{\frac{wr}{AR}}$.

For the relative velocity with which the water strikes the wheel is $V-v$: whence, because the force of the stroke is as the square of the relative velocity, we have $V^2 : (V-v)^2 :: A : F$, the force of the water to turn the wheel, when its velocity is v; and $F = A\left(\frac{V-v}{V}\right)^2$, or $v = V - V\cdot\sqrt{\frac{F}{A}}$. But the acceleration of the wheel will cease when the force of the water to turn the wheel is equal to the force of the weight which opposes it; that is, when $FR = wr$, or $F = \frac{wr}{R}$. Hence, substituting this value of F for it in the preceding value of v, there results $v = V - V\sqrt{\frac{w}{AR}}$, for the velocity of the wheel when its acceleration ceases.

This conclusion, it should be observed, is true only on the hypothesis that the water escapes from the wheel as fast as it impinges.

COR. 1. The expression $F = A\left(\frac{V-v}{V}\right)^2$, found above, is the same in effect as the expression $F = \varphi\left(\frac{W-V}{w}\right)^2$ found in art. 377. II. Only we have now made a change in the notation, to suit it for more easy recollection in the enquiry before us. Hence, pursuing similar methods to those adopted in the place just referred to, we may deduce similar practical inferences; and which, without a repetition of the investigation, may be expressed as below.

COR. 2. *If the weight* W *vary, its momentum will be the greatest possible when the wheel has acquired its uniform velocity, if* $W = \frac{4AR}{9r}$.

COR. 3. *The greatest momentum generated in the ascending weight will be* $= \frac{4}{27}AV$.

COR. 4. *When the momentum of the ascending weight* W *is a maximum, that weight will be* $\frac{4}{9}$ *of the weight which would, if suspended from the axle, balance the force of the stream.*

COR. 5. *When the momentum of the ascending weight is a maximum, the velocity of the wheel will be* $\frac{1}{3}$ *of the absolute velocity of the stream.*

Cor. 6. *When the uniform velocity of the ascending weight* W *is a maximum, the radius of the wheel will be* $= \frac{9wr}{4A}$.

For the radius of the wheel being R, and that of the axle r, the uniform velocity of the ascending weight will be found by multiplying $V - V\sqrt{\frac{wr}{AR}}$ into the fraction $\frac{r}{R}$; that is, the velocity of w will be $= \frac{r}{R}V - V\sqrt{\frac{wr^3}{AR^3}}$: which, supposing R variable, and making the fluxion $= 0$, gives $R = \frac{9wr}{4A}$.

471. **Prop.** *If the velocity of the stream be given, the greatest effect will be as the quantity of water expended.*

For, by cor. 3. last prop. the greatest effect is as $\frac{4}{27}AV$; that is, since $\frac{4}{27}$ is constant, the effect is in a ratio compounded of the force of impact and the velocity of the stream: but the force of the impact is as the quantity of water expended and velocity conjointly; consequently the effect is as the quantity of water expended and the square of the velocity; or, if the velocity be given, as the quantity of water expended.

Cor. 1. *When the expence of water is the same, the greatest effect will be as the square of the velocity.*

Cor. 2. *The expence of water being the same, the effect will be as the height of the head of water.*

For $V = \sqrt{2gh}$, whence $h \propto V^2$.

Cor. 3. *The aperture being the same, the effect will be as the cube of the velocity.*

For the effect is as Q (the quantity of water) into V^2; and when the aperture is given $Q \propto V$, whence, effect $\propto V^3$.

472. **Prop.** *If all the water which passes by an undershot wheel be supposed to impinge against it, the force of the stream will be simply in the direct ratio of the relative velocity.*

Because the number of particles which strike the wheel in a given time is given, whatever be the velocity of the wheel.

Cor. 1. *According to this hypothesis, v the velocity of the wheel will be equal to* $V - V\frac{wr}{AR}$.

For, in this case, we have $V : V - v :: A : F$, whence $v = V - \frac{VF}{A}$; which, by substituting for F its value $\frac{wr}{R}$, becomes $v = V - \frac{Vwr}{AR}$.

Cor. 2. *Retaining the same hypothesis, if* W *vary, its momentum will be* $= \frac{AR}{2r}$, *when it is a maximum.*

For since the uniform velocity of the circumference of the wheel is $= \mathrm{V} - \frac{\mathrm{V}wr}{\mathrm{AR}}$, which, multiplied into $\frac{r}{\mathrm{R}}$, gives $\frac{\mathrm{V}r}{\mathrm{R}} - \frac{\mathrm{V}r^2w}{\mathrm{AR}^2}$ for the uniform veloc. of the ascending weight w: hence its momentum is $\frac{\mathrm{V}rw}{\mathrm{R}} - \frac{\mathrm{V}r^2w^2}{\mathrm{AR}^2}$; which being a maximum we make its fluxion $= 0$, considering w as variable, whence results $w = \frac{\mathrm{AR}}{2r}$.

COR. 3. *On the same supposition, the velocity of the wheel will be half the velocity of the stream, when the effect is a maximum.*

For, by cor. 1. $v = \mathrm{V} - \frac{\mathrm{V}wr}{\mathrm{AR}}$; and in the case of the last cor. $w = \frac{\mathrm{AR}}{2r}$. Putting this for v in the former equation, it is transformed to $v = \mathrm{V} - \frac{\mathrm{VAR}r}{2\mathrm{AR}r} = \frac{1}{2}\mathrm{V}$.

COR. 4. *Still retaining the same hypothesis, the greatest momentum generated in the ascending weight will be* $= \frac{1}{4}\mathrm{AV}$.

For $\frac{1}{2}\mathrm{V} \cdot \frac{r}{\mathrm{R}} = \frac{\mathrm{V}r}{2\mathrm{R}}$ is the uniform velocity of the ascending weight; and the weight moved w is, by cor. 2. $= \frac{\mathrm{AR}}{2r}$. Consequently $\frac{\mathrm{V}r}{2\mathrm{R}} \cdot \frac{\mathrm{AR}}{2r} = \frac{1}{4}\mathrm{A}y$, the momentum.

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473. In practice, the velocity of the wheel when the machine is in its greatest perfection will be between one-third and one-half of the velocity of the stream. For the water does not all escape the instant after it has made the impact, but is confined by the channel for some time: so that the succeeding water, which would otherwise pass by the wheel inefficaciously, drives the confined water against the float-boards, and therefore acts in the same manner as if it actually impinged against the wheel. Experiments shew that when the most work is done in a given time the velocity of the wheel is much nearer the half than the third of the velocity. See the following chapter, art. 483.

The discrepances between the results of the experiments and the conclusions from the preceding theory arise, as before suggested, from our ignorance of the different circumstances which affect the motion. It may not be amiss to point out how the investigation might be conducted supposing these things to be discovered. To this end let it be considered, that the moving force will generally consist of the impact, and the weight of such portion of the water as may tend to move the wheel by its gravity: this entire moving force, whether determined by theory or by experiment, may be denoted as before by A. After the moving force which impels the circumference of the wheel has been determined, the resistance opposed to this force must be found; for on the relation between the moving force and the resistance will depend the acceleration of the machine. The resistance will arise, 1. from inertia. This may be readily estimated: for let the distance of the centre of gyration of the wheel from that of motion be ϱ, the wheel's weight W, and the other quantities as in the general notation of this chapter, then will the inertia of the wheel, which resists the communication of motion to the circumference, be expressed by $\frac{W\varrho^2}{R^2}$, (art. 310. cor. 4.). And, in a nearly similar manner, the inertia of any parts of the system may be obtained, knowing the weight, figure, and position, by some of the rules in chap. IV. of book II.... 2. If the machine be one that raises water or other weights, the weight raised, allowing for its mechanical effect on the point whose acceleration is sought, must be subducted from the moving force before found; and this will be a constant quantity. 3. There are likewise other kinds of resistance homogeneal to weight, viz. those of friction, tenacity, &c. which vary in some ratio of the velocity of the machine: and, in order to proceed with the investigation, the exact quantity of weight to which the friction is equal, when the wheel moves with a given velocity, must be known, as well as the variation of the resistances in respect of the velocities; circumstances which must be determined by experiment. If then the force equivalent to the friction, &c. be subtracted from the moving force, the remainder will shew the moving power by which the circumference may be considered as actually impelled; and this, divided by the inertia of the mass moved, will give the force which accelerates the circumference. Thus, if, when the velocity of the wheel's circumference is V, the friction is equal to the weight Q applied at the circumference, and varies in the *n*th power of the velocity, we shall have the force which accelerates the circumference, as follows:

$$\frac{A}{w+W} \times \left(\frac{V-v}{V}\right)^2 \times \frac{R^2}{\varrho^2} - \frac{wRr}{w+W\varrho^2} - \frac{R^2 Q v^n}{V^n(w+W)\varrho^2}.$$

whence, if x be the space which has been described by the circumference when the velocity is v, and $g = 32\frac{1}{6}$ feet, the principles of acceleration will give us this equation:

$$\frac{v\dot{v}}{g} = \left[\frac{A}{w+W} \times \left(\frac{V-v}{V}\right)^2 \times \frac{R^2}{\varrho^2} - \frac{wRr'}{w+W\varrho^2} - \frac{R^2\varrho v^n}{V^n(w+W)\varrho^2}\right] \times \dot{x}$$

And from this if v be determined in functions of x, and constant quantities, the velocity communicated to the wheel may be ascertained. It is only by taking all these circumstances into the account that we can produce an exact coincidence between theory and matter of fact. Further observations may be seen in Atwood's Treatise on Motion, &c. § vi.

474. PROP. *In any proposed overshot-wheel to compute the effective weight of water in the buckets.*

An overshot unloaded is considered as perfectly in equilibrio upon its axis: but when it is loaded, the equilibrium is destroyed; because the weight of the water lies all upon one moiety of the circumference, and causes it to preponderate.

Let ACB (fig. 1. pl. XVIII.) represent an overshot-wheel, of which the buckets a, b, c, B, d, e, are loaded with water; and let there be supposed an equal quantity of fluid W in each bucket. Now the centre of gravity of the bucket a being directly over the axle C of the wheel will have no tendency to give it a rotatory motion, but will merely cause it to press more firmly upon its pivots, just as it would if a weight a', equal to that of the water in the bucket a, hung directly underneath the axle. But the second bucket, whose centre of gravity is b, acts in the same manner upon the wheel as would an equal weight b' hanging freely at the point f; consequently its effort will be proportional to the product $w \times cf = w \times \sin acb$: and the same will hold with respect to the weight in the bucket e. The water in the buckets c and d act in the same manner as would the equal weights c' and d', hanging freely at the point g; their joint effort, therefore, will be proportional to $2w \times cI = w \times 2\sin acc$. And the water in the bucket B, acting at the extremity of the radius, will have its effect proportional to $w \times \sin 90 = w$R, R being the radius of the wheel. Hence, 1. if there be 12 buckets about the wheel, as in the figure, the arcs ab, bc, cB, &c. will each contain 30°: and we shall have $(0 + 2\sin 30° + 2\sin 60° + 1)w = (0 + 1 + \sqrt{3} + 1)w = 3{\cdot}7320508w$, for the effective weight of the water on the wheel, while its real weight is $6w$: so that the real weight of the water is to its effective weight as 6 to 3·7320508, or as 1 to ·622008.

2. If the number of buckets were 24, and all supposed equally full; then, pursuing a similar course, we should find $(2\sin 15 + 2\sin 30° + 2\sin 45° + 2\sin 60° + 2\sin 75° + \sin 90°) \times w = 7{\cdot}585754\,w$, for the effective weight of water on the wheel, its real weight being $12w$: hence the latter will be to the former as 12 to 7·585754, or as 1 to ·632214.

3. If we suppose the number of buckets to be so increased,

that the weight may be considered as equally diffused over the semi-circumference aBN of the wheel; then will the sum of all the efforts to turn the wheel be equal to the sum of all the distances, as CI into all the corresponding weights at e; that is, by the nature of the centre of gravity, as the semicir. aBN × Ck, k being the centre of gravity of the semicircular arc. Now by art. 118, Ck = ·63662 CB: consequently, the actual weight will be to its effective weight as arc aBN to ·63662 × arc. aBN; or as 1 to ·63662. This being the ratio to which the others approximate.

475. **Prop.** *In an overshot-wheel the machine will have the greatest rotatory velocity when the diameter of the wheel is two-thirds of the height of the water above the lowest point of the wheel.*

If the wheel ABN (fig. 1. pl. XVIII.) revolve with a velocity, such as a body would acquire in falling through the altitude x above the upper part a of the wheel, the water will fall into the buckets without any impulse, (because then $\mathrm{V} - v = 0$), and produce its effect by its weight only. Let $h = x + a\mathrm{N}$, the height of the supply of water above the lowest point N of the wheel. Now, as we have already observed in the latter part of the preceding article, the sum of all the effective forces of the water, in all the buckets of the semicircle aBN, will be equal to the semicircle aBN × Ck, k being the centre of gravity of the semicircular arc; that is, it will be = the quantity of water $a\mathrm{BN} \cdot \frac{k\mathrm{C}}{\mathrm{CB}}$, acting at the point B. But $k\mathrm{C} = \frac{\mathrm{BC} \cdot a\mathrm{N}}{\mathrm{arc}\ a\mathrm{BN}} = \frac{\mathrm{BC}^2}{a\mathrm{CB}}$; consequently, the force of all the water to turn the wheel is = quantity of water $a\mathrm{BN} \cdot \frac{\mathrm{BC}}{a\mathrm{CB}}$, acting at B = quantity of water 2BC, or aN, acting at B. Hence, if $\mathrm{BC} = \mathrm{R}$, the whole force of water $\propto \mathrm{R} \times a\mathrm{N} = \mathrm{R} \times (h - x)$. But the velocity of the wheel is $\sqrt{(2gx)}$; and, consequently, the force of the water to produce the rotatory motion of the wheel is $= \mathrm{R}\ \sqrt{2gx} - (h - x)$: this being made a maximum, will require $h^2x - 2hx^2 + x^3$ a max. or its fluxion $h^2\dot{x} - 4hx\dot{x} + 3x^2\dot{x} = 0$, whence we find $x = \frac{1}{3}h$. Consequently the height of the supply of water above the upper part of the wheel should be one-third of the whole height; and the diameter of the wheel two-thirds of that height, that the machine may have its maximum velocity.

Cor. The velocity of an overshot-wheel, when the water produces its effect by its weight only, and the machine is in its state of greatest rotation, is to the velocity of an undershot-wheel as $\sqrt{3}$ to 1, on the supposition that all the water escapes from the undershot-wheel the moment after it makes its impact

For, h being the height of the water, the velocity of the overshot-wheel will be $= \sqrt{\frac{1}{3}gh}$, and the velocity of the undershot $= \sqrt{\frac{1}{9}gh}$; consequently, the former is to the latter as $\sqrt{\frac{1}{3}}$ to $\sqrt{\frac{1}{9}}$, or as $\sqrt{3}$ to 1.

476. PROP. *The efficacy of an overshot-wheel is to that of an undershot-wheel, the height of the water, aperture, and diameter, being given, as 13 to 5 nearly.*

If A be the altitude of the column of water, whose weight is equal to the force of impact on the undershot-wheel, when quiescent, since the velocity of the wheel is equal to $\frac{1}{3}$ of that of the stream, (art. 470. cor. 5.), we shall have this analogy, $3^2 : (3-1)^2 :: A : \frac{4}{9}A$, the weight which is equivalent to the force of the stream when the machine is in its greatest perfection. Whence, if v be the velocity acquired in falling down the altitude A, the momentum of the wheel will be $= \frac{4}{9}A \times \frac{1}{3}v = \frac{4}{27}Av$ (art. 470. cor. 3.), when in its best state. In the overshot-wheel the weight of the water acting at the circumference is equivalent to $\frac{2}{3}A$, and the velocity is $\sqrt{(\frac{1}{3}gA)}$ when the machine has its most rapid rotation. But $\sqrt{\frac{1}{3}gA} = v\sqrt{\frac{1}{3}}$: consequently, the momentum of the overshot-wheel $= \frac{2}{3}A \times v\sqrt{\frac{1}{3}} = \frac{2}{3}Av\sqrt{\frac{1}{3}}$; and is to that of the undershot-wheel as $\frac{2}{3}\sqrt{\frac{1}{3}}$ to $\frac{4}{27}$, or as $\frac{2}{3}\sqrt{3}$ to 1, or as 2·5987 to 1, or nearly as 13 to 5.

COR. In theory there is no limit to the weight which a given stream of water can raise by means of a water-wheel.

For either the radius of the wheel may be increased, or that of the axle diminished, without limit. But this is far from being the case in practice.

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476. A. It must not be concluded, however, that when an overshot-wheel raises a given weight a given altitude in the least time, the maximum of *effect* is produced. Bossut has shown (Hydrodynamique, tom. i. p. 542—544.), that if Q be the quantity of water issuing in a second, and h the height due to the velocity of the circumference of the wheel, the effect of an overshot-wheel varies as $Q(2r-h)$. This will manifestly be a maximum when h vanishes. Still, the conclusion that overshot-wheels are the more powerful the slower they move, requires limitations, which must be prescribed by experiment. (Vide art. 481—3.)

Generally, to render any hydraulic machine the most perfect, or capable of producing the greatest possible effect, it is requisite, 1st. So to construct it that the fluid shall lose absolutely all its motion by its action on the machine, or at least that it shall only retain precisely the quantity necessary to ensure its escape after its action. 2dly. That it shall lose all its motion by imperceptible degrees, and without there being any violent percussion, either on the part of the fluid, or on that of the solid materials of the machine. These being ensured, the form of the machine would be but of small consequence; yet the ensuring of both conditions is, in many cases, absolutely impossible; and hence arises the necessity of judicious selection in the kind of machine to be adopted.

The intelligent reader may advantageously consult *Fabre, Essai sur les Construction des Machines Hydrauliques,* and the treatises of *Albert Euler* and *Bossut.*

CHAPTER IV.

Account of Mr. Smeaton's Experiments on Water-wheels.

477. The late ingenious and experienced engineer, Mr. *John Smeaton*, paid very great attention to the construction of mills and water-wheels; and as his experiments and observations relative to these subjects were very extensive and judicious, we trust a concise account of his enquiries will be both interesting and beneficial.

In the 51st vol. of the Phil. Trans. Mr. Smeaton has given a valuable paper on water-wheels, of which the following is an abridgment.

Having described the machines and models used for making his experiments, he observes, that with regard to *power* it is most properly measured by the raising of a weight; or, in other words, if the weight raised be multiplied by the height to which it can be raised in a given time, the product is the measure of the power raising it; and, of consequence, all those powers are equal whose products made by such multiplication are equal: for, if a power can raise twice the weight to the same height, or the same weight to twice the height, in the same time that another can, the former power may be considered as double the latter; but if a power can only raise half the weight to double the height, or double the weight to half the height, in the same time that another can, the two powers are equal. This, however, as Mr. Smeaton remarks, " must be understood only of a slow and equable motion without acceleration or retardation." Indeed this must be looked upon merely as a popular measure, or abridged representative, of easy comprehension and remembrance, and tolerably well adapted to the uses of those engaged in the construction of machines, when restricted to those cases where the power expended and work performed are of the same kind as when a heavy body descends, and by its preponderance raises another body; but it is vague, inadequate, and unfit for general adoption; for *the quantity of motion extinguished or produced, and not the product of the weight and height, is the true, unequivocal, measure of mechanical power* *really expended, or the mechanical effect actually produced; and these two are always equal and opposite.* Mr. Smeaton's measure, however, is mostly applicable to the cases in which he adopts it, and as such it was necessary to explain it, that the student might the better understand the foundation of Mr. Smeaton's rules and inferences.

478. To compute the effects of water-wheels exactly, it is necessary to know in the first place what is the real velocity of the water which impinges on the wheel. 2. The quantity of water expended in a given time: and, 3. How much of the power is lost by the friction of the machinery.

1. With regard to the velocity of the water, Mr. Smeaton determined by experiments with the machinery described in the volume referred to, that with a head of water 15 inches in height, the velocity of the wheel is 8·96 feet in a minute. The area of the head being 105·8 inches, this multiplied by the weight of a cubic inch of water equal to ·579 of an ounce avoirdupois, gives 61·26 ounces for the weight of as much water as is contained in the head upon one inch in depth; and by further calculations derived from the machinery made use of, he computes that 264·7 pounds of water descend in a minute through the space of 15 inches. The power of the water, therefore, to produce mechanical effects in this case will be 264·7 × 15, or 3970. From the result of the experiment, however, it appeared that a vast quantity of the power was lost; the effect being only to raise 9·375 pounds to the height of 135 inches: so that the power was to the effect as 3970 to 9·375 × 135 = 1266, or as 10 to 3·18.

This, according to our author, must be considered as the greatest single effect of water upon an undershot-wheel, where the water descends from an height of 15 inches; but as the force of the current is not by any means exhausted, we must consider the true proportion betwixt the power and effect to be that betwixt the quantity of water already mentioned and the sum of all the effects producible from it. This remainder of power, it is plain, must be equal to that of the velocity of the wheel itself multiplied into the weight of the water. In the present experiment, the circumference of the wheel moved with the velocity of 3·123 feet in a second, which answers to a head of 1·82 inches; and this height being multiplied by 264·7, the quantity of water expended in a minute, gives 481 for the power of the water after it has passed the wheel; and hence the true proportion betwixt the power and the effect will be as 3849 to 1266; or as 11 to 4.

As the wheel revolved 86 times in a minute, the velocity of the water must be equal to 86 circumferences of the wheel;

which, according to the dimensions of the apparatus used by Mr. Smeaton, was as 86 to 30, or as 20 to 7.—The greatest load with which the wheel would move was 9lb. 6oz.; and by 12lb. it was entirely stopped. Whence our author concludes, that the impulse of the water is more than double of what it ought to be according to theory: but this he accounts for by observing, that in his experiment the wheel was placed not in an open river, where the natural current after it has communicated its impulse to the float has room on all sides to escape, as the theory supposes, but in a conduit, to which the float being adapted, the water cannot otherwise escape than by moving along with the wheel. It is observable, that a wheel working in this manner, as soon as the water meets the float, receiving a sudden check, it rises up against the float like a wave against a fixed object, insomuch that when the sheet of water is not a quarter of an inch thick before the float, yet this sheet will act upon the whole surface of a float whose height is three inches: and consequently, were the float no higher than the thickness of the sheet of water, as the theory also supposes, a great part of the force would have been lost by the water dashing over the float. (See art. 467.)

479. Mr. Smeaton next proceeds to give tables of the velocities of wheels with different heights of water; and from the whole deduces the following conclusions: 1. The virtual or effective head being the same, the effect will be nearly as the quantity of water expended. 2. The expence of water being the same, the effect will be nearly as the height of the virtual or effective head. 3. The quantity of water expended being the same, the effect is nearly as the square of the velocity. 4. The aperture being the same, the effect will be nearly as the cube of the velocity of the water. Hence, if water passes out of an aperture in the same section, but with different velocities, the expence will be proportional to the velocity; and therefore, if the expence be not proportional to the velocity, the section of the water is not the same. 5. The *virtual* head, or that from which we are to calculate the power, bears no proportion to the head water; but when the aperture is larger, or the velocity of the water less, they approach nearer to a coincidence: and, consequently, in the large openings of mills and sluices, where great quantities of waters are discharged from moderate heads, the head of water, and virtual head determined from the velocity, will nearly agree; which is also confirmed by experience. 6. The most general proportion betwixt the power and effect is that of 10 to 3; the extremes 10 to 3·2, and 10 to 2·8. But it is observable, that where the power is greatest, the second term of the ratio is greatest also: hence

we may allow the proportion subsisting in great works to be as three to one. 7. The proportion of velocity between the water and wheel is in general about 5 to 2. 8. There is no certain ratio between the load that the wheel will carry at its *maximum*, and what will totally stop it; though the proportions are contained within the limits of 20 to 19, and 20 to 15; but as the effect approaches nearest to the ratio of 20 to 15, or of 4 to 3, when the power is greatest either by increase of velocity or quantity of water, this seems to be the most applicable to large works; but as the load that a wheel ought to have in order to work to the best advantage can be assigned by knowing the effect that it ought to produce, and the velocity it ought to have in producing it, the exact knowledge of the greatest load it will bear is of the least consequence in practice.

Mr. Smeaton, after having finished his experiments on the undershot mills, reduced the number of floats, which were originally 24, to 12; which caused a diminution in the effect, by reason that a greater quantity of water escaped between the floats and the floor than before: but on adapting to it a circular sweep of such a length that one float entered into the curve before the other left it, the effect came so near that of the former, as not to give any hopes of advancing it by increasing the number of floats beyond 24 in this particular wheel.

480. Our author next proceeds to examine the power of water when acting by its own gravity in turning an overshot-wheel: " In reasoning without experiment," says he, " one might be led to imagine, that however different the mode of application is, yet that whenever the same quantity of water descends through the same perpendicular space, the natural effective power would be equal, supposing the machinery free from friction, equally calculated to receive the full effect of the power, and to make the most of it: for, if we suppose the height of a column of water to be 30 inches, and resting upon a base or aperture of one inch square, every cubic inch of water that departs therefrom will acquire the same velocity or momentum from the uniform pressure of 30 cubic inches above it, that one cubic inch let fall from the top will acquire in falling down to the level of the aperture: one would therefore suppose that a cubic inch of water let fall through a space of 30 inches, and there impinging upon another body, would be capable of producing an equal effect by collision, as if the same cubic inch had descended through the same space with a slower motion, and produced its effects gradually. But, however conclusive this reasoning may seem, it will appear in the course of the following deductions, that the effect of the gravity of descending bodies is very different from the effect of the

stroke of such as are non-elastic, though generated by an equal mechanical power."

Having made such alterations in his machinery as were necessary for overshot-wheels, Mr. S. next gives a table of experiments with the apparatus so altered. In these the head was 6 inches, and the height of the wheel 24 inches; so that the whole descent was 30 inches; the quantity of water expended in a minute was 96⅔ pounds; which, multiplied by 30 inches, gives the power=2900: and, after making the proper calculations, the effect was computed at 1914; whence the ratio of the power to it comes to be nearly as 3 to 2. If, however, we compute the power from the height of the wheel only, the power will be to the effect nearly as 5 to 4.

481. From another set of experiments the following conclusions were deduced:

1. The effective power of the water must be reckoned upon the whole descent; because it must be raised to that height in order to be able to produce the same effect a second time. The ratios between the powers so estimated, and the effects at a *maximum*, differ nearly from 4 to 3, and from 4 to 2. Where the heads of water and quantities of it expended are the least, the proportion is nearly from 4 to 3; but where the heads and quantities are greatest, it comes nearer to that of 4 to 2; so that by a medium of the whole, the ratio is nearly as 3 to 2. Hence it appears, that the effect of overshot-wheels is nearly double to that of undershot ones; the consequence of which is, that non-elastic bodies when acting by their impulse or collision communicate only a part of their original impulse, the remainder being spent in changing their figure in consequence of the stroke. The ultimate conclusion is, that the effects, as well as the powers, are as the quantities of water and perpendicular heights multiplied together respectively.

2. By increasing the head, it does not appear that the effects are at all augmented in proportion; for, by raising it from 3 to 11 inches, the effect was augmented by less than one-seventh of the increase of perpendicular height. Hence it follows, that the higher the wheel is in proportion to the whole descent, the greater will be the effect; because it depends less upon the impulse of the head, and more upon the gravity of the water in the buckets: and if we consider how obliquely the water issuing from the head must strike the buckets, we shall not be at a loss to account for the little advantage that arises from the impulse thereof, and shall immediately see of how little consequence this is to the effect of an overshot-wheel. This, however, as well as other things, must be subject to limitation; for it is necessary that the velocity of the water should be some-

what greater than the wheel, otherwise the latter will not only be retarded by the striking of the buckets against the water, but some of the power will be lost by the dashing of the water over the buckets.

3. To determine the velocity which the circumference of the wheel ought to have in order to produce the greatest effect, Mr. Smeaton observes, that the more slowly any body descends by the force of gravity when acting upon any piece of machinery, the more of that force will be spent upon it, and consequently the effect will be the greater. If a stream of water falls into the bucket of an overshot-wheel, it will be there retained till the wheel discharges it by moving round; and, of consequence, the slower the wheel moves, the more water it will receive; so that what is lost in velocity is gained by the greater pressure of water upon the buckets. From the experiments, however, it appears, that when the wheel made about 20 turns in a minute the effect was greatest; when it made only 18¾ the motion was irregular; and when loaded so as not to admit its turning 18 times, the wheel was overpowered with the load. When it made 30 turns, the power was diminished by about $\frac{1}{20}$th, and when the number of turns were increased to 40, it was diminished by one-fourth. Hence we see, that in practice the velocity of the wheel should not be diminished further than what will procure some solid advantage in point of power; because, *cæteris paribus*, the bucket must be larger as the motion is slower; and the wheel being more loaded with water, the stress will be proportionably increased upon every part of the work. The best velocity for practice, therefore, will be that when the wheel made 30 turns in a minute, which is little more than three feet in a second. This velocity is applicable to the highest overshot-wheels as well as the lowest. Experience however determines, that high wheels may deviate further from this rule before they will lose their power, by a given aliquot part of the whole, than low ones can be permitted to do; for a wheel of 24 feet high may move at the rate of six feet *per* second; while our author has seen one of 33 feet high move very steadily and well with a velocity of little more than two feet. The reason of this superior velocity in the 24 feet wheel, may probably be owing to the small proportion that the head, requisite to give the proper velocity to the wheel, bears to the whole height.

4. The maximum load for an overshot-wheel is that which reduces the circumference of the wheel to its proper velocity; which is known by dividing the effect it ought to produce in a given time by the space intended to be described by the circumference of the wheel in the same time: the quotient will be

the resistance overcome at the circumference of the wheel, and is equal to the load required, including the friction and resistance of the machinery.

5. The greatest velocity that an overshot-wheel is capable of, depends jointly upon the diameter or height of the wheel and the velocity of falling bodies; for it is plain that the velocity of the circumference can never be greater than to describe a semi-circumference, while a body let fall from the top describes the diameter, nor even quite so great; as the difference in point of time must always be in favour of that which falls through the diameter. Thus, supposing the diameter of the wheel to be 16 feet and an inch in diameter, an heavy body would fall through this space in one second; but such a wheel could never arrive at this velocity, or make one turn in two seconds, nor could an overshot-wheel ever come near it: because, after it has required a certain velocity, great part of the water is prevented from entering the buckets, and part is thrown out again by the centrifugal force: and as these circumstances have a considerable dependence upon the form of the buckets, it is impossible to lay down any general rule for the velocity of this kind of wheels.

6. Though in theory we may suppose a wheel to be made capable of overcoming any resistance whatever, yet as in practice it is necessary to make the wheel and buckets of some certain and determinate size, we always find that the wheel will be stopped by such a weight as is equal to the effort of the water in all the buckets of a semi-circumference put together. This may be determined from the structure of the buckets themselves; but, in practice, an overshot-wheel becomes unserviceable long before this time; for when it meets with such an obstacle as diminishes its velocity to a certain degree, its motion becomes irregular; but this never happens till the velocity of the circumference is less than two feet *per* second, when the resistance is equable.

7. From the above observations we may easily deduce the force of water upon breast-wheels, &c. But, in general, all kinds of wheels where the water cannot descend through a given space unless the wheel moves with it, are to be considered as overshot-wheels; and those which receive the impulse or shock of the water, whether in an horizontal, oblique, or perpendicular direction, are to be considered as undershots. Hence when the water strikes at a certain point below the surface of the head, and after that descends in the arch of a circle, pressing by its gravity upon the wheel; the effect of such a wheel will be equal to that of an undershot whose head is equal to the difference of level between the surface of the water in

the reservoir and the point where it strikes the wheel, added to that of an overshot, whose height is equal to the difference of level between the point where it strikes the wheel and the level of the tail-water.

482. In the Philosophical Transactions, vol. 51, for the year 1759, there is another paper with experiments on mills turned both by water and wind, by Mr. Smeaton. From those experiments it appears, page 129, that the effects obtained by the overshot-wheel are frequently 4 or 5 times as great as those with the undershot-wheel, in the same time, with the same expence of water, descending from the same height above the bottoms of the wheel; or that the former performs the same effect as the latter, in the same time, with an expence of only one-fourth or one-fifth of the water, from the same head or height. And this advantage seems to arise from the water lodging in the buckets, and so carrying the wheel about by their weight. But, in page 130, Mr. Smeaton reckons the effect of overshot only double to that of the undershot-wheel. And hence he infers, in general, " that the higher the wheel is in proportion to the whole descent, the greater will be the effect; because it depends less upon the impulse of the head, and more upon the gravity of the water in the buckets. However, as every thing has its limits, so has this; for thus much is desirable, that the water should have somewhat greater velocity than the circumference of the wheel in coming thereon, otherwise the wheel will not only be retarded by the buckets striking the water, but thereby dashing a part of it over so much of the power is lost." He is further of opinion, that the best velocity for an overshot-wheel is when its circumference moves at the rate of about 3 feet in a second of time.

483. Lastly, in the Philosophical Trans. for 1776, p. 457, the same author says, " The velocity of the wheel, which according to M. Parent's determination, adopted by Desaguliers and Maclaurin, ought to be no more than $\frac{1}{3}$ of that of the water, varies at the maximum *between one-third and one half*; but in all the cases in which the most work is performed in proportion to the water expended, and which approach the nearest to the circumstances of great works when properly executed, the maximum lying much nearer one half than one-third, one half seeming to be the true maximum, if nothing were lost by the resistance of the air, the scattering of the water carried up by the wheel, &c.

M. Bossut has detailed, in the second volume of his valuable *Hydrodynamique*, a variety of experiments in relation to this subject. Most of his results correspond with the analogous determinations of Smeaton. In undershot-wheels he found, as has already been observed in art. 467, that 48 was the most efficacious number of float-boards. He also found that the best portion of the outer arc of the wheel to be immersed, was about 24° 54′ in small wheels. When the wheels were large, as from 20 to 24 feet in diameter, the arc immersed may extend from 25 to 30 degrees; but must not exceed the latter quantity.

It appears, likewise, from Bossut's experiments, that undershot-wheels are most efficacious when the velocity of the centre of impression of the float-board is about *two-fifths* of that of the stream.

With regard to the *position* of the float-boards, he found that when their number was 48, the construction was most advantageous when they were directed towards the centre. At an inclination of 8 degrees with the radius the effect was less; at an inclination of 12 degrees still less; but that at an inclination of 16 degrees, the effect was nearly the same as when the float-boards were directed to the centre. Of these results he presents the physical explication.

With respect to overshot-wheels (in reference to which, however, the experiments of this philosopher were but few), he observed that the velocity required for the greatest effect, was to the velocity with which the wheel would move if the mill performed no work, as $8\frac{3}{10}$ to $40\frac{1}{4}$, or nearly as 1 to 5.

Many other experiments tending to improve the theory of water-wheels are described by Mr. Banks in part IV. of his Treatise on Mills. See also a Memoir on the most advantageous Construction of Water-wheels by Mr. *Mallet* of Geneva, in Phil. Trans. for 1767; and two papers by *M. Lambert* in the Berlin Memoirs for 1775.

An extensive chronological catalogue of writings on the subject of Mill-work in general, may be seen under the word MILL, in our second volume.

PNEUMATICS.

BOOK V.

Preliminary Remarks on the Mechanical Properties of Atmospheric Air.

484. The term *Pneumatics,* as well as the word *Pneumatology,* is derived from a Greek word denoting *air,* or, more properly, *breath:* but in their modern acceptation the terms differ widely, the latter denoting the science of the intellectual phenomena depending upon the operations or affections of the mind of man; while the former relates to that part of natural philosophy which treats of the mechanical properties of air, and the different elastic fluids. Thus Pneumatics includes, as an important branch, the doctrine of *Aerostatics,* common air being the most extensive and universal of the fluids which possess elasticity.

Previous to our entering upon the theoretic part of this science, it will be proper to relate some of the most obvious and natural experiments tending to prove the existence of the different mechanical affections which are usually ascribed to the air as a body: to this object we shall therefore appropriate a few of the following articles.

485. I. *Air is a ponderous fluid which surrounds the earth.*

For, that it is a fluid is obvious, because its parts are easily moved, and yield to the smallest inequality of pressure: and that it is ponderous will appear from the following considerations:

1. It always accompanies this globe in its orbit round the sun, surrounding it to a certain distance, under the name of the *Atmosphere,* which indicates the being connected with the earth by its general force of gravity. It is chiefly in consequence of this that it is continually moving round the earth from east to west, forming what is called the trade-wind.

2. It is in like manner owing to the gravity of the air that it supports the clouds and vapours which we see constantly floating in it. We have seen bodies of no inconsiderable weight float, and even rise, in the air. Soap-bubbles and balloons filled

with inflammable gas rise and float in the same manner as a cork rises in water. This phenomenon proves the weight of the air in the same manner that the swimming of a piece of wood indicates the weight of the water which supports it.

3. But we are not left to these refined observations for the proof of the air's gravity. We may observe familiar phenomena, which would be immediate consequences of the supposition that air is a heavy fluid, and, like other heavy fluids, presses on the outsides of all bodies immersed in or surrounded by it. Thus, for instance, if we shut the nozzle and valve hole of a pair of bellows, after having squeezed the air out of them, we shall find that a very great force, even some hundred pounds, is necessary for separating the boards. They are kept together by the pressure of the heavy air which surrounds them, in the same manner as if they were immersed in water. In like manner, if we stop the end of a syringe after its piston has been pressed down to the bottom, and then attempt to draw up the piston, we shall find a considerable force necessary, viz. about 15 or 16 pounds for every square inch of the section of the syringe. Exerting this force, we can draw up the piston to the top, and we can hold it there: but the moment we cease acting, the piston rushes down and strikes the bottom. It is called a suction, because we feel something as it were drawing in the piston; but it is really the weight of the incumbent air pressing it in. And this obtains in every position of the syringe; because the air is a fluid, and presses in every direction. Nay, it presses on the syringe as well as on the piston; and if the piston be hung by its ring on a nail, the syringe requires force to draw it down (just as much as to draw the piston up); and if it be let go, it will spring up, unless loaded with at least 15 pounds for every square inch of its transverse section.

4. Let the air be exhausted from a glass vessel, and by means of a cock let the vessel be kept exhausted; if the vessel be weighed while it is exhausted, and then again when the air is re-admitted, there will be a manifest difference, exhibiting the weight of as much air as the vessel contained.

5. If a glass tube more than 31 inches in length, one end of which is closed up, be filled with mercury, and be held vertically, the other extremity being immersed in a vessel of the same fluid, then the mercury in the tube will descend from the upper extremity, and will remain suspended at some altitude between 28 and 31 inches from the surface of the external mercury: the suspension of the mercury is occasioned by the pressure of the external air upon the surface of the mercury in the vessel; when this pressure is removed by placing the tube and vessel under a receiver and exhausting the air, the mercury will sink

in the tube, and on re-admitting the air, will rise. This is called the *Torricellian* experiment.

The instrument here used is called a *Barometer;* because the weight of a column of mercury whose base is the orifice of the tube, and altitude equal to that of the mercury in the tube above the surface in the reservoir, is equal to the weight of a column of air extending to the top of the atmosphere, and whose base is equal to the same orifice. And since the weight of this column of quicksilver, *cæteris paribus,* is as its altitude, it follows that the weight of the air is proportional to the altitude of the mercury in the barometer.

The altitude at which the mercury is sustained in the barometer above the surface of the mercury in the reservoir is called the *standard altitude*; and will be the same in any number of tubes, whatever be their bore or their position, provided the tube is not so very slender as to expose the mercury to a sensible alteration from the capillary attraction (art. 432.). The pressure of the atmosphere is equal to about 15 lbs. avoirdupois upon every square inch, at the medium height of the mercury in the barometer: for a cubic foot of mercury weighs about 13568 ounces, and $\frac{13568}{1728} \times 29\frac{3}{4} = 233{\cdot}6$ oz. $= 14{\cdot}6$ lbs.

6. If a barometer tube, instead of being hermetically sealed at the top, be closely covered with a piece of bladder, the mercury will rise to the same height as in a common barometer; but on piercing the bladder with a needle so as to admit the air, the mercury will fall: for in this case the weight of the air presses upon the mercury in the tube, and the weight of these two together must obviously preponderate over the contrary pressure, and destroy the equilibrium.

486. II. *The pressure of the atmosphere varies at different altitudes, above the surface of the earth.*

Let a glass tube, open at both ends, be put through a cork into a large phial containing a small quantity of coloured water; let the lower end of the tube be in the water; and let the cork and tube be closely cemented to the neck of the bottle: then blow through the tube till the quantity of air within the phial is so increased that the water will rise above the neck of the phial. Let this phial be placed in a vessel of sand, to keep the air within of the same temperature: then will the water stand at different heights in the tube, according to the elevation of the place where it is set; whence it appears that the pressure of the atmosphere varies at different altitudes. And the same thing will appear more clearly in a subsequent article. In ascending the mountain of Snowden, which is 3720 feet high, the barometer sunk 3·8 inches.

Hence the proportion of the specific gravity of air near the earth's surface to that of water may be ascertained. Thus, if the difference in height of the two places where the above experiment is made be 52 feet, and that difference cause a variation of $\frac{3}{4}$ of an inch in the height of the water; it follows, that a column of water of $\frac{3}{4}$ of an inch, or $\frac{1}{16}$ of a foot, is equiponderant to a column of air of 52 feet, having the same base: consequently the specific gravity of water is to that of air as 52 to $\frac{1}{16}$, or as 832 to 1.

487. III. *Air is elastic, or capable of compression and expansion.*

This is proved by various experiment[illegible]. By the great expansion of a small quantity of air in a bladder apparently nearly empty, when the air is removed from the external parts in the receiver. 2. By the extrusion of a fluid from a glass bubble, by the expansion of a bubble of air contained in it. 3. By the expulsion of the white and yolk of an egg through a small hole in the little end, by the expansion of the air contained in the greater end. 4. By putting an almost emptied bladder into a small box, and laying a proper weight on the lid, which, on exhausting the air, will be raised up by the expansion of the air in the bladder. 5. Also a bladder filled with air, and just made to sink with a weight, will upon exhaustion soon rise by the expansion of the contained air. 6. Glass bubbles and images filled with water so as to make them just sink in that fluid, will, on exhausting the air from the surface, rise to the top of the vessel. 7. Beer, cyder, water, and porous bodies, emit great quantities of air under the exhausted receiver. 8. A shrivelled apple, when put under an exhausted receiver, will have its coat distended by the internal air so as to look smooth. 9. If the open end of a tube, whose other end is closed, be immersed perpendicularly in water, the space occupied by the air will be diminished, as the depth of the tube or the upward pressure of the water is increased: or, if the shorter leg of a bent tube be closed, and mercury poured into the longer, the air will be compressed in the shorter leg into a space continually decreasing as the quantity of pressing mercury is increased; and if some of the mercury be taken from the longer leg, the air in the shorter will expand and occupy a proportionably larger space. 10. The mercury may be raised by the expansion of a small quantity of confined air to the same height in an exhausted tube above the air-pump, as that to which it is raised in the mercurial gauge by the pressure of the atmosphere below it.

The limits of the condensation and rarefaction of air by human powers are not ascertained. Dr. Hales found, that when dry wood was put into a strong vessel, which it almost filled, and the remainder was filled with water, the swelling of the wood, occasioned by its inhibition of water, condensed the air of his gauge into the thousandth of its original bulk. He found that pease treated in the same way generated elastic air, which pressing on the air in the gauge condensed it into the fifteen hundredth part of its bulk. This is the greatest condensation that has been ascertained with precision, although in other experiments it has certainly been carried much further; but the precise degree could not be ascertained.

The main use to be made of this observation at present is, that since we have been able to exhibit air in a density a thousand times greater than the ordinary density of the air we breathe, it cannot, as some imagine, be only a different form of water; for in this state it is as dense or denser than water, and yet retains its great expansibility.

Another important observation is, that in every state of density in which we find it, it retains its perfect fluidity, transmitting all pressures which are applied to it with undiminished force, as appears by the equality constantly observed between the opposing columns of water or other fluid by which it is compressed, and by the facility with which all motions are performed in it in the most compressed states in which we can make observations of this kind. This fact is totally incompatible with the opinion of those who ascribe the elasticity of air to the springy ramified structure of its particles, touching each other like so many pieces of sponge or foot-balls. A collection of such particles might indeed be pervaded by solid bodies with considerable ease, if they were merely touching each other, and not subjected to any external pressure. But the moment such pressure is exerted, and the assemblage squeezed into a smaller space, each presses on its adjoining particles: they are individually compressed, flattened in their touching surfaces, and *before the density is doubled* they are squeezed into the form of perfect cubes, and compose a mass, which may indeed propagate pressure from one place to another in an imperfect manner, and with great diminution of its intensity, but will no more be fluid than a mass of soft clay.

488. IV. *The elastic force of the air is equal to the force of compression.*

For if the air be exhausted from an open tube whose lower part is immersed in a vessel containing mercury, the air within the vessel being prevented from escaping, then will this air by its elasticity force the mercury up the tube very nearly to the same height as it would be raised by the pressure of the atmosphere.

This proposition is sometimes proved independent of experiments in the following manner: if the force with which the air endeavours to expand itself when it is compressed were *less* than the compressing force, it would yield still further to that force; if it were greater, it could not have yielded so far. Consequently, when any force has so compressed the air that it remains at rest, the force of the air arising from its elasticity can neither be greater nor less than the compressing force: that is, it must be equal to it.

But it must be confessed that, in this case, the experimental proof is the most satisfactory.

489. V. *The elasticity of air not very different from its natural state, is inversely as the space occupied by it.*

1. Let a cylindrical tube BC (fig. 7. pl. XI.), open at one end B, be filled with mercury to an altitude equal to BD before inversion, and after the immersion of B in a basin of mercury, the air, which before occupied a space equal to CD, is dilated through a larger space as CA, and, if BN be the standard altitude (art. 485. v.), depresses the mercury from N to A. Now the elastic force of air in its natural state, or occupying a space equal to GD (E), is to the elastic force of air occupying the space BD (*e*) as the columns of mercury which they are capable of supporting (art. 488.); that is, as BN to AN: but, it is collected from experiments, that BN : AN :: AC : GD; consequently AC : GD :: E : *e*.

And the same thing may be otherwise shewn by experiments with bent tubes.

Hence, since the density of the air is inversely as the magnitude or space occupied (art. 10.), it is, therefore, as the elasticity, or as the compressing force (art. 488.): so that, putting D for the density, C the compressing force, E the force of elasticity, and S the space, we have $C \propto D \propto E \propto \frac{1}{S}$. These relations, however, are confined within very narrow limits, for it has been asserted that when the air is compressed into a space only four times less than the space it occupies in its natural state, it does not then vary inversely as the force of compression, the resistance increasing much more rapidly. *Comment. Bonon.* vol. I. p. 209, &c.

The experiments of Boyle, Mariotte, and Amontons, were not extended to very great compressions: so that they found generally that the elasticity of the air was proportioned to its density; and the law was long acquiesced in, being called the *Boylean law.* But later philosophers have carried the compression much further. Thus Sulzer compressed air into $\frac{1}{8}$ of its former dimensions: the results of his experiments are exhibited in the following table, where the column D shew the densities, and those marked E the corresponding elasticities.

1st Set.		2d Set.		3d Set.	
D	E	D	E	D	E
1·000	1·000	1·000	1·000	1·000	1·000
1·100	1·093	1·236	1·224	1·091	1·076
1·222	1·211	1·294	1·288	1·200	1·183
1·375	1·284	1·375	1·332	1·333	1·303
1·571	1·599	1·466	1·417	1·500	1·472
1·692	1·6[illegible]	1·571	1·515	1·714	1·659
1·833	1·796	1·692	1·647		
2·000	1·958	2·000	1·964	2·000	1·900
2·288	2·130				
2·444	2·375	2·444	2·392	2·400	2·241
3·143	2·936	3·143	3·078	3·000	2·793
3·666	3·391	3·666	3·575		
4·000	3·706			4·000	3·631
4·444	4·035	4·444	4·320		
4·888	4·438				
5·500	4·922	5·500	5·096		
5·882	5·522			6·000	5·297
		7·333	6·694		
				8·000	6·835

Other experiments for the same purpose were instituted by professor Robison, the results of which may be seen below.

Dry Air.		Moist Air.		Camp. Air.	
D	E	D	E	D	E
1·000	1·000	1·000	1·000	1·000	1·000
2·000	1·957	2·000	1·920	2·000	1·909
3·000	2·848	3·000	2·839	3·000	2·845
4·000	3·737	4·000	3·726	4·000	3·718
5·500	4 930	5·500	5·000	5·500	5·104
6·300	5·342	6·000	5·452	6·000	5·463
7·620	6·490	7·620	6·775	7·620	6·812

Here it appears again in the clearest manner that the elasticities do not increase so fast as the densities, and the differences are even greater than in Mr. Sulzer's experiments.

The second table contains the results of experiments made on very damp air in a warm summer's morning. In these it appears that the elasticities are almost precisely proportional to the densities + a small constant quantity, nearly 0·11, deviating from this rule chiefly between the densities 1 and 1·5, within which limits we have very nearly $D = E^{1 \cdot 0017}$. As this air is nearer to the constitution of atmospheric air than the former, this rule may be safely followed in cases where atmospheric air is concerned, as in measuring the depths of pits by the barometer.

The third table shews the compression and elasticity of air strongly impregnated with the vapours of camphire. Here the Boylean law appears pretty exact, or rat[illegible] the elasticity seems to increase a little faster than the density.

490. VI. *Heat increases the elasticity of the air, and cold diminishes it. Or, heat expands and cold condenses the air.*

This property is likewise proved by experience.

1. Tie a bladder very close with some air in it, and lay it before the fire: then as it warms it will more and more distend the bladder, and at last burst it, if the heat be continued, and increased high enough. But if the bladder be removed from the fire, as it cools it will contract again, as before. Indeed it was upon this principle that the first air-balloons were made by Montgolfier: for, by heating the air within them, by a fire underneath, the hot air distends them to a size which occupies a space in the atmosphere, whose weight of common air exceeds that of the balloon.

2. Also, if a cup or glass, with a little air in it, be inverted into a vessel of water; and the whole be heated over the fire, or otherwise; the air in the top will expand till it fill the glass, and expel the water; and part of the air itself will follow, by continuing or increasing the heat.

Many other experiments to the same effect might be adduced.

The expansion of air, though exposed to the same degree of heat, is not the same in experiments made at different times; owing to the difference of density, coldness, humidity, &c. The expansive force of hot steam may exceed the force of gunpowder more than 30 times, and indeed is irresistible when the force is intense. Hence it follows, that when air is much impregnated with water it will possess an expansive power by heat, much greater than that of pure air. Whether the degree of expansion in pure air be proportional to that of the heat by which it is produced, is not known; but it is manifest that the variation of space occupied by a portion of air exposed to different degrees of heat may be sufficient to convey a tolerable idea of the actual quantity of heat. Upon this principle, therefore, have been constructed air thermometers, to exhibit small variations of heat.

Taking an average it has been found, by experiment, that for each degree of heat measured by Fahrenheit's thermometer, mercury, water, and air, expand by the following parts of their own bulk, viz.

Mercury the	9600th	part of its own bulk.
Water . . .	6666th	
Air	435th	

In mercury the corresponding expansions for 1° gradually diminish, being expressed by ·0001177 at 2° of the thermometer, and by ·0000783 at 212°: but at 12° the expansion corresponding to a de[illegible] of variation in heat is ·0001160, and at 102° it is ·0001003, so that between these limits the variation in the measure of expansion is very trifling. Taking into the estimate the changes in the expansion, &c. the specific gravities of these fluids at different temperatures have been stated as below.

Spec. grav. of	air - -	1	when the barom. is at 29·27, and the thermom. at 53°.
	water -	836	
	mercury	11365	
Or thus,	air - - -	1	when the barom. is 29·27, and the thermom. at 55°.
	water - -	832	
	mercury -	11315	
Or thus,	air - - -	1	when the barom. is 29·5, and the thermom. is 55°, which are their mean heights in this country.
	water - -	826	
	mercury -	11227	
Or thus,	air - -	1·201 or $1\frac{1}{5}$	in the last circumstances.
	water - -	1000	
	mercury -	13592	
Or thus,	air - - -	1·222	nearly when the barom. is - - - 30, and thermometer - - - 55.
	or - -	$1\frac{2}{9}$	
	water - -	1000	
	mercury -	13600	

On this subject the student may advantageously consult General Roy's paper in the *Phil. Trans.* vol. 67. Also Sir Geo. Shuckburgh's and M. De Luc's papers in the same volume; and Mr. Dalton's Chemical Philosophy.

For the principal formulæ and results of M. Biot's researches into the dilatations of liquids and solids under different temperatures, turn to the article Thermometer in the 2d volume of this work.

In permanently elastic fluids, the law which connects the temperature with the elasticity is simple, the latter increasing in a geometrical progression when the former follows the arithmetical. If the elasticity of such a fluid, at any temperature, be denoted by unity, if the number of degrees above that temperature be x, and f the elastic force; then is

$$f = (1{\cdot}376)^{\overline{\frac{x}{180}}}.$$

$$\text{Or, } \log. f = \frac{x}{180} \times \log. 1{\cdot}376.$$

The law which connects the elasticity and the expansion of aeriform fluids, is different from that which regulates the elasticity and expansion of vapour, while the latter remains in contact with the liquid from which it is generated. But when the communication between them is cut off, the same law is observed by the expansion from heat and the augmentation of elasticity, in both kinds of fluids. According to the experiments of Gay Lussac, the expansion is at the rate of 1·376 to 1, for 180° of Fahrenheit; and for any other temperature x, it is $(1{\cdot}376)^{\frac{x}{180}}$ or $(1{\cdot}376)^{\frac{1}{180}} \times (1{\cdot}376)^{x}$, which reduces to $(1{\cdot}0018)$ $1{\cdot}376)^{x}$.

CHAPTER I.

Equilibrium of Elastic Fluids.

491. **Prop.** *To investigate equations of equilibrium for elastic fluids.*

This will be very easy, if we consider that such fluids must, from the nature of perfect elasticity (art. 389.), occupy a smaller space in proportion as the forces which compress them are greater, and restore themselves to their primitive volumes, when the action of the compressing forces cease. Let, then, P be a pressure exerted upon a quantity of M of the fluid, whose density is D; p another pressure, m the mass or volume the fluid takes in consequence of this pressure, and d the density of this mass; so shall we have these equations:

$$PM = pm, \quad MD = md, \text{ and } Pd = pD \;\ldots\ldots\; \text{(I.)}$$

When $P = p$, then $M = m$, as is obvious.

These values only give the pressures exerted upon a unit of surface; but if we drop the consideration of gravity, or any other force which may cause a variation of density in the different parts of the fluid, we may then reason from the principles of hydrostatics. This granted, the pressure p exerted upon any surface denoted by a will be

$$p = \frac{ad}{D}P, \text{ or } p = \frac{aM}{m}P \;\ldots\ldots\ldots\; \text{(II.)}$$

Pressures being commonly valued, as we have often seen, by weights, we may represent that which is exerted upon a unit of surface by the weight of a prism of the same fluid whose height will be given. Let H be the height corresponding to P, and h to p; then we have HD for the mass of this prism, and HDg its weight (g denoting the force of gravity): hence $P = HDg$; and in like manner $p = hdg$. Substituting these for P and p in the equations marked (I.) above, we have $hD = Hd$, and $hm = HM$. Whence we learn that the property of non-elastic fluids, demonstrated art, 388, obtains likewise with regard to elastic fluids.

Instead of taking the same fluid, we may employ another whose density is δ; and, by proceeding in a similar manner, get the following general equation for the pressure,

$$p = \frac{M}{m} ag\delta H = \frac{d}{D} ag\delta H \ldots\ldots \text{(III.)}$$

Many other equations might be deduced with equal facility; but those here given are among the most useful.

492. Prop. *If an elastic fluid be quiescent, and composed of particles equally repulsive at equal distances, and at unequal distances, repelling each other according to any law of the distance, its density will be uniform.*

For if the distances of any two particles from an intermediate particle be unequal, their repulsive forces must be unequal, and, of consequence, motion must ensue; which is contrary to the hypothesis: therefore the fluid must have its particles at equal distances, or be uniformly dense.

Cor. If any portion of an elastic fluid be uniformly dense, and equally compressed on all sides, it must be quiescent.

493. Prop. *If the component particles of a uniform cubical mass of a fluid repel each other, with forces varying according to any inverse or direct ratio of their distances (less than the direct duplicate) the fluid will be elastic.*

For the whole repulsive force of any surface of the fluid is as the number of particles in that surface, and the force of each, or as the number in L the length of that surface, into the number in B the breadth, into the force of each particle; or, substituting R for the whole repulsive force of the surface, I for the interval or distance between two contiguous particles, and F for the force with which they repel each other, R will vary as $\frac{F}{I^2}$. Hence, if F vary in any inverse ratio, or any direct ratio less than the duplicate ratio of I, K will vary in some inverse ratio of I, which is a necessary condition of elastic fluids.

494. Prop. *If the particles of an elastic fluid repel each other with forces varying inversely, as the* n*th power of their distances, that is as* $\frac{1}{I^n}$, *and the compressing force* C *upon any surface be equal to its whole repulsive force* R, *then will* C *vary as that power of the density* D *whose exponent is* $\frac{n+2}{3}$.

For, let a portion of the fluid be contained in a given cubic space, one of whose faces is the rectangle of L × B, the compressing force being applied to that surface. Now, the number of particles in the given square surface is as $\frac{1}{I^2}$; and by hypothesis, the force F, with which two particles repel each other, is as $\frac{1}{I^n}$; therefore the elastic force of the fluid, and of consequence the compressive force C, is as $\frac{1}{I^n} \times \frac{1}{I^2}$ or as $\frac{1}{I^{n+2}}$. But the density of the fluid contained in the given cubical space is inversely as the cube of the distance between the centres of the particles; that is, $D \propto \frac{1}{I^3}$ and $I \propto \frac{1}{D^{\frac{1}{3}}}$: whence, by substituting $D^{-\frac{1}{3}}$ for I, in the expression $C \propto \frac{1}{I^{n+2}}$, we have $C \propto D^{\frac{n+2}{3}}$.

COR. 1. Conversely, if $D^{\frac{n+2}{3}}$ vary as C, the repulsive force of each particle, or F, must vary as $\frac{1}{I^n}$.

For the quantity of matter being given, $D \propto \frac{1}{L^3} \propto \frac{1}{I^3}$ and $D^{\frac{n+2}{3}}$ will vary as $\frac{1}{I^{n+2}}$: but F varies as C divided by the number of particles in L^2, or as $C \times I^2$, or $\propto D^{\frac{n+2}{3}} \times I^2$, or $\propto I^2 \div I^{n+2}$ or $\propto \frac{1}{I^n}$.

COR. 2. Hence again we see, since $n+2$ must be always positive to make C positive, that n must be either some whole positive number, or a negative number less than 2, in order to constitute a fluid of particles which repel each other.

COR. 3. If water be supposed compressible in a very small degree (art. 331.), the particles must be kept at a distance by some repulsive force, while D remains nearly constant. Now, since $C \propto D^{\frac{n+2}{3}}$, we shall have $D \propto C^{\frac{3}{n+2}}$; in which, that $C^{\frac{3}{n+2}}$ may be nearly invariable, n must be a very great number: *hence, according to this hypothesis, the repulsive force of the particles of water varies inversely in a very high power of their distances*

COR. 4. When the density of the fluid varies as the force which compresses it, or $D \propto C$, the expression $C \propto D^{\frac{n+2}{3}}$ becomes $C \propto D^{\frac{1+2}{3}}$ and $n = 1$; whence $F \propto \frac{1}{I^n}$ becomes $F \propto \frac{1}{I}$, or the force of each particle is inversely as the interval between two contiguous particles.

Cor. 5. Hence, because the density of the air is nearly proportional to the force which compresses it (art. 489.), its constituent particles must repel one another with forces varying inversely as their distances.

Cor. 6. The density of the air varying as the compressing force, and that perpetually decreasing in ascending the atmosphere, the density and elasticity of the air also perpetually decrease.

495. Def. A *homogeneous atmosphere* is an atmosphere supposed to be of the same weight as that which actually surrounds the earth; its density being uniform, and every-where equal to the density of the air at the earth's surface.

496. Prop. *To find the altitude of a homogeneous atmosphere.*

Let H be the height of the homogeneous atmosphere, its uniform density being D, the same as the density of the air pressing upon the mercury in the basin of the barometer; h the height of the mercury in the barometer tube, and d the density of that fluid: then (art. 491.) we have $M = m$, or $HD = hd$,

$$\text{whence } H = \frac{hd}{D}.$$

Now it appears from article 490, that when the densities of air and mercury D and d are as $1\frac{2}{9}$ and 13600, the height h of the mercury in the barometer is $2\frac{1}{2}$ feet.

Hence $H = \frac{hd}{D} = \frac{13600 \times 2\frac{1}{2}}{1\frac{2}{9}} = 27818$ feet, $= 5{\cdot}268$ miles. So that the height of the homogeneous atmosphere is rather more than $5\frac{1}{4}$ miles.

Cor. *If it were not for the changes of temperature, the height* H *of the homogeneous atmosphere would be invariable, for one and the same latitude, whatever might be the height of the mercury in the barometer.*

For if d be constant, because the specific gravity of air varies as D its density, and this again as h, the height of the mercury in the tube, it follows that $\frac{h}{D}$ is invariable, and consequently $H = \frac{hd}{D}$ is constant likewise.

Another method of ascertaining the height of the homogeneous atmosphere is given in the scholium at the end of the next proposition.

497. Prop. *Supposing the force of gravity to vary as the* n*th power of the distance from the centre of the earth, and the compressive force to vary as the density, to find the relation between the density of the air and the altitude above the surface of the earth.*

Let x represent the variable distance from the surface of the earth, the radius of the earth being unity, d the density of the air at the distance x, and H the height of the homogeneous atmosphere. Now, since by hypothesis the compressing force varies as the density, the fluxion of the former will vary as the fluxion of the latter: while, at any distance, x, the fluxion of the compressing force must vary as the force of gravity, the density, and the fluxion of the altitude, conjointly; so that the fluxion of the compressing force will be to that of the density in the constant ratio of $x^n d\dot{x}$ to $-\dot{d}$, the latter fluxion having the negative sign, because the density decreases while the altitude increases. Consequently, since, by the definition of a homogeneous atmosphere, H will represent the compressing force at the surface of the earth, we have $\text{H} : 1 :: x^n d\dot{x} : -\dot{d}$, whence

$$x^n\dot{x} = -\text{H} \times \frac{\dot{d}}{d}, \text{ and } \frac{x^{n+1}}{n+1} = -\text{H}\cdot\text{hyp. log.}\, d + \text{C}.$$

Now to correct the fluent we must consider that, when $x = 1$, $d = 1$; whence we find $\text{C} = \frac{1}{n+1}$ for the value of the constant quantity: and the correct fluent is $\frac{x^{n+1}}{n+1} = \frac{1}{n+1} - \text{H. hyp. log.}\, d$. Hence

$$\frac{1-x^{n+1}}{n+1} = \text{H}\cdot\text{hyp. log.}\, d.$$

which is the general equation expressing the relation between the altitude and the density.

Cor. 1. When the force of gravity varies inversely as the square of the distance, $n = -2$, and $\frac{1-x^{n+1}}{n+1} = \text{H}\cdot\text{hyp. log.}\, d$, becomes $\frac{1}{x} - 1 = \text{H}\cdot\text{hyp. log.}\, d$. So that if x increase in harmonic progression, $\frac{1}{x}$ will decrease in arithmetic progression, and consequently hyp. log. d will decrease in arithmetic progression.

Cor. 2. If the force of gravity be supposed constant, $n = 0$, and $1 - x = \text{H}\cdot$ hyp. log. d. Consequently, if x increase in arithmetic progression, since $1 - x$ will then decrease in arithmetic progression, the hyp. log. of d will decrease in arithmetic progression.

Cor. 3. Since the hyperbolic logarithms are to the common logarithms in a constant ratio, viz. that of 1 to ·43429448, &c. it follows that when x increases in arithmetic progression the common logarithms of the densities will decrease in arithmetic progression, and the densities themselves in geometrical progression, on the supposition of equal gravity.

Cor. 4. Hence, retaining the same hypothesis, different altitudes above the earth's surface will vary as the negative logarithms of the densities or weights of air at those altitudes. So that if D and d denote the densities at the heights H and h, since $\text{H} \propto -\log. \text{D}$, and $h \propto -\log. d$, the difference $\text{H} - h \propto -\log. \text{D} - (-\log. d) \propto \log. \text{D} - \log. d$, or $\propto \log. \frac{\text{D}}{d}$. And if $\text{H} = 0$, or $\text{D} =$ the density at the earth's surface, then any altitude above the surface is as the log. of $\frac{\text{D}}{d}$. Or generally the log. of $\frac{\text{D}}{d}$ varies as the altitude of the one place above the other, whether the lower place be at the surface of the earth or not: and upon this property is founded the method of finding the heights of mountains, &c. by the barometer.

SCHOLIUM.

498. The properties comprised in the preceding corollaries are sometimes proved by means of the logarithmic curve; and as a description of the relation which obtains between the ordinates of that curve and the density of the air will furnish an agreeable illustration of what is done above, such description is subjoined.

Let ARQ (fig. 3. pl. XVIII.) represent a section of a terrestrial hemisphere by a plane through its centre O, mOAM a vertical line, AE a horizontal line through A, a point on the earth's surface. Let this line AE represent the density of the air at A; and let DH parallel to AE be taken to AE, as the density at D to the density at A: then it is manifest that if a logistic or logarithmic curve EHN be drawn, having AM for its axis, and passing through the points E and H, the density of the air at any other point C in the vertical AM will be represented by CG, the ordinate to the curve in that point: for it is a known property of this curve, that if portions AB, AC, AD, of its axis be taken in arithmetical progression, the ordinates AE, BF, CG, will be in geometrical progression.

Another property of this curve is, that if EK or HS touch the curve in E or H the subtangent AK or DS is a constant quantity, being the modulus of the system of logarithms which the particular curve represents. And a third property is, that the infinitely extended area, AENM is equal to the rectangle AELK of the ordinate at A and the subtangent; and, in like manner, the area MDHN above DH is equal to the rectangle SD·DH or KA·DH; so that the area lying above or beyond any ordinate is proportional to that ordinate. (See Emerson on Curve Lines, book I. § 5. Hutton's Dictionary art. LOGARITHMIC *Curve*, or the Treatise on Curve Lines at the end of Robertson's Conic Sections.)

Now these geometrical properties of this curve are analogous to the chief circumstances in the varying density of the atmosphere, *on the supposition of equal gravity.* The area MBFN, for instance, represents the whole quantity of the elastic fluid which is above B: for BF is the density at B, and BC is the thickness of the stratum between B and C; whence BFGC will be as the quantity of air in that stratum: in like manner CGHD will be as the quantity of aerial matter in the stratum whose thickness is CD; and the same of their sums, or of MBFN, the whole area. Also, as each ordinate is proportional to the area above it, so each density and the quantity of air in each stratum, is proportional to the quantity of air above it. And again, as the whole area AENM is equal to the rectangle AELK, so the whole air of variable density above A might be contained in a column KA of the same base, if, instead of being compressed by its own weight, it were without weight, and compressed by an external force equal to the pressure of the air at the surface of the earth. In this case it would be of the uniform density AE which it has at the surface of the earth, and would constitute the *homogeneous atmosphere,* (art. 495.)

Hence we learn that the height of the homogeneous atmosphere is the subtangent of the curve whose ordinates are as the densities of the air at different heights, on the supposition of equal gravity. This curve is generally called the *atmospherical logarithmic*; and its subtangent, or the height of the homogeneous atmosphere, is the modulus of the system of logarithms to which the curve corresponds.

We have already (art. 496.) shewn how to find the height of the homogeneous atmosphere by a single observation upon the barometer: but the same thing may be effected, perhaps, more accurately in a different manner. When the mercury and the air are of the temperature 32° of Fahrenheit's thermometer, and the barometer on the sea-shore stands at 30 inches, if we take it to a place 936 feet higher it will fall to 29 inches. Now in all logarithmic curves having equal ordinates the portions of the axes intercepted between the corresponding pairs of ordinates are proportional to the subtangents—(Emerson on Log. Curve, prop. viii. cor. 3.): and the subtangent of the curve belonging to our common tables is ·43429448, and the difference of the logarithms of 30 and 29 (which is the portion of the axis intercepted between the ordinates 30 and 29) is ·0147233; whence we have ·0147233 : ·43429448 : : 935 : 27600 feet, or nearly $5\frac{1}{4}$ miles. This determination of the height of the homogeneous atmosphere does not quite correspond with the former; but a trifling error in measuring the altitudes of the mercury in the barometer at the two stations would cause all the difference.

499. All that has gone before in this scholium proceeds on the supposition of *equal* gravity, or the hypothesis of Cor. 2; whereas we know that the weight of a particle of air decreases as the square of its distance from the earth's centre increases, conformably to Cor. 1. Hence we see that in order that a superior stratum may produce an equal pressure at the surface of the earth it must be denser, because a particle of it gravitates less. The density, therefore, at equal elevations, must be greater than on the supposition of equal gravity, and the law of diminution of density must be different.

Make $\mathrm{OD} : \mathrm{OA} :: \mathrm{OA} : \mathrm{O}d$;

$\mathrm{OC} : \mathrm{OA} :: \mathrm{OA} : \mathrm{O}c$;

$\mathrm{OB} : \mathrm{OA} :: \mathrm{OA} : \mathrm{O}b$, &c.;

so that $\mathrm{O}d$, $\mathrm{O}c$, $\mathrm{O}b$, OA, may be reciprocals to OD, OC, OB, OA; and through the points A, *b*, *c*, *d*, draw the perpendiculars AE, *bf*, *cg*, *dh*, making them proportional to the densities in A, B, C, D; and suppose CD to be exceedingly small, so that the density may be supposed uniform through the whole stratum. Thus we have

$$\mathrm{OD}\cdot\mathrm{O}d = \mathrm{OA}^2, = \mathrm{OC}\cdot\mathrm{O}c$$

$$\text{and } \mathrm{O}c : \mathrm{O}d :: \mathrm{OD} : \mathrm{OC};$$

$$\text{and } \mathrm{O}c : \mathrm{O}c - \mathrm{OD} :: \mathrm{OD} : \mathrm{OD} - \mathrm{OC},$$

$$\text{or } \mathrm{O}c : cd :: \mathrm{OD} : \mathrm{DC};$$

$$\text{and } cd : \mathrm{CD} :: \mathrm{O}c : \mathrm{OD};$$

or, because OC and OD are ultimately in the ratio of equality, we have

$$cd : \mathrm{CD} :: \mathrm{O}c : \mathrm{OC} :: \mathrm{OA}^2 : \mathrm{OC}^2,$$

and $cd = \mathrm{CD}\cdot\frac{\mathrm{OA}^2}{\mathrm{OC}^2}$, also $cd\cdot cg = \mathrm{CD}\cdot cg\cdot\frac{\mathrm{OA}^2}{\mathrm{OC}^2}$. But $\mathrm{CD}\cdot cg\cdot\frac{\mathrm{OA}^2}{\mathrm{OC}^2}$ is as the pressure at C arising from the absolute weight of the stratum CD: for this weight is as the bulk, as the density, and as the gravitation of each particle jointly; and CD expresses the bulk, *cg* the density, and $\frac{\mathrm{OA}^2}{\mathrm{OC}^2}$ the gravitation of each particle. Therefore $cd\cdot cg$ is as the pressure on C arising from the weight of the stratum DC. But $cd\cdot cg$ is evidently the element of the curvilineal area A*mn*E, formed by the curve E*fghn* and the ordinates AE, *bf*, *cg*, &c. *mn*. Hence the sum of all the elements, such as *cdhg*, that is, the area *cmng* below *cg*, will be as the *whole* pressure on C, arising from the gravitation of all the air above it; but, by the nature of air, this whole pressure is as the density which it produces, that is, as *cg*. Therefore the curve E*gn* is of such a nature that the area lying below or beyond any ordinate *cg* is proportional to that ordinate. This is the property of the logarithmic curve, and E*gn* is a logarithmic curve.

Moreover, this curve is equal to EGN. For, let B continually approach to A, and ultimately coincide with it. It is evident that the ultimate ratio of BA to A*b*, and of BF to *bf*, is that of equality; and if EFK, E*fk*, be drawn, they will contain equal angles with the ordinate AE, and will cut off equal subtangents AK, A*k*. The curves EGN, E*gn*, are therefore equal, but in opposite positions.

Lastly, if OA, O*b*, O*c*, O*d*, &c. be taken in arithmetical progression decreasing, their reciprocals OA, OB, OC, OD, &c. will be in harmonical progression increasing, as is well known: but, from the nature of the logarithmic curve, when OA, O*b*, O*c*, O*d*, &c. are in arithmetical progression, the ordinates AE, *bf*, *cg*, *dh*, &c. are in geometrical progression. Therefore when OA, OB, OC, OD, &c. are in harmonical progression, the densities of the air at A, B, C, D, &c. are in geometrical progression; and thus may the density of the air at all elevations be discovered. Thus to find the density of the air at K, the top of the homogeneous atmosphere, make OK : OA : : OA : OL, and draw the ordinate LT, so shall LT express the density at K.

Dr. Halley was the first who observed the relation between the density of the air and the ordinates of the logarithmic curve, or common logarithms. This he did on the supposition of equal gravity; and his discovery is acknowledged by Sir Isaac Newton in *Princip.* ii. *prop.* 22. *schol.* Halley's dissertation on the subject is in No. 181 of the Phil. Trans. Newton, with his usual sagacity, extended the same relation to the true state of the case, where gravity is as the square of the distance inversely; and showed that when the distances from the earth's centre are in harmonic progression the densities are in geometric progression. He shows indeed, in general, what progression of the distance, on any supposition of gravity, will produce a geometrical progression of the densities, so as to obtain a set of lines OA, O*b*, O*c*, O*d*, &c. which will be logarithms of the densities. The subject was afterwards treated in a more familiar manner by Cotes in his *Hydros. Lect.* and in his *Harmonia Mensurarum;* also by Brooke Taylor, *Method. Increment.;* Wolf, in his *Aerometria;* Herman, in his *Phoronomia*, &c.; and, on account of the simplicity and elegance of this method, it now obtains a place in almost every treatise on pneumatics.

500. Prop. *To determine the actual density of the atmospheric air at any altitude above the earth's surface.*

This may be done with the aid of the equation in cor. 1. art. 497. but with greater facility by means of the atmospherical logarithmic, thus. By the property in Emerson's Treatise, referred to in art. 498. say, as the height of the homogeneous atmosphere is to the modulus of Briggs's system, so as the given altitude in feet to a fourth number, which in the common tables is the logarithm of the ratio of the density of the air at the earth's surface to its density expressed by unity at the proposed height, on the supposition of equal gravity. But, if we attend to the variation of gravity, the process will be this. Suppose C to be the point at which the density is required; make, as directed in the last article, OC : OA : : OA : Oc, or OC : OA : : AC : Ac, and Ac thus obtained will be the height to which the density is to be calculated by the preceding analogy, on the hypothesis of equal gravity.

Let us take for an example the height of 7 miles, and conceive the radius of the earth to be 4000 miles. Then shall we have $OC\ (=4007) : OA\ (=4000) : : AC\ (=7) : Ac = \frac{4000 \times 7}{4007}$ $= 6{\cdot}98777$ miles $= 36895\frac{1}{2}$ feet. Wherefore, taking 27600 feet for the height of the homogeneous atmosphere, we shall have

$$27600 : {\cdot}43439448 : : 36895\tfrac{1}{2} : {\cdot}5806957,$$

which is the common logarithm of 3·80799, or 3·808 nearly. Consequently the density of the air at the earth's surface is to its density at the altitude of 7 miles as $3\frac{4}{5}$ to 1 nearly, allowing for the diminution of the force of gravity.

This result agrees nearly with experiments. Thus Mr. Cotes inferred from the French experiment at the Puy de Dome, that at the altitude of 7 miles the air was rather more than 4 times rarer than at the surface of the earth: but from the experiments of Mr. Caswell, at Snowden, he concluded that at the same altitude of 7 miles the air was not quite four times rarer than at the surface. And Sir Isaac Newton, in the last edition of his Optics, states it as 4 times rarer at the height of $7\frac{1}{2}$ miles: which, properly reduced, gives 3·86 for the comparative rarity at 7 miles.

501. Returning to the hypothesis of equal gravity, we may readily find an equation for the altitude, which shall include the changes in temperature. Thus, let D and d represent the density of the air at any two places, measured by the column of mercury in the barometer: then, since the difference of altitude a (art. 497. cor. 4.) is always as log. $\frac{D}{d}$, assume h, so

that $a = h \times \log. \frac{D}{d}$, where h will be of one constant value for all altitudes: and to determine that value let a case be taken in which we know the altitude a corresponding to a known density d; as, for instance, take $a = 1$ foot, or 1 inch, or some such small altitude; then, because the density D may be measured by the pressure of the atmosphere, or the uniform column of 27600 feet, when the temperature is 55°; therefore 27600 feet will denote the density D at the lower place, and 27599 the less density d at 1 foot above it; consequently $1 = h \times \log. \frac{27600}{27599}$; which, by the nature of logarithms, is nearly $= h \times \frac{\cdot 43439448}{27600} = \frac{h}{63551}$ nearly; and hence $h = 63551$ feet; which gives, for any altitude in general, this theorem, viz. $a = 63551 \times \log. \frac{D}{d}$, or $= 63551 \times \log. \frac{M}{m}$ feet, or $10592 \times \log. \frac{M}{m}$ fathoms; where M is the column of mercury which is equal to the pressure or weight of the atmosphere at the bottom, and m that at the top of the altitude a; and where M and m may be taken in any measure, either feet or inches, &c.

This formula is adapted to the mean temperature of the air 55° of Fahrenheit's thermometer: but it has been found by the experiments of Sir Geo. Shuckburgh and Gen. Roy, that for every degree of the thermometer different from 55° the altitude a will vary by its 435th part: hence if we would change the factor h from 10592 to 10000, because the difference 592 is the 18th part of the whole factor 10592, and because 18 is nearly the 24th part of 435; therefore the change of temperature answering to the change of the factor h is 24°, which reduces the 55° to 31°. So that $a = 10000 \times \log. \frac{M}{m}$ fathoms is the easiest expression for the altitude, and answers to the temperature of 31°, or very nearly the freezing point: and for every degree above that the result must be increased by so many times its 435th part, and diminished when below it.

This method was first given by Dr. Hutton, under the article *Atmosphere*, in his Mathematical Dictionary.

It is obvious that the relation between the altitudes of places and the heights of mercury in the barometer, is not exactly the same in all latitudes. For according as the action of gravity is greater or less, a column of mercury of given height will weigh more or less, and consequently the air subjected to that pressure will be more or less compressed. We have seen (art. 286.) that if gravity at the earth's surface under the parallel of 45°, be measured by unity, that at any other latitude λ, will be expressed by $1 - \cdot 002837 \cos 2\lambda$: the density of the air at the earth's surface will vary in the same ratio. This will cause a small correction of the computed altitude, in different latitudes, never exceeding $\frac{1}{352}$ of such altitude. Thus, at 0°... corr $+\frac{1}{352}$, 10°... $+\frac{1}{375}$, 20°... $+\frac{1}{460}$, 30°... $+\frac{1}{705}$, 40°... $+\frac{1}{2030}$, 45°... 0, 50°... $-\frac{1}{2030}$, 60°... $-\frac{1}{705}$, 70°... $-\frac{1}{460}$, 80°... $-\frac{1}{375}$, 90°... $-\frac{1}{352}$. For intermediate latitudes the correction may readily be computed.

A correct and extensive table of the comparative rarity and expansion, density and compression, of air at different altitudes, assuming the homogeneous atmosphere at 29725 feet, and allowing for the variation of gravity, is inserted at p. 253 of Helsham's Lectures.

CHAPTER II.

On the Admeasurement of Altitudes by the Barometer and Thermometer.

502. Pascal and Descartes hinted that if a barometer were carried to a higher situation the quicksilver would sink in the tube; and the fact was verified by M. Perier. But the admeasurement of altitudes by means of the barometer was first distinctly suggested by Dr. Halley, in a paper in No. 181 of the Philosophical Transactions, to which we have before referred (art. 499.). He there states that, according to experiments shewn before the Royal Society, when the barometer stood at 30 inches, and in a mean state of heat and cold, the specific gravity of air was to that of water as 1 to 800; and then proceeds as follows: "By the like trials the weight of mercury to water is as 13½ to 1, or very near it; so that the weight of mercury to air is as 10800 to 1, and a cylinder of air of 10800 inches or 900 feet is equal to an inch of mercury; and were the air of an equal density like water, the whole atmosphere would be no more than $5\frac{1}{10}$ miles high; and in the ascent of every 900 feet the barometer would sink an inch. But the expansion of the air increasing in the same proportion as the incumbent weight of the atmosphere decreases, that is, as the mercury in the barometer sinks, the upper parts of the air are much more rarefied than the lower, and each space answering to an inch of quicksilver grows still larger, so that the atmosphere must be extended to a much greater height. Now, on these principles, to determine the height of the mercury at any assigned height in the air, and *è contra,* having the height of the mercury given to find the height of the place where the barometer stands, are problems not more difficult than curious." He then points out a method of solution, and gives tables of the heights of mercury at different altitudes, and *vice versa;* which need not be copied in this place, as they depend upon erroneous estimates of the specific gravities of mercury and air.

He also mentions the condensation and rarefaction by heat and cold, and the effect of aqueous and other vapours; but was of opinion that these would compensate each other.

503. It appears, then, that Dr. Halley's method cannot apply to every case, since it depends upon the supposition of the relative density of air and mercury being only affected by *a change of pressure.* But since all bodies are expanded by heat, and as they are not all *equally* expanded by it, it follows that a change of temperature may change the relative gravity of mercury and air, even although both undergo the same change of temperature: and since the air may be warmed or cooled when the mercury is not, or may change its temperature independent of it, still greater variations of specific gravity may be expected. Hence many corrections are required to adjust the barometrical method to the various attending circumstances, and it was not till long after Dr. Halley's time that the method was turned to any real use. The chief improvements and adjustments are due to M. *de Luc*, who had favourable opportunities of making numerous observations among the high hills of Switzerland, and who availed himself of these opportunities in almost every variety of circumstances. The result of his experiments and enquiries was published at Geneva in a treatise on the barometer and thermometer, and in a paper in the 57th volume of the Philosophical Trans. Other curious and valuable papers on this subject by Dr. Maskelyne, Dr. Horsley, Sir George Shuckburgh, and General Roy, are inserted in the different volumes of these Transactions: and an interesting disquisition is given by M. Biot, in the 3d volume of his " Astronomie Physique."

Some other important deductions, in a recent paper of M. Biot's, may be seen in the article THERMOMETER, in our second volume.

The most accurate experiments for ascertaining the expansion of mercury are those of General Roy (vol. 67, Philosophical Transactions). These are contained in the following table; where the first column expresses the temperature by Fahrenheit's thermometer, the second expresses the bulk of the mercury, and the third the expansion of an inch of mercury for an increase of one degree in the adjoining temperatures,

TABLE I.

Temp.	Bulk of ☿.	Expans. for 1°
212°	30·5117	0·0000763
202	30·4888	0·0000787
192	30·4652	0·0000810
182	30·4409	0·0000833
172	30·4159	0·0000857
162	30·3902	0·0000880
152	30·3638	0·0000903
142	30·3367	0·0000923
132	30·3090	0·0000943
122	30·2807	0·0000963
112	30·2518	0·0000983
102	30.2223	0·0001003
92	30·1922	0·0001023
82	30·1615	0·0001043
72	30·1302	0·0001063
62	30·0984	0·0001077
52	30·0661	0·0001093
42	30·0333	0·0001110
32	30·0000	0·0001127
22	29·9662	0·0001143
12	29·9319	0·0001160
2	29·8971	0·0001177
0	29·8901	

The General also made experiments on the expansion of the mercury in a real barometer: and in some rules for barometrical admeasurement, it will be necessary to know the *logarithmic* difference to the expansion for the mean temperature of the two barometers which are used. These logarithmic differences are contained in the following table, which is carried as far as 112°, beyond which it is not probable that any observations will be made. The number for each temperature is the difference between the logarithms of 30 inches, of the temperature 32, and 30 inches expanded by that temperature.

Table II.

Temp.	Log. diff.	Dec. of Fath.	Ft.	In.
112°				
102	0·0000427	·427	2	7
92	0·0000436	·436	2	7
82	0·0000444	·444	2	8
72	0·0000453	·453	2	9
62	0·0000460	·460	2	9
52	0·0000468	·468	2	10
42	0·0000475	·475	2	10
32	0·0000482	·482	2	11
22	0·0000489	·489	2	11
12	0·0000497	·497	3	0
0	0·0000504	·504	3	0

General Roy compared a mercurial and an air thermometer, each of which was graduated arithmetically, that is, the units of the scales were equal bulks of mercury, and equal bulks (perhaps different from the former) of air. He found their progress as in the following table.

Table III.

Merc.	Diff.	Air.	Diff.
212		212·0	
192	20	194·4	17·6
172	20	176·2	18·2
152	20	157·4	18·8
132	20	138·0	19·4
112	20	118·0	20·0
92	20	97·2	20·8
72	20	75·6	21·6
52	20	53·0	22·6
32	20	31·4	21·6
12	20	11·4	20·0

The subsequent table contains the expansion of 1000 parts of air, nearly of the common density, by heating it from 0 to 212. The first column contains the height of the barometer; the second contains this height augmented by the small column of mercury in the tube of the manometer, and therefore expresses the density of the air examined; the third contains the total expansion of 1000 parts; and the fourth contains the expansion for 1°, supposing it uniform throughout.

TABLE IV.

Barom.	Density of Air examined.	Expansion of 1000 parts by 212°.	Expansion by 1°.
29·95	31·52	483·89	2·2825
30·07	30·77	482·10	2·2741
29·48	29·90	480·74	2·2676
29·90	30·73	485·86	2·2918
29 96	30·92	489·45	2·3087
29·90	30·55	476·04	2·2455
29·95	30·60	487·55	2·2998
30·07	30·60	482·80	2·2774
29·48	30·00	489·47	2·3087
Mean	30·62	484·21	2·2840

Hence it appears, that the mean expansion of 1000 parts of air of the density 30·62 by one degree of Fahrenheit's thermometer is 2·284, or that 1000 becomes 1002·284.

If this expansion be supposed to follow the same rate that was observed in the comparison of the mercurial and air thermometer, we shall find that the expansion of a thousand parts of air for one degree of heat at the different intermediate temperatures will be as in the following table.

TABLE V.

Temp.	Total Expansion.	Expansion for 1°.
212	484·210	2·0099
192	444·011	2·0080
172	402·452	2·1475
152	359·503	2·2155
132	315·193	2·2840
112	269·513	2·3754
92	222·006	2·4211
82	197·795	2·5124
72	172·671	2·5581
62	147·090	2·6037
52	121·053	2·5124
42	95·929	2·4211
32	71·718	2·3297
22	48·421	2·2383
12	26·038	2·1698
0		

If we would have a mean expansion for any particular range, as between 12° and 92°, which is the most likely to comprehend all the geodætical observations, we need only take the difference of the bulks 26·038 and 222·006 = 195·968, and divide this by the interval of temperature 80°, and we obtain 2·4496, or 2·45 for the mean expansion for 1°, whence a complete table may readily be constructed.

504. Having thus stated the most material circumstances which are to be taken into the account in this kind of admeasurement, and given such tables as will be useful in the computation, we shall proceed to exhibit the most approved precepts and rules.

In order that the observations may be carefully and properly made, the person who undertakes them should be provided with two portable barometers of the best construction (both filled with mercury of the same specific gravity), on which, by means of a nonius properly adapted to the scale, he may read off the height of the mercurial columns to the 200th part of an inch: each barometer should be fitted up with an attached thermometer, set in the wooden frame in the same manner as the barometer tube is. The ball of each thermometer had best be nearly of the same diameter as the barometer tube. Besides these he must also be provided with two other thermometers, detached from the barometers. Of these barometers, one with its attached and detached thermometers is to be placed in the shade at the top of the eminence whose height is required, while the other remains below. Let them continue in their places at least a sufficient time for the detached thermometer to acquire the temperature of the air, that is to say, till the contained fluid is stationary. Then the observer on the eminence must note down the height of the mercurial column in the barometer, as well as the temperatures exhibited by the attached and detached thermometers; and at *the same time* the other observer must make like observations upon the instruments below. If, in this manner, three or four sets of observations be taken at each station after short intervals of time, and the mean of the results furnished by these sets respectively be taken as the true altitude (following one of the subsequent rules), the probability of error will be much diminished. It is also advisable to make the observations in serene weather, between 11 and 12 o'clock. For it has been found that the computed heights are too *small* when the observations have been made near sunrise or sunset, or when the wind blows fresh from the south; and that, on the contrary, the computed results are too *great*, when the observations are made about 3 o'clock in a hot summer's day, or during a brisk wind from the north or east.

In practice, however, it will often be necessary to deviate from the preceding directions, and such deviations may be pretty well guarded against mistake. In cases where better instruments cannot be had, any well-made portable barometer so graduated as to shew the true change in the altitude of the mercury, may afford such observations as ought not to be neglected. A small error in the position of the point from whence the scale of inches commences, will not sensibly affect the result; for, that result depends chiefly (art. 501.) upon log. $\frac{M}{m}$ or log. M − log. m, which will not be much affected by a small and equal change in M and m. Suppose, for example, M was 29·36 inches, and m 27·84 inches, then log. M − log. $m =$ 230869; but if M were 29·11, and $m = 27{\cdot}59$, each less by a quarter of an inch, we should have log. M − log. $m = 232905$, which would not cause a difference in the result of more than 1 fathom in a hundred. And an error of $\frac{1}{4}$ of an inch in the position of the zero is far greater than may be reasonably expected to occur. If an hour or more can be allowed for the mercury in the barometer to acquire the temperature of the surrounding air, which is shewn by the detached thermometer, then the attached thermometer may be dispensed with. A single barometer may supply the place of two, if the observations can be made first at the base, then at the summit, and again at the base, in a moderate space of time: and if the two sets of observations at the base correspond pretty nearly, it may be presumed that the density of the air below has undergone no material change during the operations. The necessary observations being made, the altitude of the object at whose top and bottom the instruments were placed, may be ascertained by one or other of the following rules.

I. *Dr. Robison's method.*

505. In this method no tables are required; it will be sufficiently exact for most purposes, and is not difficult to remember. It was deduced from these considerations.

1. The height through which we must rise in order to produce any fall of the mercury in the barometer, is inversely proportional to the density of the air, that is, to the height of the mercury in the barometer.

2. When the barometer stands at 30 inches, and the air and quicksilver are of the temperature 32°, we must rise through 87 feet to produce a depression of $\frac{1}{10}$ of an inch.

3. But if the air be of a different temperature, this 87 feet

must be increased or diminished by about 0·21 of a foot, for every degree of difference of the temperature from 32°.

4. Every degree of difference of the temperatures of the mercury at the two stations makes a change of 2·833 feet, or 2 feet 10 inches in the elevation.

Hence the following rule:

I. Take the difference of the barometric heights in tenths of an inch. Call this D.

II. Multiply the difference d between 32° and the mean temperature of the air by ·21, and take the sum or difference of this product and 87 feet. This is the height through which we must rise to cause the barometer to fall from 30 inches to 29·9; and may be called h.

III. Let m be the mean between the two barometric heights:

Then $\frac{30\text{D}h}{m}$ is the approximated elevation very nearly.

IV. Multiply the difference δ of the mercurial temperatures by 2·833 feet, and add this product to the approximated elevation if the upper barometer has been the warmest, otherwise subtract it: then will the resulting sum or difference be the corrected elevation.

Or, this rule may be expressed by the following formula: where d is the difference between 32° and the mean temperature of the air, D is the difference of barometric heights in tenths of an inch, m is the mean barometric height, δ the difference between the mercurial temperatures, and E is the correct elevation.

$$\text{E} = \frac{30(87 \pm 0{\cdot}21d)\text{D}}{m} \pm \delta \times 2{\cdot}833.$$

For an example, suppose that the mercury in the barometer at the lower station was at 29·4 inches, its temperature 50° of Fahrenheit's thermometer, and the temperature of the air 45°: the height of the mercury at the upper station 25·19 inches, its temperature 46°, and the temperature of the air 39°.

$$\text{Here D} = 294 - 251{\cdot}9 = 42{\cdot}1$$
$$h = 87 + (10 \times {\cdot}21) = 89{\cdot}1$$
$$m = \tfrac{1}{2}(29{\cdot}4 + 25{\cdot}19) = 27{\cdot}295$$

$\frac{30\text{D}h}{m}$ = approximate elevation = . . . 4123·24

Correction for temp. of mercury 4 × 2·833 = 11·33

Corrected elevation in feet . . . . . 4111·91

Ditto in fathoms . . . . 685·32

II. *M. de Luc's method.*

506. I. Subtract the logarithm of the barometrical height at the upper station from the logarithm of that at the lower, and count the index and first four decimal figures of the remainder as fathoms, the rest as a decimal fraction. This may be called the *elevation.*

II. Note the different temperatures of the mercury at the two stations, and the mean temperature. Multiply the logarithmic expansion corresponding to this mean temperature (in table II. art. 503.) by the difference of the two temperatures, and subtract the product from the elevation if the barometer has been coldest at the upper station, otherwise add it. This difference, or sum, will be the *approximated elevation.*

III. Observe the difference of the temperatures of the air at the two stations by a detached thermometer, also the mean temperature, and its difference of temperature from 32°. Multiply this difference by the expansion of air for the mean temperature, and multiply the approximated elevation by 1 ± this product, according as the air is above or below 32°. The last product is the correct elevation in fathoms and decimals.

Taking the same example as before, we have

☿al heights.	Temp. ☿.		Mean.	Temp. Air.	Mean.
29·4	50	4	48	45	42
25·19	46			39	

I.	Log. of 29·4		1·4683473
	Log. of 25·19		1·4012282
	Elevation in fathoms		671·191
II.	Expans. for 48° (tab. II.) . . .	·473	
	Multiply by (50 − 46) . . .	4	
		. .	1·892
	Approximated elevation		669·299
III.	Expans. of air at 42° . . .	00·00238	
	Multiply by (42 − 32) . . .	10	
		0·0238	

Hence 669·299 × 1·0238 = 685·228, the correct elevation.

III. *Sir George Shuckburgh's method.*

507. I. Reduce the barometric heights to what they would be if they were of the temperature 32°.

II. The difference of the logarithms of the reduced barometrical heights will give the approximate elevation.

III. Correct the approximated elevation as before.

Same example.

I. Mean expans. for 1° from tab. I. is 0·000111.
18° × 0·000111 × 29·4 = ... 0·059
Subtract this from . . . . 29·4

Reduced barometric height . . . . 29·341
Expans. from tab. I. is 0·000111.
14° × 0·000111 × 25·19 . . . . 0·039
Subtract from . . . . 25·190

Reduced barometric height . . . . 25·151
II. Log. 29·341 . . . . 1·4674749
Log. 25.151 . . . . 1·4005553

Approximated elevation . . . . . 669·196
III. This multiplied by 1·0238 gives . . 685·125

Remark 1. If 0·000101 be supposed the mean expansion of mercury for 1° as Sir George Shuckburgh determines it, the reduction of the barometric heights will be had sufficiently exact by multiplying the observed heights of the mercury by the difference of its temperatures from 32, and cutting off four more decimal places; thus 29·4 × $\frac{18}{10000}$ gives for the reduced height 29·347 and 25·19 × $\frac{14}{10000}$ gives 25·155, and the difference of their logarithms gives 669·4 fathoms for the approximated elevation, which differs from the one given above by no more than 15 inches.

Remark 2. If 0·0024 be taken for the expansion of air for one degree, the correction for this expansion will be had by multiplying the approximated elevation by 12, and this product by the sum of the differences of the temperatures from 32°, counting that difference as negative when the temperature is below 32, and cutting off four places; thus 669·196 × 12 × (13 + 07) × $\frac{1}{10000}$ = 16·061, which added to 669·196 gives 685·257, differing from the former only 9 inches.

IV. *Dr. Hutton's method.*

508. I. Observe the height of the barometer at the bottom of any height or depth intended to be measured, with the temperature of the quicksilver by means of a thermometer attached to the barometer, and also the temperature of the air in the shade by a detached thermometer.

II. Let the same thing be done also at the top of the said height or depth, and at the same time, or as near the same time as may be. And let those altitudes of barometer be reduced to the same temperature, if it be thought necessary, by correcting either the one or the other; that is, augment the height of the mercury in the colder temperature, or diminish that in the warmer, by its $\frac{1}{9600}$ part for every degree of difference of the two. The altitudes of mercury so corrected being what we denoted by M and *m*, in art. 501.

III. Take the difference of the common logarithms of the two heights of the barometer, corrected as above if necessary, cutting off three figures next the right hand for decimals, the rest being fathoms in whole numbers.

IV. Correct the number last found for the difference of temperature of the air, as follows: Take half the sum of the two temperatures for the mean one; and for every degree which this differs from the temperature 31°, take so many times the $\frac{1}{435}$ part of the fathoms above found, and add them if the mean temperature be above 31°, but subtract them if the mean temperature be below 31°; and the sum or difference will be the true altitude in fathoms; or, being multiplied by 6, it will be the altitude in feet. (*Math. Dict.* art. *Barometer.*)

Same example.

Thermometers.		Barometers.
Detached.	Attached.	
45	50	29·4 lower.
39	46	25·19 upper.
Mean 42	Diff. 4	

As 9600 : 4 : : 29·4 : ·0123
corr. ·0123

Mean 42 $M = 29·3877$ . . . . log. 4681656
Stand. 31 $m = 25·19$ 4012282

Diff. 11 As 435 : 11 : : 669·374 : 16·924
16·924

The altitude sought is 686·298 fathoms.

509. Such are the most approved rules for the determination of altitudes by the barometer. By a comparison of the four methods it will be seen that no two of the results differ more than a 600th part; so that in point of accuracy we know not which should be preferred. In practice, perhaps, the first and

fourth may be most expeditious; and, in general, an arithmetical mean between the results of these two will not be far from the truth.*

As to the advantages of the barometric compared with the geometrical method of measuring elevations, we shall state them chiefly in the language of Mr. Nicholson, (Natural Philosophy, vol. ii.). First the instruments are neither very expensive, nor even difficult for an ingenious philosopher to make in any country where he can procure quicksilver and glass tubes; but the geometrical method requires instruments of considerable price, which cannot at all be accurately constructed by the most ingenious person who is destitute of the tools, and unacquainted with the artifices necessary to render them correct. Secondly, the barometers require no other adjustment than to observe previously whether they agree, and, if they do not to allow for their difference. The barometrical observations are likewise easily made; whereas, on the contrary, the previous adjustment and subsequent use of instruments for measuring angles require a degree of precision and skill not usually obtained without practice. Thirdly, the error *of observation* in the barometrical method for all elevations is nearly a constant quantity, never amounting to so much as half a fathom for a mistake of the 500th of an inch; but any error either in the measurement of lines or angles proportionally affects the result; so that the greater the elevation required to be measured, the larger the quantity of error. Fourthly, the barometrical observations require no particular circumstances of advantage either in the figure or situation of the mountains, &c. to be measured, nothing more being required than that both stations be accessible. These observations, and the computation, are performed after the same method in all cases; but, in the geometrical method, if the horizontal distance of the two stations be considerable, or if there be not a convenient plain for measuring a fundamental base, the operation becomes very complicated, and the probability of error is multiplied.

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* Besides the above, which are appropriated to English measures and Fahrenheit's thermometer, we may here exhibit a few formulæ adapted to the centigrade thermometer and French measures. Let T and t be the temperature of the air, at the lower and upper stations, indicated by the centigrade thermometer, H and h the heights of the mercury in the barometer at those stations, and M and m the temperatures of the mercury in the respective barometers; then, according to Laplace, the altitude A, in metres, of the upper above the lower station, will be expressed by $\text{A} = 18393 \left(1+\frac{2(\text{T}+t)}{1000}\right) \log. \frac{\text{H}}{h+\frac{\text{M}-m}{5412}h}$.

This reduced to English measures gives for the altitude in fathoms; $\text{A} = 10050 \left(1+\frac{2(\text{T}+t)}{1000}\right) \log. \frac{\text{H}}{h(1+ \cdot 00018(\text{M}-m))}$.

M. Biot presents the following formula, comprehending the allowance for changes in latitude, and not erring more than 1 metre in 1500: namely, $\text{A} = 18393 \left\{1+\cdot 002837 \cos 2\lambda\right\} \left\{1+\frac{2(\text{T}+t)}{1000}\right\} \log. \frac{\text{H}}{h}$.

And lastly, M. Prony, in the "Connaissance des Tems pour l'an 1816," presents a very commodious expeditious theorem, approximating to the true result usually within 1 in 1200; viz. $\text{A}=\text{K}\theta q$. Of these three factors K is constant, being $=15969$ metres; the multipliers of this number are 18336, the double modulus, and the constant part of the term in which $\log. \frac{\text{H}}{e}$ is found repeated, the variable part of that term being here neglected. The second factor $\theta = 1 + \frac{2(\text{T}+t)}{1000}$. And the third $q=\frac{\text{H}-e}{\text{H}+e}$, where $e=h\,[1+\cdot 000185\,(\text{T}-t)]$. The quantity q is the first term of the following series, $\log. \frac{\text{H}}{e} = 2\text{M}\,q\,(1+\frac{1}{3}q^2+\frac{1}{5}q^4+\frac{1}{7}q^6+\&c.)$ M being $=\cdot 434294$. When the altitude exceeds 1000 metres, the second term of the series must be taken into the account. The demonstration of this formula is given at p. 312 of the above cited volume.

For a simple practical rule by Sir H. C. Englefield, the reader may consult Nicholson's Journal, N° 55, N. S.

After all, it must not be disguised that the principles of the geometrical method are established and sure, and that an extreme degree of exactness may often be obtained in this way by good instruments in the hands of a skilful observer: whereas the modifications of the atmosphere with regard to the effect which exhalations of various kinds, and the greater or less abundance of the electric matter, may have in expanding the air without changing its temperature, are not yet sufficiently known to render the corrections altogether so perfect as might be wished. These remain to be ascertained more accurately by future observations: mean while it should be remembered, that the elevations determined by the barometer are most to be depended upon when the extreme temperatures of the column of air do not greatly differ, and when the air is cold and dry.

The various circumstances affecting the density of the atmosphere and likely to occasion errors in barometrical admeasurements, such as the expansion from various degrees of heat, the change in the relation between the density and the compressing force, the variation in the force of gravity, the solution of moisture in the air, the diminution of the weight of mercury in the upper barometer, &c. have all been made the subjects of investigation in an elaborate paper by Mr. Professor Playfair, published in the Transactions of the Royal Society of Edinburgh, vol. I.

M. *Laplace* has proposed a method of combining the observations of the barometer with geographical measures, to determine in a more decisive manner the position of different places. This position, so far as it depends upon geographical admeasurement, is shewn by the intersection of two rectangular co-ordinates; one of which exhibits the distance from the first meridian or the longitude, and the other the distance from the equator, or the latitude. He supposes a third co-ordinate perpendicular to the two preceding ones, by which may be measured the vertical distance between the same point of intersection and the level of the sea. Thus he takes, for France, the level at Brest, where the mean height of the barometer is very nearly 76 centimetres, or about 29·92 of our standard inches. Then making in each place a great number of barometrical observations during a year or two, the mean between all these observations gives the elevation of the place proposed above the level of the sea. It is recommended to choose in each respective tract of country, for the level to which all the observations are to be referred, the mean height of the nearest river. Such a course of observations carefully made at different places with accurate barometers, might furnish results very interesting in the topography of different countries. Much, in this way, has already been done in various parts of the continent; and one of the tables in the 3d vol. of the "Trigonometrical Survey of England and Wales," furnishes numerous important results of this kind.

CHAPTER III.

On the Motion of Air when the Equilibrium of Pressure is removed.

510. In the present chapter we shall consider air (taken for a representative of elastic fluids in general) as acted upon by gravity equally and in parallel directions; and in a series of propositions enquire into the circumstances of its passage from a vessel into a vacuum, or from one vessel to another, in which the air is of less density.

Prop. *To determine with what velocity air will rush into a void space when impelled by its weight alone.*

This is manifestly analogous to the hydraulic problem of water flowing through an orifice in the bottom or side of a vessel (art. 439.): and the manner of invertigation there adopted will immediately apply to the present instance. For when the moving force, and the matter to be moved, vary in the same proportion, the velocity will continue the same; that is, since $F \propto BV$ (art. 216.) when $F \propto B$, V is constant. If, therefore, there be similar vessels of air, water, oil, or any other fluid, all of the height of a homogeneous atmosphere (art. 495.), they will be discharged through equal and similar holes with the same velocity: for, in whatever proportion the quantity of matter moving through the hole be varied by a change of density, the pressure which forces it out, acting in circumstances perfectly similar, varies in the same proportion. Hence it follows, that *air rushes from the atmosphere into a void with the velocity which a heavy body would acquire by falling from the top of a homogeneous atmosphere.*

Let us take, then, for the height of the homogeneous atmosphere $H = 27818$ feet (art. 496.); and if we put $g = 32\frac{1}{6}$ feet, as we have heretofore done, we shall have $V = \sqrt{(2gH)} = 1339$ feet nearly $= 8\sqrt{H}$ nearly, the velocity sought: no regard being here paid to the velocity which the air acquires after its issuing into the previous void by its continual expansion.

SCHOLIUM.

511. The first solution to this problem we recollect was given by Dr. Papin in the Phil. Trans. No. 184. See also the New Abridgment, part x. p. 334. It is deduced from this proposition, viz. that of different liquors under the same pressure, those specifically lighter must acquire a greater celerity, and their different velocities will be to one another as the roots of the specific gravities of the said liquors. "If, therefore," says Dr. Papin, "we would know what is the velocity of air when driven by any degree of pressure whatever, we need only find what would be the velocity of water under the same pressure, and then take the square roots of the specific gravities of these two fluids, because, as much as the square root of the specific gravity of water exceeds the square root of the specific gravity of the air, so much in proportion will the velocity of the air exceed that of the water. For example, when I would compute what would be the velocity of a bullet shot by the pneumatic engine, described in Phil. Trans. No. 179, I should first compute what was the velocity of the air itself that drove the bullet: I therefore observe that on this occasion the air sustains a pressure much about the same as that of water when its head is 32 feet high; now such water would spout out with a sufficient velocity to ascend 32 feet perpendicular, and therefore it has the velocity of 45 feet in a second. It remains, therefore, only to know the proportion of the gravity of air to that of water: this we have found not to be always the same; because the height, the heat, and the moisture of the atmosphere, are variable; yet we may say in general, that the ratio of the specific gravities of water and air is about 840 to 1. Taking, then, their square roots, which are 29 and 1, we may conclude that the velocity of air must exceed that of water 29 times; and so multiplying 45, the velocity of water, by 29, we shall find that the velocity of the air, driven by the whole pressure of the atmosphere, is about 1305 feet in a second."

This number, as Dr. Hutton remarks, is too small; for the mean pressure of the atmosphere is now known to be about that of a column of water of $33\frac{3}{4}$ feet, and this will give for its velocity $46\frac{12}{25}$ feet, instead of 45; which, multiplied by the 29, gives nearly 1348 feet: a determination which differs from that in the preceding article by about its 150th part.

512. Prop. *To determine the velocity with which the air of the atmosphere will rush into a space containing rarer air.*

Whatever the density of the rarer air may be, its elasticity, which varies as its density, will balance a proportional part of the pressure of the atmosphere; and it is the excess of this last only which is the moving force, the matter to be moved is the same as before. Let D, then, be the natural density of the air, and δ the density of the air contained in the vessel into which it is supposed to run: let P be the pressure of the atmosphere, and therefore equal to the force which impels it into a void, and π the force with which this rarer air would run into a void. Then (art. 491.) we have $P\delta = \pi D$, and $\pi = \frac{P\delta}{D}$. Also the moving force in the present case is $P - \pi$, or $P - \frac{P\delta}{D}$. Lastly, let V be the velocity of air of density D rushing into a void, and v the velocity with which it will rush into the rarefied air of density δ. Now, in this as well as other fluids, the pressures are as the squares of the velocities of efflux; therefore (art. 469. cor. 2.) $P : P - \frac{P\delta}{D} :: V^2 : v^2$, and by reduction we find $v = V \times \sqrt{\left(1 - \frac{\delta}{D}\right)}$; an expression for the velocity sought, not considering the resistance which the air of density D will experience from the inertia of that in the vessel, which it must displace in its motion.

COR. 1. Hence, it appears that there will always be a current into the vessel while δ is less than D.

COR. 2. Hence also, we learn the gradual diminution of the velocity as the vessel fills; for, because δ gradually increases, $1 - \frac{\delta}{D}$, and consequently the value of v continually diminishes.

513. PROP. *To determine the time* t *in seconds, in which the air of the atmosphere will flow into a given vessel from its state of vacuity, till the air in the vessel has acquired any proposed density* δ.

Let H be the height in feet due to the velocity V (art. 244.), s the solid content or capacity of the vessel in cubic feet, and a the area or section of the aperture in square feet, D representing, as before, the natural density of the air. Now, since the quantity of air to fill the vessel will depend upon its density and the capacity of the vessel conjointly, we may express it by Ds when the air is in its ordinary state, and by δs when it has acquired the density δ. To find the rate at which the vessel fills we must take the fluxion of the quantity δs, which will be $s\dot{\delta}$, because s is invariable. The velocity of influx at the first instant is V or $\sqrt{(2gH)}$ (art. 510.), and when the air in the vessel has acquired the density δ, that is, at the end of the time t, the velocity is $\sqrt{(2gH)} \times \sqrt{\left(1 - \frac{\delta}{D}\right)}$

or $\sqrt{\left(2g\mathrm{H}\cdot\frac{\mathrm{D}-\delta}{\mathrm{D}}\right)}$. Hence the rate of influx, which may be measured by the indefinitely small quantity of air which will enter during the time $\dot{t}$ with this velocity, will be denoted by $\sqrt{(2g\mathrm{H})}\times\sqrt{\frac{\mathrm{D}-\delta}{\mathrm{D}}}\times\mathrm{D}a\dot{t}=a\dot{t}\sqrt{[2g\mathrm{DH}(\mathrm{D}-\delta)]}$. Making these two values of the rate of influx equal, we have $a\dot{t}\sqrt{[2g\mathrm{DH}(\mathrm{D}-\delta)]}=s\dot{\delta}$, or $\dot{t}=\frac{s}{a\sqrt{(2g\mathrm{DH})}}\times\frac{\dot{\delta}}{\sqrt{(\mathrm{D}-\delta)}}$. Taking the fluents, we have $t=\frac{s}{\frac{1}{2}a\sqrt{(2g\mathrm{DH})}}\times\sqrt{(\mathrm{D}-\delta)}+\mathrm{C}$.

To determine the constant quantity C we must recollect that when $t=0$, $\delta=0$, and $\sqrt{(\mathrm{D}-\delta)}=\sqrt{\mathrm{D}}$: hence $\mathrm{C}=\sqrt{\mathrm{D}}$; and the correct fluent is

$$t=\frac{s}{a\sqrt{(\frac{1}{2}g\mathrm{DH})}}\times[\sqrt{\mathrm{D}}-\sqrt{(\mathrm{D}-\delta)}].$$

Cor. When $\delta=\mathrm{D}$ the motion ceases, in which case t becomes $\frac{s\sqrt{\mathrm{D}}}{a\sqrt{(\frac{1}{2}g\mathrm{DH})}}$ or $\frac{s}{a\sqrt{(\frac{1}{2}g\mathrm{H})}}$, or $\frac{s}{4a\sqrt{\mathrm{H}}}$ nearly, for the time of completely filling the vessel.

514. To illustrate this by an example in numbers, let us suppose the capacity of the vessel to be 8 cubic feet, or nearly a wine hogshead, and that the orifice by which the air of the ordinary density (which we shall make $=1$) enters, is an inch square, or $\frac{1}{144}$ of a foot. Here $4\sqrt{\mathrm{H}}=4\sqrt{27818}=668$, and $t=\frac{8}{\frac{1}{144}668}=\frac{1152}{668}=1''\cdot7245$. If the hole be only $\frac{1}{100}$ of a square inch, or the side $\frac{1}{10}$ of an inch, the time of completely filling the vessel will be nearly $172\frac{1}{2}$ seconds, or rather less than 3 minutes.

If we make the experiment with a hole cut in a thin plate, we shall find the time greater nearly in the ratio of 62 or 63 to 100. As is likewise the case with water flowing through small orifices (art. 458. 517.), and for similar reasons.

In like manner we can find the time necessary for bringing the air in the vessel to $\frac{3}{4}$ of the ordinary density. For the only variable part of the correct fluent above is $\sqrt{(\mathrm{D}-\delta)}$, which in this case becomes $\sqrt{(1-\frac{3}{4})}=\sqrt{\frac{1}{4}}=\frac{1}{2}$, whence $\sqrt{\mathrm{D}}-\sqrt{(\mathrm{D}-\delta)}=\frac{1}{2}$. And if the hole is a square, each side being $\frac{1}{10}$ of an inch, the time is $\frac{1}{2}$ of $172\frac{1}{2}$, that is, $86\frac{1}{4}$ seconds.

515. Prop. *Let the air in the vessel* ABCD (fig. 4. pl. XVIII.) *be compressed by a weight acting on the cover* AD, *which is move-*

able down the vessel, to determine the velocity of the air expelled at the aperture C.

The density D of the external air corresponds to its pressure P: and, since the immediate effect of the external pressure is to compress the air in the vessel and give it another density, let the additional pressure on the cover of the vessel be p, and the density of the air in the vessel be d; then shall we have $\text{P} : \text{P} + p :: \text{D} : d$; whence $p = \text{P} \cdot \frac{d-\text{D}}{\text{D}}$. Now, because the pressure which expels the air is the difference between the force which compresses the air in the vessel and that which compresses the external air, the expelling force is p; and because the quantities of motion are the forces which similarly produce them, we shall have $\text{P} : \text{P} \cdot \frac{d-\text{D}}{\text{D}} :: \text{MV} : mv$, where M and m express the quantities of matter expelled, V the velocity with which air rushes into a vacuum, and v the velocity required. But because the quantities of aerial matter which issue from the same orifice in an instant are as the densities and velocities jointly, we shall have $\text{MV} : mv :: \text{DVV} : dvv$. Consequently $\text{P} : \text{P} \cdot \frac{d-\text{D}}{\text{D}} :: \text{DV}^2 : dv^2$. And hence we deduce $v = \text{V}\sqrt{\frac{d-\text{D}}{d}}$.

COR. Another expression for the velocity may be obtained without considering the density: for since $\text{P} : \text{P} + p :: \text{D} : d$, therefore $d = \frac{\text{D}(\text{P}+p)}{\text{P}}$, and $d - \text{D} = \frac{\text{D}(\text{P}+p)}{\text{P}} - \text{D} = \frac{\text{D}(\text{P}+p) - \text{DP}}{\text{P}} = \frac{\text{D}p}{\text{P}}$; whence $\frac{d-\text{D}}{d} = \frac{\text{DP}}{\text{P}} \div \frac{\text{D}(\text{P}+p)}{\text{P}} = \frac{p}{\text{P}+p}$. Substituting this value of $\frac{d-\text{D}}{d}$ for it in the final value of v, it becomes $v = \text{V} \cdot \sqrt{\frac{p}{\text{P}+p}}$, which is both a simple and convenient expression.

516. PROP. *To enquire into the effect of the air's elasticity, when ascertaining the velocity with which the air will issue from a vessel into a vacuum.*

Let ABCD (fig. 4. pl. XVIII.) be a vessel containing air of any density D. This air is in a state of compression, and if the compressing force be removed it will expand, and its elasticity will diminish with the density. Now its elasticity in any state is measured by the force which keeps it in that state (art. 488.); and the force which keeps common air in its ordinary density is the pressure of the atmosphere, which (art. 485.) is at a medium the same with the weight of a column of mercury 29½ inches high. If, therefore, we suppose that this air, instead of being confined by the top of the vessel, is pressed down by a moveable piston carrying a cylinder of mercury of the same base, and 29½ inches high, its elasticity will balance this pressure just as it does the pressure of the atmosphere: and, since it is a fluid and propagates through every part and in every direction the pressure exerted on any one part, it will press on any small portion of the vessel by its elasticity, in the same manner as when loaded with this column. Hence, if this small portion of the vessel be removed, and a passage made into the void, the air will begin to flow out with the same velocity as it would flow out when impelled by its weight alone, or with the velocity acquired by falling from the top of the homogeneous atmosphere; that is, a velocity of 1339 feet nearly (art. 510.). But as soon as any air has passed through the orifice, the density of that remaining in the vessel is diminished, and its elasticity is diminished, consequently the expelling force is diminished: but the matter to be moved is diminished in the very same proportion as the density, the capacity of the vessel remaining unchanged: therefore, since the density and elasticity follow the same law, the quantity of matter moved will vary as the moving force, and the velocity will continue the same from the beginning to the end of the efflux.

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517. Hence, since the velocity with which the air issues out of such a vessel is *constant*, we may readily compare the velocity of theory in art. 510. with that found by experiment. But for this purpose we shall here describe the simple apparatus and experiments of Mr. Banks, in which a change of density is avoided. (Banks on the Power of Machines, p. 10.) A (fig. 8. pl. XVIII.) is a cask of known capacity, into the top of which is screwed an aperture *a* of a known area. The tube T*d*, recurve at *d*, is soldered or screwed into the top of the said cask. The hole *a* is stopped, and water poured into the tube T till it is full; at which time a quantity of water will have passed out of the hole at *d*, and condensed the air in the cask till its spring is equal to the weight of the water in the tube. At this time a cock placed over the tube T, sufficiently large to supply water as fast as it can descend into the vessel A, must be opened to keep the tube constantly filled: for this purpose one person must attend it, and another must open the aperture *a*, which need only have been closed by a finger, and he must measure the seconds from the moment that the finger is removed till the water flies out at the jet. Hence, from knowing the capacity of the vessel and the area of the jet, the velocity may be ob-

tained. If the tube Td should be continued nearly to the bottom of A, while A was filling with water, the length of the compressing column would be gradually diminished, and of consequence the pressure would be constantly changing: to avoid any irregularity this might produce, the open end of the tube is as near the top of the cask as is consistent with a free passage for the water.

Experiments. The vessel contained 15lb. 6oz. of water; from which we find its capacity 425·088 cubic inches. The area of the aperture a, through which the air is discharged, is ·0046 inches.

Exp. I. { The altitude of T above the cask 30 inches;
Time of expelling the air 33′, by several trials.

Exp. II. { The altitude of T 6 feet;
Time of expelling the air 21·3′, by several trials.

In the first experiment 425·088, the capacity of the cask, being divided by ·0046, the area of the hole, gives 92410·4 inches, for the length of the stream driven out in 33′. Hence $\frac{92410 \cdot 4}{12 \times 33} = 233 \cdot 3$ feet, the velocity per second.

From the second experiment we deduce, by a similar process, 361·6 feet, for the velocity per second: and, to shew the correspondence of this with the first, say, as $\sqrt{2\frac{1}{2}}$ (the head) :233·3 :: $\sqrt{6}$ (the head): 361·8 feet; differing from the former by only a fifth of a foot.

To compare the velocity thus found by experiment with that assigned by theory (art. 510.), we may say as $\sqrt{6} : 361 \cdot 6 :: \sqrt{33}$ (the height of a column of water equivalent in pressure to the atmosphere): 845·2 feet, the velocity with which the atmosphere would begin to enter into a vacuum. Making the allowance spoken of in art. 514. we shall have $1339 \times \cdot 63 = 843 \cdot 57$, agreeing as nearly with the experimental result as can reasonably be expected.

518. PROP. *To find the quantity of aerial matter which will be expelled from the orifice* C *of the vessel* ABCD (fig. 4.) *during any time* t, *and the density of the remaining air at the end of that time.*

In the element of time $\dot{t}$ there issues (by art. 513.) the capacity of air $a\dot{t}\sqrt{(2g\text{H})}$, the velocity V being constant (art. 516.), and consequently the *quantity* of air $ad\dot{t}\sqrt{(2g\text{H})}$. On the other hand, the quantity of air at the beginning of the efflux was $s\text{D}$, s being, as before, the capacity of the vessel; and when the air has acquired the density d, the quantity in the vessel is sd, and $s\text{D} - sd$ is the quantity expelled: consequently the quantity discharged in the element of time $\dot{t}$ must be the fluxion of $s\mathrm{D} - sd$; that is, $-s\dot{d}$. Hence we have the equation $ad\dot{t}\surd(2g\mathrm{H}) = -s\dot{d}$, and $\dot{t} = \frac{-s\dot{d}}{ad\surd(2g\mathrm{H})} = \frac{s}{a\surd(2g\mathrm{H})} \times -\frac{\dot{d}}{d}$. The fluent of this is $t = \frac{s}{a\surd 2g\mathrm{H}} \cdot$ hyp. log. d. Now this fluent must be so taken that when $t=0$, $d=\mathrm{D}$, or hyp. log. $\frac{\mathrm{D}}{d} =$ hyp. log. $1 = 0$. So that the correct fluent is $t = \frac{s}{a\surd(2g\mathrm{H})} \cdot$ hyp. log. $\frac{\mathrm{D}}{d} = \frac{s}{8a\surd\mathrm{H}} \cdot$ hyp. log. $\frac{\mathrm{D}}{d}$, nearly.

Cor. Hence it follows that the whole air of a vessel situated as in the proposition will not flow out of it into a void, in any finite time whatever.

419. Prop. *To determine the time when the vessel, instead of discharging its air into a void, emits it into air of a less density, that density remaining invariable during the efflux; as may be supposed the case when a vessel holding condensed air emits it into the surrounding atmosphere.*

Let the initial density of the air in the vessel be δ, and that of the atmosphere D. Then it is manifest that the expelling force is $\mathrm{P} - \frac{\mathrm{PD}}{\delta}$ (art. 512.); and that after the time t it is $\frac{\mathrm{P}d}{\delta} - \frac{\mathrm{PD}}{\delta}$. We have therefore $\frac{\mathrm{P}\delta - \mathrm{PD}}{\delta} : \frac{\mathrm{P}d - \mathrm{PD}}{\delta} :: \mathrm{MV} : mv :: \delta\mathrm{V}^2 : dv^2$: whence, by reduction, we have $v = \mathrm{V}\sqrt{\frac{\delta(d-\mathrm{D})}{d(\delta-\mathrm{D})}}$. From which equation we may learn, that when $d=\mathrm{D}$ the motion will be at an end; and that if $\delta=\mathrm{D}$ there can be no efflux.

Now, to find the relation between the time and the density, let H, as before, be the height due to the velocity V. Hence the height due to the velocity of efflux v must be $\mathrm{H} \cdot \frac{\delta(d-\mathrm{D})}{d(\delta-\mathrm{D})}$, and the small parcel of air which will flow out in the element of time $\dot{t}$ will be $ad\dot{t}\sqrt{\left(2g\mathrm{H} \cdot \frac{\delta(d-\mathrm{D})}{d(\delta-\mathrm{D})}\right)}$. And another expression for the same is, as in the preceding article, $-s\dot{d}$. Making these two values equal, we soon deduce the fluxionary equation $\dot{t} = \frac{s\surd(\delta - \mathrm{D})}{a\surd(2g\delta\mathrm{H})} \times \frac{-\dot{d}}{\surd(d^2 - d\mathrm{D})}$. The fluent of this, so corrected that when $t=0$, $d=\delta$, is

$$t = \frac{s}{a} \times \sqrt{\frac{\delta - D}{2g\delta H}} \times \text{hyp. log.} \left(\frac{\delta - \frac{1}{2}D + \sqrt{(\delta^2 - \delta D)}}{d - \frac{1}{2}D + \sqrt{(d^2 - dD)}} \right).$$

COR. When d becomes $= D$ the efflux is complete, and the expression for the time will then become

$$t = \frac{s}{a} \times \sqrt{\frac{\delta - D}{2g\delta H}} \times \text{hyp. log} \left(\frac{\delta - \frac{1}{2}D + \sqrt{(\delta^2 - \delta D)}}{\frac{1}{2}D} \right).$$

520. PROP. *Let the capacities of two vessels containing airs of different densities be* A *and* B; *if they communicate by a tube whose section is* a, *there will be a current from the vessel containing the denser air into that containing the rarer: it is proposed to find an expression for the time of efflux.*

Let P be the elastic force of the air in the vessel A, Q its density, and V its velocity; D being the density of the air in the vessel whose capacity is B. After the time t let the density of the air in A be q, its velocity v, the density of the air in B having become δ. The force expelling from A will then be $P - \frac{PD}{Q}$ at the first instant, and at the end of the time t it will be $\frac{Pq - P\delta}{Q}$. So that we shall have $\frac{PQ - PD}{Q} : \frac{Pq - P\delta}{Q} :: QV^2 : qv^2$; whence arises $v = V \cdot \sqrt{\frac{Q(q - \delta)}{q(Q - D)}}$; and we see that the motion will cease when $\delta = q$.

Now the capacities of the vessels being A and B respectively, we have for a second equation, including the densities, $AQ + BD = Aq + B\delta$; from which we find $\delta = \frac{A(Q - q) + BD}{B}$. This value of δ being substituted in the preceding value of v, transforms it to $v = V \cdot \sqrt{\frac{Q[B(q - D) - A(Q - q)]}{qB(Q - D)}}$; an equation which gives the relation between the velocity v and the density q.

In order to find the time when the air in A is reduced to the density q, it will be convenient to abridge the work by some substitutions: thus, make $Q(B + A) = M$, $BQ(D + Q) = N$, $BQ - BD = R$, and $\frac{N}{M} = m$. Then, proceeding as in the propositions immediately before this, we obtain the fluxionary equation

$$aq\dot{t}\sqrt{\left(2gH \cdot \frac{M(q - N)}{Rq}\right)} = \text{flux. of } (AQ - Aq) = -A\dot{q}.$$

This gives $\dot{t} = \frac{A\sqrt{R}}{a\sqrt{(2gMH)}} \times \frac{-\dot{q}}{\sqrt{(q^2 - mq)}}$; the fluent of which, so corrected that $t = 0$ when $q = Q$, is as below:

$$t = \frac{A}{a} \times \sqrt{\frac{R}{2gMH}} \times \text{hyp. log.} \left(\frac{Q - \frac{1}{2}m + \sqrt{(Q^2 - mQ)}}{q - \frac{1}{2}m + \sqrt{(q^2 - mq)}} \right)$$

In a manner not widely different from some of the propositions in this chapter might various other problems relative to the passage of air from one vessel to another be solved: but such problems are not of very frequent use in the common applications of Pneumatics to practice, and are therefore not inserted here. Some particular cases are considered in the article *Pneumatics*, Encyclo. Britan. as well as in Bossut's "Hydrodynamique" part ii. ch. 11 and 12, to which works the student, who wishes to investigate the subject further, is referred for other information.

CHAPTER IV.

On the Theory of Air Pumps, and Pumps for raising Water.

521. THE *Air-pump* is a machine formed for exhausting the air out of a proper vessel, and so to make what is called a vacuum: it is one of the most useful philosophical instruments whose operations depend upon the properties of the air. By the help of this machine the chief propositions relative to the weight and elasticity of the air are proved experimentally, in a simple and satisfactory manner. This machine is constructed in various ways, one of the best of which will be described in the second volume of this work. At present we shall describe one of a portable and convenient form; being well adapted to most of the purposes for which air-pumps are employed.

EFGH (fig. 1. pl. I. vol. II.) is a square table of wood; AA are two strong barrels or tubes of brass, firmly retained in their position by the cross-piece TT, which is pressed on them by screws OO, fixed on the tops of the brass pillars N, N. These barrels communicate with a cavity in the lower part D. At the bottom within each barrel is fixed a valve, opening upwards; and in each barrel a piston works, having a valve likewise opening upwards. The pistons are moved by a cog-wheel in the piece TT, turned by the handle B, of which wheel the teeth catch in the racks of the pistons C, C. PQ is a circular brass plate, having near its centre the orifice K of a concealed pipe that communicates with the cavity: in the piece D at V is a screw that closes the orifice of another pipe, for the purpose of admitting the external air when required. LM is a glass receiver, out of which the air is to be exhausted, and which has obtained the name of *receiver*, because it receives or holds the subjects on which the experiments are to be made. This receiver is placed on the plate PQ, which is previously covered with a wet sheep-skin, or smeared with wax, to prevent the air from insinuating under the edge of the glass.

When the handle B is turned, one of the pistons is raised, and the other depressed; consequently a void space is left between the raised piston and the lower valve in the correspondent barrel: the air contained in the receiver LM communicating with the barrel by the orifice K immediately raises the lower valve by its spring, and expands into the void space; and thus a part of the air in the receiver is extracted. The handle then, being turned the contrary way, raises the other piston, and performs the same act in its correspondent barrel: while, in the mean time, the first-mentioned piston being depressed, the air by its spring closes the lower valve, and, raising the valve in the piston, makes its escape. The motion of the handle being again reversed, the first barrel again exhausts, while the second discharges the air in its turn: and thus, during the time the pump is worked, one barrel exhausts the air from the receiver, while the other discharges it through the valve in its piston. Hence it is evident that the vacuum in the receiver of this air-pump (and the same may be said of all others) can never be perfect; that is, the air can never be entirely exhausted: for it is the elasticity of the air in the receiver that raises the valve, and forces air into the barrel; and the barrel at each exsuction can only take away a certain part of the remaining air, which is in proportion to the quantity before the stroke, as the capacity of the barrel to the sum of the capacities of the barrel, receiver, and communicating pipe.

522. Now, if we suppose no vapour from moisture, &c. to rise in the receiver, the degree of exhaustion after any number of strokes of the piston may be determined by knowing the respective capacities of the barrel and of the receiver, including the pipe of communication, &c. For, as we have seen above that every stroke diminishes the density in a constant proportion, namely, as much as the whole content exceeds that of the cylinder or barrel; the exhaustion will go on in a geometrical progression, the ratio of which is the same as that which the sum of the receiver and barrel together bears to that of the receiver: and this ratio of exhaustion will be followed until the elasticity of the included air is so far diminished by its rarefaction as to render it too feeble to push up the valve of the piston.

Let, then, the capacity of the barrel, receiver, and pipe of communication together be expressed by $b+r$, and that of the barrel alone by b, and let 1 represent the primitive density of the air in the pump: so shall we have

$b+r : r :: 1 : \frac{r}{b+r}$ = the density after 1 stroke of the piston,

$b+r : r :: \frac{r}{b+r} : \frac{r^2}{(b+r)^2}$ = density after 2 strokes,

$b+r : r :: \frac{r^2}{(b+r)^2} : \frac{r^3}{(b+r)^3}$ = density after 3 strokes;

$$\left.\begin{array}{l}\text{and the } n\text{th power of} \\ \text{the ratio } \frac{r}{b+r}\text{, that is}\end{array}\right\} \frac{r}{(b+r)^n} = d, \text{ density after } n \text{ strokes,}$$

From which we may easily find the density after any number of strokes, when the ratio of $b : r$, and consequently that of $b+r$ to r, is given.

From the same formula, $\frac{r^n}{(b+r)^n} = d$, we may also derive another for finding the number of strokes of the piston necessary to rarefy the air a number of times, or to give it a certain density d, the primitive density being 1. For the above equation, expressed logarithmically, is $n \times \log. \frac{r}{b+r} = \log. d$; or $n \times (\log. r - \log. b+r) = \log. d$: consequently, $n = \frac{\log. d}{\log. r - \log. (b+r)}$: in which expression d will be a fraction. If the number of times which the air is rarefied be expressed by N an integer, then the logarithmic equation will be $n = \frac{\log. N}{\log. (b+r) - \log. r}$.

A further reduction of the same theorem will furnish us with the proportion between the capacities of the receiver and the barrel, when the air is rarefied to the fractional density d by a definite number n of strokes of the piston. For since $\frac{r^n}{(b+r)^n} = d$, if we take the nth root of both members of the equation we shall have $\frac{r}{b+r} = \sqrt[n]{d}$. Thus, if d be equal $\frac{1}{177147}$, and the number of strokes $n = 11$; we shall find $\frac{\log. d}{11} = \log. \frac{1}{3}$; so that $r : b+r :: 1 : 3$, and $b : r :: 2 : 1$.

523. For the numerous uses of the air-pump the reader may consult the popular treatises on natural philosophy: we shall merely observe in this place, that the specific gravity of air may be accurately ascertained by means of this machine. The method is as follows: To the neck of a glass bottle, made in the form of a Florence flask, adapt a cap and valve opening outwards; screw it on the pump, and exhaust it to a known degree, which will be shewn by the gauge attached to the pump for that purpose: then, from the weight of the bottle before and after exhaustion, we have the weight of the exhausted air; and from the ratio of the height of the mercury in the gauge to the standard altitude we know the proportion which the exhausted part bears to the whole air originally in the vessel, whose weight is therefore known. Subtracting this weight from the weight of the vessel when full of air, there will remain the weight of the vessel itself: fill it with water and weigh it, and subduct the weight of the vessel from this weight; the remainder is the weight of a bulk of the same magnitude with the air which fills the vessel, and whose weight was also previously ascertained.

Following this method, it has been found by a mean of several experiments, that the specific gravity of air is to that of water as 1·222 to 1000, very nearly, when the barometer stands at 30 inches, and in the mean temperature of 55° of Fahrenheit's thermometer. This agrees with the result already given in art. 490.

Pumps for raising Water.

524. The term *Pump* is generally applied to a hydraulic machine for raising water by means of the pressure of the atmosphere. Of pumps there are a great many different sorts, some of the best of which will be described in the second volume of this work: at present we shall only speak of three or four of the most common, and shall give merely such a general description of their construction as will enable the student to understand the principles on which their operation depends.

The four kinds of pumps of which we shall now treat are, the *sucking pump*, the *lifting pump*, the *forcing pump*, and the *centrifugal pump:* of these the first three have some parts in common, and particularly the pistons and suckers; they will therefore be treated in a rather connected way: the properties of the centrifugal pump will be considered separately.

The *piston* is a body ABCD of circular base (figs. 5. 6. 13. pl. XVIII.), which may be moved through the interior part of the tube or body of the pump, filling it exactly as it moves along. The sucker E is moveable about a joint in such a manner as either to permit or to prevent the passage of the water, according as it presses upwards or downwards. In figs. 5. 6. there are likewise suckers in the pistons. FGHK (figs. 5. 13.) is another tube joined to the body of the pump, and is generally called the *pipe* or *sucking pipe:* its lower extremity is immersed in the water, of which we suppose RS is the horizontal surface.

525. The *sucking* pump is represented in fig. 5. In this

pump if we suppose a power P applied to the handle of the piston so as to raise it from I to C, the air contained in the space DVKHGFC tends by its spring to occupy the space that the piston leaves void: it therefore forces up the valve E, and enters into the body of the pump, its elasticity diminishing in proportion as it fills a greater space. Hence it will exert on the surface GH of the water a less effort than is made by the exterior air in its natural state upon the surrounding parts of the same surface RG, HS: and the excess of pressure on the part of the exterior air will cause the water to rise in the pipe GK to a certain height HN, such that the weight of the column GN, together with the spring of the super-incumbent air, shall just be a counterpoise to the pressure of the exterior air. At that time the sucker E closes of itself; and if the piston be lowered, the air contained between the piston and the base IV of the body of the pump having its density augmented as the piston is lowered, will at length have its density, and consequently its elasticity, greater than that of the exterior air: this difference of elasticity will constitute a force which will push the sucker L in the piston upwards, and some air will escape till the exterior and interior air are reduced to the same density. The sucker L then falls again: and if we again elevate the piston, the water will be raised higher in FGHK, for the same reason as before. Thus, after a certain number of strokes of the piston, the water will reach the body of the pump; where, being once entered, it will be forced at each stroke of the piston through the spout X: for the water above the piston will then press upon the sucker, and keep it shut whilst the piston is rising; so that a cylinder of water whose height is equal to the stroke OT of the piston (or the vertical distance through which it passes) will be raised by each upward motion and forced through the aperture X, provided it is of an adequate magnitude.

526. The *lifting pump* is represented in fig. 6. pl. XVIII. Its manner of operation is this: The piston PCD is here placed below the horizontal surface RS of the water, and when it is caused to descend it produces a vacuum between the sucker E (which is pushed down by the external air) and the base CD of the piston. The weight of the water, together with that of the exterior air about R and S, presses up the sucker L, and the water passes into the body of the pump: and when the water ceases to enter, the weight of the sucker L closes it. Then, if the piston be raised, it raises all the water above it, forces up the sucker E, and introduces the water into the part IVYX. When the piston is raised to its highest position, the sucker E is made to close by the super-incumbent water, and retains the fluid there until by a fresh stroke of the piston more water is forced up-

wards through the sucker E; that which was before in the upper part of the pump being expelled through a proper orifice or spout in the neighbourhood of X, in order to make way for a new supply. And so the operation is continued, and water delivered at every stroke of the piston.

527. The *forcing pump* unites in some measure the properties of the other two. The piston ABCD (fig. 13. pl. XVIII.) which here has no sucker, being elevated, rarefies the air in the space DGHVOC, and the water rises towards K: the subsequent descent of the piston forces some of the air in this space through the valve L: the next ascent of the piston closes the valve L, and raises the water in GK; and so on till the water passes through the sucker E and enters the space DIVOC. Then the piston being pushed down closes the sucker E, and some of the condensed air is forced through the valve L. A further stroke raises more water into the space DIVOC, and expels more air through L. At length the water reaches L, and the subsequent strokes raise it into the tube MO*mn*; from whence it is carried off by a spout, as in the other pumps. Or, if this pump be closed at *mn*, excepting a narrower pipe *ps*, then when the water is raised by the process just described to *or*, above the bottom *s* of the tube, the elastic force of the compressed air in the space *morn* will compel the water to issue from the aperture *p* in a continued stream or jet; thus forming an artificial fountain.

528. Let us now enquire into the fundamental properties of these machines. By means of the *lifting* pump, water may be elevated to any height we please, provided we employ a sufficient force. But the estimation of this force requires various considerations. We must have regard to the dimensions of the piston, the barrel of the pump, the height to which the water is to be raised, and the velocity with which it is elevated; besides the effects of friction, &c. At present, however, we shall not examine these particulars in all their extent; but shall confine ourselves to one of them. Now it is certain that the power necessary to raise the water to any proposed height must at least be capable of sustaining in equilibrio the pressure experienced by the base of the piston when it is kept at rest, and the fluid has attained the required height. This pressure, then, we proceed to estimate.

In general the power must be, at least, capable of sustaining the weight of a column of water which has for its base that of the piston, and for its altitude the distance between the surface RS of the water in the reservoir and the upper surface XY of that in the pump. For when the base DC (fig. 6.) of the piston is below the surface RS of the water in the reservoir, it is manifest that the power has not to sustain the pressure of the water contained between RS and DC; because that pressure is counterbalanced by that of the water surrounding the lower part of the pump, and which is transmitted by means of the inferior orifice of the pipe. The power, therefore, has only to sustain the pressure exerted upon the surface DC by the fluid comprised between RS and XY; which pressure (art. 386.) is equal to the weight of a column of water whose base is CD and altitude the vertical distance between RS and XY.

When the piston is above R′S′, the surface of the water in the reservoir, then it is evident the water contained between DC and R′S′ does not press the piston downwards. But, as in that case it can only be sustained above R′S′ by the pressure of the air upon the water surrounding the pump, and as this pressure is only capable of sustaining in equilibrium the contrary pressure of the air upon the surface XY, it follows that the surface DC of the piston is surcharged by a weight equivalent to the column which has DC for its base and CR′ for its altitude. And this pressure, joined to that which is exerted upon DC by the super-incumbent fluid between DC and XY, makes the whole pressure upon the piston, as before, equal to that of a column of water whose base is DC, and height the distance between XY and R′S′.

529. The *sucking pump* requires in its theory the aid of other principles. To judge of its effect a mere evaluation of the power will not suffice: we must enquire if under the proposed circumstances the water can possibly be raised to the piston, and made to pass through the sucker L; for in some cases the water will never pass a certain altitude, how many strokes soever we give to the piston. To understand this, conceive that the water has been actually raised to T (fig. 5. pl. XVIII.), and that the situation of the piston in the figure is the lowest which can be given to it: and for greater simplicity, suppose that the pump is of the same internal diameter throughout. It is obvious that the air comprised in the space CDTZ is of the same density and elasticity as the exterior air (at least dropping the consideration of the weight of the sucker L and the friction attending its motion); for if its spring were *less* the water would rise higher than ZT, and if it were *greater* it would raise the sucker L, and mix with the exterior air till both became of the same density. Suppose now that the play of the piston, or the distance through which it is raised or lowered at each stroke, is DO: then when the base CD is raised to QO, the air which previously occupied the space CDTZ will tend to expand and fill the space QOTZ; and if the water did not rise would actually be so expanded. Its elastic force would then be less than that of the natural air,

in the ratio of CDTZ to QQTZ (art. 489.), or of DT to OT. If, therefore, this elastic force, together with the weight of the column of water whose height is ZR, constitute a pressure equal to that of the atmosphere, or equal to the weight of a column of water of equal base and height at a medium 33 feet, there will be an equilibrium, and the water will not rise further: if this joint pressure is greater than that of 33 feet of water, the water cannot be retained so high; but if it is less than the column of 33 feet, the water will continue to rise in the pump.

530. From these considerations we may readily investigate a general theorem.

Let a, the altitude or vertical distance from the point O to the surface RS of the water in the reservoir, $p =$ OD the play of the piston, and x the distance OT: then we have DT $= x - p$, and ST the height of the point T will be $a - x$. Since the air contained in CDTZ has the same density and elasticity as the exterior air, its force may be measured by a column of water of the same base ZT and 33 feet high; and because when this air is so expanded as to fill the space QOTZ the elastic force will be less in the ratio of DT to OT, we shall have (rejecting the base of the column, as equally affecting every part of the process) this latter force expressed by the fourth term of this proportion, $x : x - p :: 33 : \frac{33}{x}(x - p)$. But the force which the water comprised between ZT and RS exerts in opposition to the exterior pressure of the air, is measured by the height $a - x$: consequently, the elastic force of the air in the space QOTZ, together with the weight of the water between ZT and RS, will be expressed by $\frac{33(x-p)}{x} + a - x$. Now in order that the water may always rise, this joint pressure must be less than the weight of a column of water of 33 feet by some variable quantity, which we will call y: so that the following equation must always obtain, viz. $\frac{33(x-p)}{x} + a - x = 33 - y$. The value of x deduced from this equation is ambiguous, being thus expressed:

$$x = \tfrac{1}{2}a + \tfrac{1}{2}y \pm \surd\,[(\tfrac{1}{2}a + \tfrac{1}{2}y)^2 - 33p].$$

Now, when the water stops and does not rise any further, y vanishes, and the equation becomes $x = \frac{1}{2}a \pm \surd\,(\frac{1}{4}a^2 - 33p)$; of which the two values are real, so long as $\frac{1}{4}a^2$ is greater than $33p$. Hence we conclude that *when one-fourth of the square of the greatest height of the piston above the surface of the water in the reservoir is greater than 33 times the play of the piston, there are always two points in the sucking pump where the water may stop* *in its motion;* and the pump must be reputed *bad* when the lowest point to which the piston can be brought is found between these two points.

But if $33p$ be greater than $\frac{1}{4}a^2$, the two values of x, when y is supposed $= 0$, become imaginary: so that in a pump so constructed it is impossible that y should vanish; that is, the pressure of the exterior air always prevails, and the water is not arrested in its passage. Hence we conclude, secondly, that *in order that the sucking pump may infallibly produce its effect, the square of half the greatest elevation of the piston above the water in the reservoir must always be less than 33 times the play of the piston.*

531. This general rule may also be easily deduced geometrically thus: Suppose the sucker or valve E be placed at the surface RS of the water (fig. 5.), the tube to be of uniform bore, and YS to be the height of a column of water whose pressure is equal to that of the atmosphere; that is, YS $= 33$ feet. Conceive the water raised by working to N: then the weight of the column of water SN, together with the elasticity of the air above it, exactly balances the pressure of the atmosphere YS. But the elasticity of the air in the space OM, (QO being the highest and CD the lowest situation of the piston,) is proportional to $\mathrm{YS}\cdot\frac{\mathrm{DN}}{\mathrm{ON}}$; and, consequently, in the case where the limit obtains, and the water rises no further, it will be $\mathrm{YS} = \mathrm{NS} + (\mathrm{YS}\cdot\frac{\mathrm{DN}}{\mathrm{ON}})$. Transposing NS, we have $\mathrm{YS} - \mathrm{NS}\ (= \mathrm{NY}) = \mathrm{YS}\cdot\frac{\mathrm{DN}}{\mathrm{ON}}$; whence $\mathrm{ON} : \mathrm{DN} :: \mathrm{YS} : \mathrm{YN}$; or, *dividendo*, $\mathrm{ON} - \mathrm{DN}\ (= \mathrm{DO}) : \mathrm{ON} :: \mathrm{YS} - \mathrm{YN}\ (= \mathrm{NS}) : \mathrm{YS}$; consequently $\mathrm{DO}\cdot\mathrm{YS} = \mathrm{ON}\cdot\mathrm{NS}$. Hence we see, that if SO, the distance of the piston in its highest position from the water, and OD, the length of the semi-stroke, or the play of the piston, be given, there is a certain determinate height, as SN, to which the water can be raised by the difference of the pressures of the exterior and interior air: for YS is to be considered as a constant quantity, and, of course, when OD is given, $\mathrm{ON}\cdot\mathrm{NS}$ is given likewise. To ensure, therefore, the delivery of water by the pump, the stroke must be such that the rectangle $\mathrm{OD}\cdot\mathrm{YS}$ may be greater than any rectangle that can be made of the parts of SO; that is, greater than the square of $\frac{1}{2}$ SO, by a well-known theorem.

Hence we deduce a practical maxim of the same import as the preceding, i.e. *No sucking pump can raise water effectually unless the play of the piston in feet be greater than the square of the greatest height of the piston, divided by* 132.

532. Resuming the equation $\frac{33(x-p)}{x} + a - x = 33 - y$, and finding thence the value of y, we have $y = \frac{x^2 - ax + 33p}{x}$. Now let AB (figs. 9. 10. pl. XVIII.) represent the greatest height of the piston above the surface of the water in the reservoir, and AD the play of the piston: suppose the different portions AP of the line AB to represent the successive values of x, and lay down upon the perpendiculars PM the values of y which correspond to these assumed values of x: so shall we have a curve MMO (fig. 9.) which, while $\frac{1}{4}a^2$ is greater than $33p$, will cut AB in two points I and I′, in such manner that the ordinates PM will lie on different sides of AB: the ordinates which are below AB shewing the positive values of y, and those which are above AB the negative values. We see, therefore, that so long as $\frac{1}{4}a^2$ is greater than $33p$ the pressure of the exterior air is strongest, until the water has attained the height BI′. At this point I′ it will stop (abstracting from the consideration of the motion acquired), because the value of y is $= 0$. But if the water by the motion it has acquired continues to rise till it reaches some point between I′ and I, it will not stop there, but will descend, if the sucker does not oppose its descending motion; because the value of y being there negative indicates that the pressure of the exterior air is weaker than the united pressures of the water and the internal air. If the water reach the point I it will stop there, for the same reason as it would at the point I′: but if once it gets above I, there is then no reason to fear that it will descend; for all the ordinates PM between I and A being positive, shew that in that portion of the pump the pressure of the external air exceeds the combined efforts of the internal air and water.

533. When, on the contrary, the value of $\frac{1}{4}a^2$ is less than that of $33p$, the curve (fig. 10.) will no where intersect the axis AB; all the ordinates are positive, and consequently the pressure of the external air is always the strongest. This confirms and illustrates what has been laid down in art. 530.

If the sucking pump were to be placed so high above the usual surface of the earth (as at the top of a high mountain), or so low beneath it (as in a deep mine), that the pressure of the atmosphere would be sensibly different from the assumed mean pressure equivalent to 33 feet of water, we must then in all the preceding investigation change the co-efficient 33 to that which would express the height in feet of the corresponding column of water. And these equivalent columns may always be ascertained by means of the height of the mercurial column in the barometer: the analogy being this;—as 29½ inches, the mean altitude of the mercurial column, to 33 feet, the mean height of the column of water; so is any other mercurial column in inches to its corresponding column of water in feet.

534. In the preceding calculus the pump has been supposed of uniform bore throughout: when this is not the case the solution is rendered somewhat more complex, but not difficult. To calculate the effort of the internal air when the water has not reached the body of the pump, having only attained the height HN, for example (fig. 5.), we must use this proportion: as the space QOVNMIQ.: the space CDVNMIC :: 33 feet : a fourth term, which being added to the weight of the column of water whose height is NH, ought again to be equal to $33 - y$, as before. Besides, when the sucking pipe FG is of a smaller diameter than the body of the pump, if the conditions which we have before specified obtain, the pump cannot fail to produce the proper effect; for the air is dilated with more facility in this latter case than when the whole is of the same internal diameter. We need only add on this point, that, if the length of the stroke in a uniform pump, which is requisite to render the machine effectual, be greater than can conveniently be made, it may be diminished *by contracting the diameter of the sucking pipe in the subduplicate ratio of the diminution of the length of the stroke.*

535. As to the effort of which the power ought to be capable to sustain the water at a determinate height YH (fig. 5.), it will be measured according to what we have said respecting the lifting pump (art. 528.), by the weight of a column of water whose base is equal to CD, and height that of XY above RS. Here, too, we drop the consideration of friction and the weight of the piston.

536. The velocity of the water flowing from the sucking pipe into the barrel should be equal to the velocity with which the piston moves. For if it be greater, less work will be done than the pump is competent to effect; and if it be less, a vacuum will be produced below the piston, which will therefore be moved upwards with great difficulty. If V be the velocity of the water in the sucking pipe, d the diameter of that pipe, D the diameter of the barrel or body of the pump, and v the velocity of the piston; then $\mathrm{V}\cdot\frac{d^2}{\mathrm{D}^2}$ will be the velocity of the water in the barrel, and we must have $v = \mathrm{V}\cdot\frac{d^2}{\mathrm{D}^2}$, if the machine be perfect. If h be the height of a column of water whose weight is equivalent to the pressure of the atmosphere, a the altitude of the water in the sucking pipe, x any other height to which it ascends in following the piston, $g = 32\frac{1}{6}$ feet; then will the moving

force $\propto h-x$, the quantity of matter moved $\propto x$, and, consequently, the accelerating force $\propto \frac{h-x}{x}$. Hence we have $v\dot{v} = g\left(\frac{h\dot{x}}{x} - \dot{x}\right)$; and, taking the fluents, there results v the velocity of the water $= \sqrt{[2g \times (h \cdot \text{hyp. log. } x - v)]}$. But when $x=a$, $v = 0$; consequently the correct fluent will be

$$v = \sqrt{\left\{2g \times h\left[\text{hyp. log.} \frac{x}{a} - (x-a)\right]\right\}}.$$

It may just be added here, that the measure we have all along given of the external force is only what is necessary for *balancing* the pressure of the water in the rising pipe. But in order that the pump may perform work it must *surmount* this pressure, and cause the water to issue at x with such a velocity that the required quantity of water may be delivered in a given time. This requires force, even although there were no opposing pressure; which would be the case if the main were horizontal. The water fills it, but it is at rest. In order that a gallon, for instance, may be delivered in a second, the whole water in the horizontal main must be put in motion with a certain velocity. This requires force. We must therefore always distinguish between the state of equilibrium and the state of actual working. It is the equilibrium only that we have considered, and no more is necessary for understanding the operation of the different species of pumps.

On the Centrifugal Pump.

537. The *centrifugal pump* consists of a vertical tube, and a horizontal arm of equal bore, or sometimes of two horizontal arms of less bore than the vertical pipe: that of which we shall speak has only one horizontal arm. In fig. 3. pl. XV. KL may represent the vertical tube, the end K of which is immersed in the water of the reservoir, and LI the horizontal arm communicating with the former. There is a conical valve at K opening upwards, and one at the end of the horizontal arm opening outwards. The whole machine being filled with water and turned swiftly round upon pivots at K and L, the arm LI retaining its horizontal position, the water will, when the motion is properly regulated, continue to be discharged from the moveable extremity I of the arm LI in an uninterrupted stream.

In order to investigate the most useful theorems relative to the operation of this machine, put $a =$ the length of the arm LI in feet, $l =$ the length of the leg KL in feet, $t =$ the time of a revolution in seconds, $g = 32\frac{1}{6}$ feet, the measure of the force of gravity, and $\pi = 3{\cdot}141593$ the circumference of a circle whose diameter is unity. Then, since the centrifugal force is as the velocity it generates in a unit of time, we shall have the centrifugal force of a particle $\dot{x}$ of the fluid at the distance x from L equal to $\dot{x}\left(\frac{2\pi x}{t}\right)^2 \div x = \frac{4\pi^2 x\dot{x}}{t^2}$, the fluent of which is $\frac{2\pi^2 x^2}{t^2} =$ the centrifugal motive force of the column x: which when $x = a$ becomes $\frac{2\pi^2 a^2}{t^2}$, for the whole centrifugal motive force of the water in the arm LI. Now the pressure of a column whose length is a will, *cæt. par.* vary as ga; hence, it will be $ga : a :: \frac{2\pi^2 a^2}{t^2} : \frac{2\pi^2 a^2}{gt^2} =$ the length of a column of water whose pressure is equivalent to the centrifugal force. If from this we deduct the altitude l of the vertical leg, the remainder $\frac{2\pi^2 a^2}{gt^2} - l$ will be the length of a column whose action would expel the water from the orifice I, with the same force as the whirling motion will occasion; and the column moved will be $a + l$, which will also denote the space through which the fluid is accelerated. Consequently the circumstances of the case before us are the same as if a constant head of water of the height $\frac{2\pi^2 a^2}{gt^2} - l$ impelled a column of water horizontally at the bottom, of the depth $a + l$. Hence the accelerative force φ is $\left(\frac{2\pi^2 a^2}{gt^2} - l\right) \div (a + l)$, and the space s is $a + l$; so that the velocity generated will, by the rules of dynamics, be $v = \sqrt{2\varphi g s} = \sqrt{\left[2g\cdot\left(\frac{2\pi^2 a^2}{gt^2} - l\right)\right]}$ $= 8{\cdot}0208\sqrt{\left(\frac{2\pi^2 a^2}{gt^2} - l\right)}$. Or, if we adapt the theorem to practical purposes, the co-efficient 8·0208 must (art. 458.) be multiplied by ·62 or ·63, and the equation will become $v = 5\sqrt{\left(\frac{2\pi^2 a^2}{gt^2} - l\right)}$, very nearly.

In this investigation we have paid no regard to the rotatory motion of the fluid moleculæ in the vertical tube, but have considered it as though it were indefinitely narrow.

538. It is manifest that l must never exceed 33 feet, because a greater column cannot be supported by the pressure of the atmosphere: so that this machine cannot raise water higher than the common sucking pump. Besides this, the time of a revolution of the arm has limits, between which alone the pump is effective, so that the investigation must be carried a little further.

When the centrifugal force is barely equal to the weight to be raised, the expression under the radical vanishes, and $v = 0$, or the water is stationary in the pump. In this case we evidently have $\frac{2\pi^2a^2}{gt^2} = l$; from which we find $t = \pi a\sqrt{\frac{2}{gl}} = \cdot 78336\frac{a}{\sqrt{l}}$, for the time of a revolution in seconds, when no work is done.

If the machine be effectual, the velocity of the stream can never exceed the velocity which a heavy body would acquire in falling down the difference between 33 feet and the height of the vertical leg: for a greater velocity than that would cause a vacuity in the machine which would not be supplied by the pressure of the atmosphere. Now, by the laws of falling bodies, the velocity acquired in falling through the height $33 - l$ is $\sqrt{[2g(33 - l)]}$, and, by what is done above, the velocity of efflux is $\sqrt{\left(\frac{2\pi^2a^2}{gt^2} - l\right)}$. Making these values equal, we have $\frac{2\pi^2a^2}{gt^2} - l = 33 - l$. From which we deduce $t = \pi a\sqrt{\frac{2}{33g}} = \cdot 78336\frac{a}{\sqrt{33}} = \cdot 13636a$.

If $l = 33$, then will the last value of t be the same as the preceding, and no discharge can take place; agreeably to the observations at the beginning of this article. In all other cases, having found the time of revolution when the water is stationary, we have only to diminish that time in the ratio of $\sqrt{33}$ to $\sqrt{l}$, and we shall have the time of revolution when the work done is *the greatest this pump will admit of.*

CHAPTER V.

On the Resistance of Fluids to Bodies moving in them.

539. THE force with which bodies moving in fluid media, as water, air, &c. are impeded and retarded in their motions, is usually termed the RESISTANCE of fluids: and as all our machines move either in water or in air, or both, it becomes a matter of importance in the theory of mechanics to enquire into the nature of this kind of force.

We know by experience that force must be applied to a body in order that it may move through a fluid, such as air or water; and that a body projected with any velocity is gradually retarded in its motion, and generally brought to rest. The analogy of nature makes us imagine that there is a force acting in the opposite direction, or opposing the motion; and that this force resides in, or is exerted by, the fluid. And the phenomena resemble those which accompany the known resistance of active beings, such as animals. Therefore we give to this supposed force the metaphorical name of RESISTANCE. We also know that a fluid in motion will hurry a solid body along with the stream, and that it requires force to maintain it in its place. A similar analogy makes us suppose that the fluid exerts force, in the same manner as when an active being impels the body before him; therefore we call this the IMPULSION *of a fluid.* And as our knowledge of nature informs us that the mutual actions of bodies are in every case equal and opposite, and that the observed change of motion is the only indication, characteristic, and measure, of the changing force, the forces are the same (whether we call them impulsions or resistances) when the relative motions are the same, and therefore depend entirely on these relative motions. The force, therefore, which is necessary for keeping a body immoveable in a stream of water, flowing with a certain velocity, is the same with what is required for moving this body with this velocity through stagnant water.

A body in motion appears to be resisted by a stagnant fluid, because it is a law of mechanical nature that force must be employed in order to put any body in motion. Now, the body cannot move forward without putting the contiguous fluid in motion, and force must be employed for producing this motion. In like manner, a quiescent body is impelled by a stream of fluid, because the motion of the contiguous fluid is diminished by this solid obstacle; the resistance, therefore, or impulse, no way differs from the ordinary communications of motion among solid bodies, at least in its nature; although it may be far more difficult to reduce the various circumstances to accurate computation, or to obtain all the requisite data on which to found the calculus.

540. The resistance which a body suffers from the fluid medium through which it is impelled depends on the velocity, form, and magnitude of the body, and on the inertia and tenacity of the fluid. For fluids resist the motion of bodies through them, 1. by the inertia of their particles; 2. by their tenacity, *i. e.* the adhesion of those particles: 3. by the friction of the body against the particles of the fluid. In perfect fluids the latter causes of resistance are very inconsiderable, and therefore are not taken into the account; but the former is always very considerable, and obtains equally in the most perfect as in the most imperfect fluids. And that the resistance varies with the velocity, shape, and magnitude of the moving body is sufficiently obvious.

We must carefully distinguish between *resistance* and *retardation:* resistance is the quantity of *motion*, retardation the quantity of *velocity*, which is lost; therefore, the retardations are as the resistances applied to the quantities of matter; and in the same body the resistance and retardation are proportional.

541. Prop. *To determine the force of fluids in motion, or the resistance of fluids against bodies moving in them.*

1. In fluids uniformly tenacious the resistance is as the velocity with which the body moves. For, since the cohesion of the particles of the fluid is always the same in the same space, whatever be the velocity, the resistance from this cohesion will be as the space described in a given time; that is, as the velocity.

2. In a fluid whose particles move freely without disturbing each other's motions, and which flows in behind as fast as a plane body moves forward, so that the pressure on every part of the body is the same as if the body were at rest, the resistance will be as the density of the fluid.

3. On the same hypothesis the resistance will be as the square of the velocity. For the resistance must vary as the number of particles which strike the plane in a given time, multiplied into the force of each against the plane; but both the number and

the force is as the velocity, and consequently the resistance is as the square of the velocity.

This proof supposes that after the body strikes a particle the action of that particle entirely ceases: whereas the particles, after they are struck, must necessarily diverge, and act upon the particles behind them; thus causing some difference between theory and experiment. This hypothesis, however, on account of its simplicity, is generally retained and corrected afterwards by deductions from actual experiments.

This ratio of the square of the velocity may be otherwise derived, thus.

It is evident that the resistance to a plane, moving perpendicularly through an infinite fluid, at rest, is equal to the pressure or force of the fluid on the plane at rest, and the fluid moving with the same velocity, and in the contrary direction, to that of the plane in the former case. But the force of the fluid in motion must be equal to the weight or pressure which generates that motion; and which, it is known, is equal to the weight or pressure of a column of the fluid, whose base is equal to the plane, and its altitude equal to the height through which a body must fall, by the force of gravity, to acquire the velocity of the fluid: and that altitude is, for the sake of brevity, called the altitude due to the velocity. So that, if a denote the area of the plane, v the velocity, and n the specific gravity of the fluid; then, the altitude due to the velocity v being $\frac{v^2}{2g}$, the whole resistance, or motive force m, will be $a \times n \times \frac{v^2}{2g} = \frac{anv^2}{2g}$; g being $32\frac{1}{6}$ feet. And hence, *cæteris paribus*, the resistance is as the square of the velocity.

4. If the direction of the motion, instead of being perpendicular to the plane, as above supposed, be inclined to it in any angle, then the resistance to the plane, in the direction of the motion, as assigned above, will be diminished in the triplicate ratio of radius to the sine of the angle of inclination, or in the ratio of 1 to s^3, where s is the sine of inclination.

For AB (fig. 7. pl. XVIII.) being the direction of the plane, and BD that of the motion, ABD the angle whose sine is s; the number of particles or quantity of the fluid which strikes the plane will be diminished in the ratio of 1 to s; and the force of each particle will likewise be diminished in the same ratio: so that, on both these accounts, the whole resistance will be diminished in the ratio of 1 to s^2; that is, in the duplicate ratio of radius to the sine of ABD. But further, it must be considered that this whole resistance is exerted in the direction BE perpendicular to the plane; and any force in direction BE is to its effect in direction AE parallel to BD as AE to BE, or as 1 to s. Consequently, on all these accounts, the resistance in the direction of the motion is diminished in the ratio of 1 to s^3. And if this be compared with the result of the preceding step, we shall have for the whole resistance, or the motive force on the plane, $m = \frac{anv^2s^3}{2g}$.

5. If w represent the weight of the body whose plane face a is resisted by the absolute force m, then the retarding force $f = \frac{m}{w} = \frac{anv^2s^3}{2gw}$.

6. And, if the body be a cylinder whose face or end is a, and diameter d, or radius r, moving in the direction of its axis; then, because $s = 1$, and $a = \pi r^2 = \frac{1}{4}\pi d^2$, where $\lambda = 3{\cdot}141593$, the resisting force m will be $= \frac{\pi nd^2v^2}{8g} = \frac{\pi nr^2v^2}{2g}$; and the retarding force $f = \frac{\pi nd^2v^2}{8gw} = \frac{\pi nr^2v^2}{2gw}$.

7. This is the value of the resistance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of motion. But were its face a conical surface, or an elliptic section, or any other figure every where equally inclined to the axis, the sine of inclination being s; then, the number of particles of the fluid striking the face being still the same, but the force of each, opposed to the direction of motion, diminished in the duplicate ratio of radius to the sine of inclination, the resisting force m would be $\frac{\pi nd^2v^2s^2}{8g} = \frac{\pi nr^2v^2s^2}{2g}$.

But if the body were terminated by an end or face of any other form, as a spherical one, or such like, where every part of it has a different inclination to the axis; then a further investigation becomes necessary, as in the following proposition,

542. Prop. *To determine the resistance of a fluid to any body moving in it, having a curved end, as a sphere, a cylinder with a hemispherical end, &c.*

1. Let BEAD be a section through the axis CA of the solid, moving in the direction of that axis. To any point of the curve draw the tangent EG, meeting the axis produced in G: also draw the perpendicular ordinates EF, ef indefinitely near to each other; and draw ae parallel to CG. Fig. 11. pl. XVIII.

Putting CF $= x$, EF $= y$, BE $= z$, $s =$ sine $\angle$ G to radius 1; then $2\pi y$ is the circumference whose radius is EF, or the circumference described by the point E, in revolving about the axis

CA; and $2\pi y \times$ Ee, or $2\pi y\dot{z}$, is the fluxion of the surface, or it is the surface described by Ee, in the said revolution about CA; which is the quantity represented by a in art. 4. of the last problem: hence $\frac{nv^2s^3}{2g} \times 2\pi y\dot{z}$, or $\frac{\pi nv^2s^3}{g} \times y\dot{z}$ is the resistance on that ring, or the fluxion of the resistance to the body, whatever the figure of it may be: the fluent of which will be the resistance required.

2. In the case of a spherical shape; putting the radius CA or CB $= r$, we have $y = \sqrt{(r^2 - x^2)}$, $s = \frac{\text{EF}}{\text{EG}} = \frac{\text{CF}}{\text{CE}} = \frac{x}{r}$, and $y\dot{z}$ or EF $\times$ Ee $=$ CE $\times$ $ae = r\dot{x}$; therefore the general fluxion $\frac{\pi nv^2}{g} \cdot s^3 y\dot{z}$ becomes $\frac{\pi nv^2}{g} \cdot \frac{x^3}{r^3} \cdot r\dot{x} = \frac{\pi nv^2}{gr^2} \cdot x^3\dot{x}$; the fluent of which, or $\frac{\pi nv^2}{4gr^2} x^4$, is the resistance to the spherical surface generated by BE. And when x or CF is $= r$ or CA, it becomes $\frac{\pi nv^2r^2}{4g}$ for the resistance on the whole hemisphere; which is also equal to $\frac{\pi nv^2d^2}{16g}$, where $d = 2r$ the diameter.

3. But the perpendicular resistance to the circle of the same diameter d or BD, by art. 6. of the preceding problem, is $\frac{\pi nv^2d^2}{8g}$; which, being double the former, shews that *the resistance to the sphere is just equal to half the direct resistance to a great circle of it, or to a cylinder of the same diameter.*

4. Since $\frac{1}{6}\pi d^3$ is the magnitude of the globe; if N denote its density or specific gravity, its weight w will be $= \frac{1}{6}\pi d^3\text{N}$, and therefore the retardive force f or $\frac{m}{w} = \frac{\pi nv^2d^2}{16g} \cdot \frac{6}{\pi\text{N}d^3} = \frac{3nv^2}{8g\text{N}d}$; which is also $= \frac{v^2}{2gs}$ by art. 243. Hence then $\frac{3n}{4\text{N}d} = \frac{1}{s}$, and $s = \frac{\text{N}}{n} \cdot \frac{4}{3}d$; which is the space that would be described by the globe, while its whole motion is generated or destroyed by a constant force which is equal to the force of resistance, if no other force acted on the globe to continue its motion. And if the density of the fluid were equal to that of the globe, the resisting force is such, as, acting constantly on the globe without any other force, would generate or destroy its motion in describing the space $\frac{4}{3}d$, or $\frac{4}{3}$ of its diameter, by that accelerating or retarding force.

5. Hence the greatest velocity that a globe will acquire by descending in a fluid, by means of its relative weight in the

fluid, will be found by making the resisting force equal to that weight. For, after the velocity is arrived at such a degree that the resisting force is equal to the weight that urges it, it will increase no longer, and the globe will afterwards continue to descend with that velocity uniformly. Now, N and n being the separate specific gravities of the globe and fluid, $N - n$ will be the relative gravity of the globe in the fluid, and therefore $w = \frac{1}{6}\pi d^3 (N - n)$ is the weight by which it is urged; also $m = \frac{\pi n v^2 d^2}{16g}$ is the resistance; consequently $\frac{\pi n v^2 d^2}{16g} = \frac{1}{6}\pi d^3 (N - n)$ when the velocity becomes uniform: from which equation is found $v = \sqrt{\left(2g \cdot \frac{4}{3}d \cdot \frac{N n}{n}\right)}$, for the said uniform or greatest velocity.

By comparing this value of v with those in arts. 216 and 243, it will appear that the greatest velocity is equal to the velocity generated by the accelerating force $\frac{N-n}{n}$ in describing the space $\frac{4}{3}d$, or equal to the velocity generated by gravity in freely describing the space $\frac{N-n}{n} \cdot \frac{4}{3}d$.—If $N = 2n$, or the specific gravity of the globe be double that of the fluid, then $\frac{N-n}{n} = 1 =$ the natural force of gravity; and then the globe will attain its greatest velocity in describing $\frac{4}{3}d$, or $\frac{4}{3}$ of its diameter.—It is further evident, that if the body be very small it will soon acquire its greatest velocity, whatever its density may be.

Exam. If a leaden ball of 1 inch diameter descend in water, and in air of the same density as at the earth's surface, the three specific gravities being as $11\frac{1}{3}$, and 1, and $\frac{3}{2500}$. Then $v = \sqrt{(2 \cdot 32\frac{1}{6} \cdot \frac{4}{36} \cdot 10\frac{1}{3})} = \frac{1}{9}\sqrt{(31 \cdot 193)} = 8{\cdot}5944$ feet is the greatest velocity per second the ball can acquire by descending in water. And $v = \sqrt{(4 \cdot \frac{193}{12} \cdot \frac{4}{36} \cdot \frac{34}{3} \cdot \frac{2500}{3})}$ nearly $= \frac{50}{9}\sqrt{\frac{34 \cdot 193}{3}} = 259{\cdot}82$, is the greatest velocity it can acquire in air.

But if the globe were only $\frac{1}{100}$ of an inch diameter, the greatest velocities it could acquire would be only $\frac{1}{10}$ of these, namely $\frac{86}{100}$ of a foot in water, and 26 feet nearly in air; and if the ball were still further diminished the greatest velocity would also be diminished, and that in the subduplicate ratio of the diameter of the ball.

SCHOLIUM.

543. It appears from the third step of the preceding article, that the resistance to the motion of a cylinder moving in the direction of its axis is double to that of a globe of equal dia-

meter; and, in experiments, when the bodies move slow, this will nearly hold in water, but more accurately in air; because its particles move more freely than those of water, and less disturb each other's motions: but when the motion is more rapid, considerable aberrations will occur; both from the mutual disturbance of the particles, and from the fluid not flowing in so fast behind as the body moves forward: in the air, also, a new cause of aberration will arise, from the condensation of the fluid before the body. Sir Isaac Newton supposes, that in a continuous non-elastic fluid, infinitely compressed, the resistances of a sphere and cylinder of equal diameters are equal: but this appears to be an error in theory as well as in fact; for the lemma (Lemma 5. book ii. § 7. *Princip.*) on which he founds his inference, has been justly called in question. When the motion is slow in water, the fluid may be conceived to be nearly of that nature which Newton supposes; yet the resistances are almost as coincident with theory as when the motion is in air: thus M. Borda found the resistance of a sphere moving in water to be to that of its greatest circle as 1 to 2·508, and in air the resistances were as 1 to 2·45. The experiments of Dr. Hutton in air give the resistances as 1 to $2\frac{1}{3}$, at a mean.

The reason that experiments give the ratio of the resistances greater than that of 2 to 1 seems to be this; in theory it is supposed that the action of every particle of the fluid ceases the instant it makes its impact on the solid; but this is not actually the case, as we have before observed (art. 541.): and since the particles, after impact on the sphere, slide along the curved surface, and hence escape with more facility than along the face of the cylinder, the error will be greater in the cylinder; that is, the greater resistance will exceed theory more than the less. It is also to be observed, that the difference between the resistances of the globe and cylinder, in water, is greater than in air; which is directly contrary to what might be inferred from Newton's reasoning, which supposes them equal in a continuous fluid, but in the ratio of 1 to 2 in a rare fluid.

544. LEMMA. If a given angle be divided into two parts, the product, or solid, contained under the square of the sine of the one part and the sine of the other will be a *maximum* when the tangent of the former is double the tangent of the latter, or, when the sine of the difference of the parts is one-third of the sine of the whole given angle.

This is a particular case of Prob. v. p. 502, *Simpson's Fluxions*; and is well known to mathematicians.

545. PROP. *Suppose that a plane* ABC (fig. 12. pl. XVIII.) *moving with a velocity and direction represented by* BD, *is acted upon by a fluid whose particles move with a velocity represented*

by DB, *and in directions parallel to that line; it is proposed to determine the angle of inclination* ABD *so that the effect of the fluid may be the greatest possible.*

Since a particle impinging on the plane at B moves through the space DB in the time that the plane itself would pass from the position *abc* to ABC, it is manifest that the distance D*e* of the said particle from the plane (produced) at the beginning of that time will be the measure of the relative celerity with which the particles of the fluid approach the plane in a direction perpendicular to it; and consequently (art. 469.) that the force of the stream in that direction will $\propto$ De^2: whence, by the resolution of forces, the efficacy in the proposed direction BH will $\propto$ De^2 × sin ABH $\propto$ De^2 × sin *ab*H.

Now the angle *b*BD being given, as well as the sides B*b*, BD, containing that angle, the remaining angle B*b*D will be known, as well as the side D*b*: of consequence, D*e* being the sine of the angle D*be* to the given radius B*b*, the effect De^2 × sin *ab*H will be a maximum, when $\sin^2$ D*be* × sin *ab*H is a maximum; that is (by the Lemma), when sin (D*ba* ∽ *ab*H) = $\frac{1}{3}$sin B*b*D: whence the difference being given, the angles themselves will be known. The geometrical construction is very simple: thus, having from the centre *b* with any radius described the arc *mr*, on *rb* (produced if necessary) let fall the perpendicular *mp*; take *pq* = $\frac{1}{3}$*mp*, and draw *qs* parallel to *pr*, cutting the circle in *s*: then bisect the arc *ms* by the line *bae*, and the thing required is done. For the sine *sv* of *sr*, that is, of the difference of the angles D*ba*, *ab*H, is $\frac{1}{3}$ of *mp*, the sine of the whole given angle B*b*D, as it ought to be.

546. To obtain a general theorem expressed algebraically, let the velocity *b*B of the plane be put $= v$, and that of the fluid $= \mathrm{V}$; also let the angle DB*b* be called B: and having drawn BFL perpendicular to the plane, or to *b*F*e*, put *b*F $= x$, and BF $= y$. Then, because FB and FL are tangents of the angles F*b*B, F*b*L, to the common radius *b*F, it follows (from the lemma) that FL = 2BF = $2y$; whence, if LR and DQ be perpendicular to HQ, we have, by similar triangles, B*b* : BF : : BL : BR; that is, $v : y : : 3y : \frac{3y^2}{v}$ = BR; and consequently *b*R = BR − B*b* $= \frac{3y^2 - v^2}{v}$. Likewise, B*b* : *b*F : : BL : LR; that is, $v : x : : 3y : \frac{3xy}{y}$ = LR. But DQ = V sin B, and *b*Q = V cos B − v, we have again, by similar triangles, DQ : *b*Q : : LR : *b*R, or V sin B : V cos B − $v : : \frac{3xy}{v} : \frac{3y^2 - v^2}{v}$. Multiplying the means and extremes of this analogy, we obtain $3y^2 - v^2 = \frac{\mathrm{V}\cos\mathrm{B} - v}{\mathrm{V}\sin\mathrm{B}} \cdot 3xy$.

Substituting in this equation for v^2 its equal x^2+y^2, completing the square and reducing, we at length find

$$\frac{x}{y}=\sqrt{\left[2+\tfrac{9}{4}\left(\frac{\text{V}\cos\text{B}-v}{\text{V}\sin\text{B}}\right)^2\right]}-\frac{3(\text{V}\cos\text{B}-v)}{2\text{V}\sin\text{B}}.$$

And this equation manifestly expresses the natural tangent of the angle bBF, or the cotangent of the required angle FbH.

547. COR. 1. If the given angle DBb be a right angle (as is the case when the wind strikes against the sails of a windmill), then is $\sin\text{B}=1$, and $\cos\text{B}=0$, the expression for the tangent of bBF (which is here equal to the angle of inclination ABD) will become $\sqrt{\left(2+\frac{9v^2}{4\text{V}^2}\right)}+\frac{3v}{2\text{V}}$. This, if v be taken $=0$, or the plane be supposed at rest, will be barely $=\sqrt{2}$, answering to an angle of 54° 44′. But if the velocity of the plane be supposed $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, or $\frac{1}{1}$, of that of the medium or stream, then the angle ABD, found from this theorem, will be equal to 58° 14′, 61° 27′, 63° 26′, 66° 58′ or 74° 19′ respectively: so that the greater the velocity of the plane, the greater also will be the angle of inclination.

Hence it appears, that the sails of a windmill, in order that the effect may be the greatest, ought to be more turned towards the wind in the extreme parts where the motion is swiftest than in the parts nearer to the axis of motion; in such a manner that the tangent of the axle formed by the direction of the wind and the sail may every-where be equal to the expression $\sqrt{\left(2+\frac{9v^2}{4\text{V}^2}\right)}+\frac{3v}{2\text{V}}$; the velocity v being proportional to the distance from the axis of motion, and increasing till, at the extremity of the sail, it is sometimes equal to V, or even exceeds it.

548. COR. 2. If the angle DBA, which the direction of the stream makes with the plane, be given, instead of the angle DBH or DBb; it will then appear that the effect will in that case be a maximum when sin ABH (the angle made by the plane and the direction of its motion): sin DBA : : ⅓BD : Bb.

For the force in the direction FB varying as De^2, its effect in the direction BH will $\propto \text{D}e^2\cdot\frac{\text{BF}}{\text{B}b}\propto\frac{\text{D}e^2\cdot\text{E}e}{\text{B}b}$. Now DB, B$b$, and the angle DBE, being given, DE is thence given. And it is well known (see Simpson's Geom. theor. 17. of max. and min.) that the solid of the square of one part of a line into the other part is a maximum, when the former part is the double of the latter. Consequently De must be $=2$Ee; so that Ee, or its equal BF, will be ⅓DE.

But, sin B*b*F : rad. : : BF ($=\frac{1}{3}$DE) : B*b*,
and rad. : sin DBA : : BD : DE,
whence, *componendo*, sin B*b*F : sin DBA : : $\frac{1}{3}$BD : B*b*.

549. Cor. 3. The proportion in the preceding corollary can only obtain when B*b* is equal to or greater than $\frac{1}{3}$DE. For, when B*b* is less than $\frac{1}{3}$DE, E*e* (which is always less than B*b*) cannot be equal to $\frac{1}{3}$DE; but will approach the nearest to it when BF coincides with B*b*, that is, when the angle F*b*H or ABH is of 90°; and in this case the effect will be a maximum when the direction of the motion is perpendicular to the plane. If the given angle DBA be a right angle (which appears to be the most advantageous, because then DE = DB), it follows that sin ABH will be to radius as $\frac{1}{3}$ of the velocity of the stream to the velocity of the plane or sail. Hence, *if the force of the wind be capable of producing a degree of velocity in a ship greater than $\frac{1}{3}$ of its own velocity, it is evident that the ship may run swifter upon an oblique course than when she sails directly before the wind.* If the velocity be to that of the wind as 1 to 3, and the course be 109° 28′, the force of the wind upon the vessel to promote its motion will be greater than the force in a direct course of 180°, in the ratio of $\sqrt[3]{32}$ to $\sqrt[3]{27}$, or of 3·1748 to 3. See *Maclaurin's Fluxions*, art. 919.

550. Prop. *To determine the relations of velocity, space, and time, of a ball moving in a fluid, in which it is projected with a given velocity.*

1. Let $a =$ the first velocity of projection, x the space described in any time t, and v the velocity then. Now, by step 4, art. 542. the accelerative force $f = \frac{3nv^2}{8g\text{N}d}$; where N is the density of the ball, n that of the fluid, and d the diameter. Therefore the general equation $v\dot{v} = gf\dot{s}$ becomes $v\dot{v} = \frac{-3nv^2}{8\text{N}d}\dot{x}$; and hence $\frac{\dot{v}}{v} = \frac{-3n}{8\text{N}d}\dot{x} = -b\dot{x}$, putting b for $\frac{3n}{8\text{N}d}$. The correct fluent of this is $\log. a - \log. v$, or $\log. \frac{a}{v} = bx$. Or, putting $c =$ 2·718281828, the number whose hyp. log. is 1, then is $\frac{a}{v} = c^{bx}$, and the velocity $v = \frac{a}{c^{bx}} = ac^{-bx}$.

2. The velocity v at any time being the c^{-bx} part of the first velocity, therefore the velocity lost in any time will be the $1 - c^{-bx}$ part, or the $\frac{c^{bx}-1}{c^{bx}}$ part of the first velocity.

Exam. 1. If a globe be projected with any velocity in a medium of the same density with itself, and it describe a space equal to $3d$ or 3 of its diameters. Then $x = 3d$, and $b = \frac{3n}{8\text{N}d} = \frac{3}{8d}$; therefore $bx = \frac{9}{8}$, and the velocity lost is $\frac{c^{x} - 1}{c^{bx}} = \frac{2{\cdot}08}{3{\cdot}08}$, or nearly $\frac{2}{3}$ of the projectile velocity.

EXAM. 2. If an iron ball of 2 inches diameter were projected with a velocity of 1200 feet per second; to find the velocity lost after moving through any space, as suppose 500 feet of air: we should have $d = \frac{2}{12} = \frac{1}{6}$, $a = 1200$, $x = 500$, $\text{N} = 7\frac{1}{3}$, $n = {\cdot}0012$; and therefore $bx = \frac{3nx}{8\text{N}d} = \frac{3.12.500.3.6}{8.22.10000} = \frac{81}{440}$, and $v = \frac{1200}{c^{\frac{81}{440}}} = 998$ feet per second; having lost 202 feet, or nearly ^{1} of its first velocity.

EXAM. 3. If the earth revolved about the sun, in a medium as dense as the atmosphere near the earth's surface; and it were required to find the quantity of motion lost in a year. Then, since the earth's mean density is about $4\frac{1}{2}$, and its distance from the sun 12000 of its diameters, we have $24000 \times 3{\cdot}1416 = 75398$ diameters $= x$, and $bx = \frac{3.75398.12.2}{8.10000.9} = 7{\cdot}5398$; hence $\frac{c^{bx} - 1}{c^{bx}} = \frac{1880}{1881}$ parts are lost of the first motion in the space of a year, and only the $\frac{1}{1881}$ part remains.

3. To find the time t; we have $\dot{t} = \frac{\dot{s}}{v} = \frac{\dot{x}}{v} = \frac{c^{bx}\dot{x}}{a}$. Now, to find the fluent of this, put $z = c^{bx}$; then is $bx = \log. z$, and $b\dot{x} = \frac{\dot{z}}{z}$, or $\dot{x} = \frac{\dot{z}}{bz}$; consequently $\dot{t}$ or $\frac{c^{bx}\dot{x}}{a} = \frac{z\dot{x}}{a} = \frac{\dot{z}}{ab}$; and hence $t = \frac{z}{ab} = \frac{c^{bx}}{ab}$. But as t and x vanish together, and when $x = 0$, the quantity $\frac{c^{bx}}{ab}$ is $= \frac{1}{ab}$; therefore, by correction, $t = \frac{c^{bx} - 1}{ab} = \frac{1}{bv} - \frac{1}{ba} = \frac{1}{b}\left(\frac{1}{v} - \frac{1}{a}\right)$ the time sought; where $b = \frac{3n}{8\text{N}d}$, and $v = \frac{a}{c^{bx}}$ the velocity.

551. PROP. *To determine the relations of space, time, and velocity, when a globe descends, by its own weight, in an infinite fluid.*

The foregoing notation remaining, viz. d = diameter, N and n the density of the ball and fluid, and v, s, t, the velocity, space, and time, in motion; we have $\frac{1}{6}\pi d^3$ = the magnitude of the ball, and $\frac{1}{6}\pi d^3\,(\text{N} - n)$ = its weight in the fluid, also $m =$

$\frac{\pi n d^2 v^2}{16g}$ = its resistance from the fluid; consequently $\frac{1}{6}\pi d^3(\mathrm{N}-n) - \frac{\pi n d^2 v^2}{16g}$ is the motive force by which the ball is urged; which being divided by $\frac{1}{6}\pi\mathrm{N}d^3$, the quantity of matter moved, gives $f = 1 - \frac{n}{\mathrm{N}} - \frac{3nv^2}{8g\mathrm{N}d}$ for the accelerative force.

2. Hence $v\dot{v} = gf\dot{s}$, and $\dot{s} = \frac{v\dot{v}}{gf} = \frac{\mathrm{N}v\dot{v}}{g(\mathrm{N}-n) - \frac{3n}{8d}v^2} = \frac{1}{b} \times \frac{v\dot{v}}{a-v^2}$, putting $b = \frac{3n}{8\mathrm{N}d}$, and $\frac{1}{a} = \frac{3n}{g \cdot 8d(\mathrm{N}-n)}$, or $ab = g$ nearly; the fluent of which is $s = \frac{1}{2b} \times$ log. of $\frac{a}{a-v^2}$, an expression for the space s, in terms of the velocity v, when s and v begin together.

3. To determine v in terms of s, put $c = 2{\cdot}718281828$; then, since the log. of $\frac{a}{a-v^2} = 2bs$, therefore $\frac{a}{a-v^2} = c^{2bs}$, or $\frac{a-v^2}{a} = c^{-2bs}$; and hence $v = \sqrt{(a-ac^{-2bs})}$, the velocity sought.

4. The greatest velocity is to be found, as in step. 5 of art. 542. by making f or $1 - \frac{n}{\mathrm{N}} - \frac{3nv^2}{8g\mathrm{N}d} = 0$, which gives $v = \sqrt{\left(g \cdot 8d \cdot \frac{\mathrm{N}-n}{3n}\right)} = \sqrt{a}$. The same value is also obtained by making the fluxion of v^2, or of $a-ac^{-2bs}$, $= 0$. And the same value of v is obtained by making s infinite, for then $c^{-2bs} = 0$. But this velocity $\sqrt{a}$ cannot be attained in any finite time, and it only denotes the velocity to which the general value of v or $\sqrt{(a-ac^{-2bs})}$ continually approaches. It is evident, however, that it will approximate towards it the faster, the greater b is, or the less d is; and that, the diameters being very small, the bodies descend by nearly uniform velocities, which are directly in the subduplicate ratio of the diameters.

5. To find the time t; we have $\dot{t} = \frac{\dot{s}}{v} = \sqrt{\frac{1}{a}} \times \frac{\dot{s}}{\sqrt{(1-c^{-2bs})}}$ Then, to find the fluent of this fluxion, put $z = \sqrt{(1-c^{-2bs})} = \frac{v}{\sqrt{a}}$, or $z^2 = 1 - c^{-2bs}$; hence $z\dot{z} = b\dot{s}c^{-2bs}$, and $\dot{s} = \frac{z\dot{z}}{bc^{-2bs}} = \frac{1}{b} \cdot \frac{z\dot{z}}{1-z^2}$; consequently $\dot{t} = \frac{1}{b\sqrt{a}} \cdot \frac{\dot{z}}{1-z^2}$, and therefore the fluent is $t = \frac{1}{2b\sqrt{a}} \times \text{log.}\ \frac{1+z}{1-z} = \frac{1}{2b\sqrt{a}} \times \text{log.}\ \frac{1+\sqrt{(1-c^{-2bs})}}{1-\sqrt{(1-c^{-2bs})}} = \frac{1}{2b\sqrt{a}} \times$ log. $\frac{\sqrt{a}+v}{\sqrt{a}-v}$, which is the general expression for the time.

Note. If the globe be so light as to ascend in the fluid, it is only necessary to change the signs of the first two terms in the value of f, or the accelerating force, by which it becomes $f = \frac{n}{N} - 1 - \frac{3nv^2}{8gNd}$; and then to proceed in all respects as before.

For more examples, see Dr. *Hutton's Select Exercises;* whence several articles in this chapter were taken.

The theory of the resistance of fluids when considered in its utmost extent, is very intricate and perplexing. Besides the propositions on this subject in *Newton's Principia,* lib. 2, the reader may be referred to the theory of *John Bernoulli* in his Dissertation on the Communication of Motion, and that of M. *D'Alembert* in his *Hydrodynamica.* The latter theory is genuine and unexceptionable, but extremely intricate, requiring a minute acquaintance with the most abstruse analysis; and it furnishes only one *new* proposition that is of any practical utility. The theory of M. Georges Juan in his *Examen Maritime,* and that of M. *La Grange,* a sketch of which may be seen in *Prony's Architecture Hydraulique,* include all the considerations that ought to be comprised in such discussions: but they present numerous difficulties inseparable from the analysis itself, and furnish but few practical results.

CHAPTER VI.

Experiments on the Resistance of Fluids.

552. In the preceding chapter we have given some of the most useful propositions in the ordinary theory of the resistance of fluids, and have referred to the works of other authors in which more rigorous theories are exhibited. But in all theories respecting the motions and actions of fluids, there are assumptions which depend on the judgment of the investigator; so that it is always proper to put every such theory to the test of experiments. If it be highly desirable that these branches of science be reduced to rules capable of being employed in practice with certainty, and if, as is our opinion, this can never be completely attained by theory alone, it becomes a matter of great importance to subject these points to an extensive and varied series of experiments; to discuss these experiments with care, and to compare them with the theory adopted, in order to discover where it is deficient, and to supply the deficiencies. A multiplicity of facts attentively analysed, and reduced as far as possible to general laws, may correct the results of theory, or compose of themselves the outlines of a kind of theory which may be readily adapted to the usual occasions of practice. Under these impressions we have already stated the results of experiments on the effluence of fluids, and the motion of water wheels (chaps. 2 and 4, book IV.); and shall now exhibit the most curious and important results which have been deduced from experiments on the resistance of fluids.

553. Experiments on this subject are by no means numerous; at least such as can be depended on for the foundation of any practical application. The first that have this character are those published by Mr. Robins in 1742, in his treatise on gunnery. They were repeated with some additions by the Chevalier Borda, and some account of them published in the Memoirs of the Academy of Sciences in 1763. In the Philosophical Transactions of the Royal Society of London, vol. lxxiii. there are some experiments of the same kind on a larger scale by Mr. Edgeworth.

In all these experiments the resistances were found to be very nearly in the proportion of the squares of the velocities; but they were found considerably greater than the weight of the column of air whose height would produce the velocity in a falling body. Mr. Robins's experiments on a square of 16 inches, describing 25·2 feet per second, indicate the resistance to be to this weight nearly as 4 to 3. Borda's experiments upon the same surface, state the disproportion to be still greater.

The resistances are found not to be in the proportion of the surfaces, but increase considerably faster. Surfaces of 9, 16, 36, and 81 inches, moving with one velocity, had resistances in the proportion of 9, 17½, 42¾, and 104¾.

Now as this deviation from the proportion of the surfaces increases with great regularity, it is most probable that it continues to increase in surfaces of still greater extent; and these are the most generally to be met with in practice in the action of wind on ships and mills.

Borda's experiments on 81 inches shew that the impulse of wind moving one foot per second, is about $\frac{1}{500}$ of a pound on a square foot. Therefore, to find the impulse on a foot corresponding to any velocity, divide the square of the velocity by 500, and we obtain the impulse in pounds. Mr. Rouse of Leicestershire made many experiments, which are mentioned with great approbation by Mr. Smeaton. His great sagacity and experience in the erection of windmills oblige us to pay a considerable deference to his judgment. These experiments confirm our opinion, that the impulses increase faster than the surfaces. The following table was calculated from Mr. Rouse's observations, and may be considered as pretty near the truth.

Velocity in Ft.	Impulse on a Ft. in Pounds.
0	0·000
10	0·129
20	0·915
30	2·059
40	3·660
50	5·718
60	8·234
70	11·207
80	14·638
90	18·526
100	22·872
110	27·675
120	32·926
130	38·654
140	44·830
150	51·462

If we multiply the square of the velocity in feet by 16, the product will be the impulse or resistance on a square foot in grains, according to Mr. Rouse's numbers.

554. The greatest deviation from the theory occurs in the oblique impulses. Mr. Robins compared the resistance of a wedge, whose angle was 90°, with the resistance of its base, and instead of finding it less in the proportion of $\sqrt{2}$ to 1, as determined by the theory, he found it greater in the proportion of 55 to 68 nearly; and when he formed the body into a pyramid, of which the sides had the same surface and the same inclination as the sides of the wedge, the resistance of the base and face were now as 55 to 39 nearly: so that here the same surface with the same inclination had its resistance reduced from 68 to 39 by being put into this form. Similar deviations occur in the experiments of the Chevalier Borda; and it may be collected from both, that the resistances diminish more nearly in the proportion of the sines of incidence than in the proportion of the squares of those sines.

The irregularity in the resistance of curved surfaces is as great as in plain surfaces. In general, the theory gives the oblique impulses on plane surfaces much too small, and the impulses on curved surfaces too great. The resistance of a sphere does not exceed the fourth part of the resistance of its great circle, instead of being its half; but the anomaly is such as to leave hardly any room for calculation. It would be very desirable to have the experiments on this subject repeated in a greater variety of cases, and on larger surfaces, so that the errors of the experiments may be of less consequence.

Mr. Robins having proved that in very great changes of velocity the resistance does not accurately follow the duplicate ratio of the velocity, lays down two positions, which he supposes may be of some service, particularly in the practice of artillery, till a more complete and accurate theory of resistance, and the changes of its augmentation, may be discovered. The first of these is, that till the velocity of the projectile surpass 1100 or 1200 feet per second, the resistance may be esteemed in the duplicate ratio of the velocity. The second is, that when the velocity exceeds 1100 or 1200 feet in a second, then the absolute quantity of the resistance will be nearly 3 times as great as it should be from a comparison with the smaller velocities. We shall soon see, however, that there is no *abrupt* change in the law of resistance; but that it is slow and continual from the smallest to the greatest velocities.

555. Dr. *Hutton*, late of the Royal Military Academy, has likewise prosecuted the subject of the resistance of the air experimentally, and that to a considerable extent. His experiments were made not only with the whirling machine, invented by Mr. Robins, but with cannon balls of different weights, from 1lb. to 6lb.; also with figures of various shapes, and with planes set at a variety of angles of inclination to the path of motion. From these curious experiments the doctor has ascertained the resistance of bodies to all velocities, from 1 to 2000 feet per second; the bodies being different, and their faces at different angles of inclination. Some of his general tables and conclusions deduced from those experiments are as follow.

TABLE I. *Resistances of different bodies.*

Veloc. per sec.	Small hemis. flat side.	Large hemis.		Cone.		Cylinder.	Whole globe.	Resis. as the power of the veloc.
		flat side.	round side.	vertex.	base.			
feet.	oz.	oz.	oz.	oz.	oz.	oz.	oz.	
3	·028	·051	·020	·028	·064	·050	·027	
4	·048	·096	·039	·048	·109	·090	·047	
5	·072	·184	·063	·071	·162	·143	·068	
6	·103	·211	·092	·098	·225	·205	·094	
7	·141	·284	·123	·129	·298	·278	·125	
8	·184	·368	·160	·168	·382	·360	·162	
9	·233	·464	·199	·211	·478	·456	·205	
10	·287	·573	·242	·260	·587	·565	·255	
11	·349	·698	·292	·315	·712	·688	·310	2·052
12	·418	·836	·347	·376	·850	·826	·370	2·042
13	·492	·988	·409	·440	1·000	·979	·435	2·036
14	·573	1·154	·478	·512	1·166	1·145	·505	2·031
15	·661	1·336	·552	·589	1·346	1·327	·581	2·031
16	·754	1·538	·634	·673	1·546	1·526	·663	2·033
17	·853	1·757	·722	·762	1·763	1·745	·752	2·038
18	·959	1·928	·818	·858	2·002	1·986	·848	2·044
19	1·073	2·998	·922	·959	2·260	2·246	·949	2·047
20	1·196	2·542	1·033	·1069	2·540	2·528	1·057	2·051
Mean propor. Nos.	140	288	119	126	291	285	124	2·040
1	2	3	4	5	6	7	8	9

In this table are contained the resistances to several forms of bodies, when moved with several degrees of velocity, from 3 feet per second to 20. The names of the bodies are at the tops of the columns, as also which end went foremost through the air; the different velocities are in the first column, and the resistances on the same line, in their several columns, in avoirdupois ounces and decimal parts. So on the first line are contained the resistances when the bodies move with a velocity of 3 feet in a second, viz. in the 2d column for the small hemisphere, of $4\frac{3}{4}$ inches diameter, its resistance ·028 oz. when the flat side went foremost; in the 3d and 4th columns the resistances to a larger hemisphere, first with the flat side, and next the round side foremost, the diameter of this, as well as all the following figures being $6\frac{5}{8}$ inches, and therefore the area of the great circle $=32$ sq. inches, or $\frac{2}{9}$ of a square foot; then in the 5th and 6th columns are the resistances to a cone, first its vertex and then its base foremost, the altitude of the cone being $6\frac{5}{8}$ inches, the same as the diameter of its base: in the 7th column the resistance to the end of the cylinder, and in the 8th that against the whole globe or sphere. All the numbers shew the real weights which are equal to the resistances; and at the bottoms of the columns are placed proportional numbers, which shew the mean proportions of the resistances of all the figures to one another with any velocity. Lastly, in the 9th column are placed the exponents of the power of the velocity which the resistances in the 8th column bear to each other, viz. which that of the 10 feet velocity bears to each of the following ones, the medium of all of them being as the 2·04 power of the velocity; that is, very little above the square or second power of the velocity, so far as the velocities in this table extend.

556. From this table the following inferences are easily deduced.

I. That the resistance is nearly in the same proportion as the surfaces; a small increase only taking place in the greater surfaces, and for the greater velocities. Thus, by comparing together the numbers in the 2d and 3d columns for the bases of the two hemispheres, the areas of which bases are in the proportion of $17\frac{}{4}$ to 32, or 5 to 9 very nearly, it appears that the numbers in those two columns expressing the resistances, are nearly as 1 to 2, or 5 to 10, as far as the velocity of 12 feet; but after that the resistances on the greater surface increase gradually more and more above that proportion.

II. The resistance to the same surface with different velocities, is, in these slow motions, nearly as the square of the velocity; but gradually increases more and more above that proportion as the velocity increases. This is manifest from all the columns; and the index of the power of the velocity is set down in the 9th column, for the resistances in the 8th, the medium being 2·04; by which it appears that the resistance to the sam body is, in these slow motions, as the 2·04 power of the velocity, or nearly as the square of it.

III. The round ends, and sharp ends, of solids, suffer less resistance than the flat or plane ends, of the same diameter; but the sharper end has not always the less resistance. Thus, the cylinder and the flat ends of the hemisphere and cone have more resistance than the round or sharp ends of the same; but the round side of the hemisphere has less resistance than the sharper end of the cone.

IV. The resistance on the base of the hemisphere, is to that on the round, or whole sphere, as $2\frac{1}{3}$ to 1, instead of 2 to 1, as the theory gives that relation. Also the experimented resistance on each of these, is nearly ^{1} more than the quantity assigned by the theory.

V. The resistance on the base of the cone is to that on the vertex nearly as $2\frac{3}{10}$ to 1; and in the same ratio is radius to the sine of the angle of inclination of the side of the cone to its path or axis. So that, in this instance, the resistance is directly as the sine of the angle of incidence, the transverse section being the same.

VI. When the hinder parts of different bodies are of different forms, the resistances are different, though the fore parts be exactly alike and equal; owing probably to the different pressures of the air on the hinder parts. Thus, the resistance to the fore part of the cylinder is less than on the equal flat surface of the cone, or of the hemisphere; because the hinder part of the cylinder is more pressed or pushed by the following air than those of the other two figures; also, for the same reason, the base of the hemisphere suffers a less resistance than that of the cone, and the round side of the hemisphere less than the whole sphere.

VII. If d be the diameter of any ball in inches, and v the velocity in feet with which it moves in the air, then will the resistance it experiences in avoirdupois pounds be denoted by $(\cdot000007565\,v^2 - \cdot00175\,v)\,d^2$. Thus an iron ball of 2·78 inches diameter, or 3lbs. weight, moving with a velocity of 1800 per second, would be resisted by a force equivalent to 176lbs., or more than 58 times its own weight.

557. Table II. *Resistances both by experiment and theory, to a globe of* 1·965 *inches diameter.*

Veloc. per sec. in feet.	Resist. by exper. oz.	Resist. by theory. oz.	Ratio of exper. to theory.	Resist. as the power of the veloc.
5	0·006	0·005	1·20	
10	0·024½	0·020	1·23	
15	0·055	0·044	1·25	
20	0·100	0· 79	1·27	
25	0·157	0·123	1·28	2·022
30	0·23	0·177	1·30	2·055
40	0·42	0·314	1·33	2·068
50	0·67	0·491	1·36	2·075
100	2·72	1·964	1·38	2·059
200	11	7·9	1·40	2·041
300	25	18·7	1·41	2·039
400	45	31·4	1·43	2·039
500	72	49	1·47	2·044
600	107	71	1·51	2·051
700	151	96	1·57	2·059
800	205	126	1·63	2·067
900	271	159	1·70	2·077
1000	350	196	1·78	2·086
1100	442	238	1·86	2·095
1200	546	283	1·90	2·102
1300	661	332	1·99	2·107
1400	785	385	2·04	2·111
1500	916	442	2·07	2·113
1600	1051	503	2·09	2·113
1700	1186	568	2·08	2·111
1800	1319	636	2·07	2·108
1900	1447	709	2·04	2·104
2000	1569	786	2·00	2·098

In the first column of this table are contained the several velocities, gradually from 0 up to the great velocity of 2000 feet per second, with which a ball or globe moved. In the 2d column are the experimented resistances in avoirdupois ounces. In the 3d column are the correspondent resistances, as computed by the theory. In the 4th column are the ratios of these two resistances, or the quotients of the former divided by the latter. And in the 5th or last, the indexes of the power of the velocity which is proportional to the experimented resistance; which are found by comparing the resistance of 20 feet velocity with each of the following ones.

From the 2d, 3d, and 4th columns it appears, that at the beginning of the motion the experimented resistance is nearly equal to that computed by theory; but that as the velocity increases, the experimented resistance gradually exceeds the other more and more, till at the velocity of 1300 feet the former becomes just double the latter; after which the difference increases a little further, till about the velocity of 1600 or 1700, where that excess is the greatest, and is rather less than $2\frac{1}{10}$; after this, the difference decreases gradually as the velocity increases, and at the velocity of 2000 the former resistance again becomes just double the latter.

From the last column it appears that near the beginning, or in slow motions, the resistances are nearly as the square of the velocities; but that the ratio gradually increases, with some small variation, till at the velocity of 1500 or 1600 feet it becomes as the $2\frac{1}{9}$ power of the velocity nearly, which is its highest ascent; and after that it gradually decreases again, as the velocity goes higher. And similar conclusions have also been derived from experiments with larger balls or globes.

558. TABLE III. *Resistance to a plane, set at various angles of inclination to its path.*

Angle with the path.	Experim. resistances. oz.	Resist. by this formula. $\cdot 84s^{1\cdot 84ac}$	Sines of the angles to radius ·840.
0°	·000	·000	·000
5	·015	·009	·073
10	·044	·035	·146
15	·082	·076	·217
20	·133	·131	·287
25	·200	·199	·355
30	·278	·278	·420
35	·362	·363	·482
40	·448	·450	·540
45	.534	·535	·594
50	·619	·613	·643
55	·684	·680	·688
60	·729	·736	·727
65	·770	·778	·761
70	·803	·808	·789
75	·823	·826	·811
80	·835	·836	·827
85	·839	·839	·838
90	·840	·840	·840

In the 2d column of this table are contained the actual experimented resistances in ounces to a plane of 32 square inches, or $\frac{2}{9}$ of a square foot, moved through the air with a velocity of exactly 12 feet per second, when the plane was set so as to make, with the direction of its path, the corresponding angles in the first column.

And from these Dr. Hutton deduced this formula, viz. $\cdot 84s.^{1\cdot 842c}$, which brings out very nearly the same numbers, and is a general theorem for every angle, for the same plane of $\frac{2}{9}$ of a foot, and moved with the same velocity of 12 feet in a second of time; where s is the sine, and c the cosine of the angles of inclination in the first column.

If a theorem be desired for any other velocity v, and any other plane whose area is a, it will be thus: $\frac{1}{38}av^2s^{1\cdot 842c}$, or more nearly $\frac{1}{42}av^{2\cdot 04}s^{1\cdot 842c}$; which exhibits the resistance very nearly to any plane surface whose area is a, moved through the air with the velocity v, in a direction making with that plane an angle, whose sine is s and cosine c.

If the fluid be water, or any other whose density is different from that of air, the formula for the resistance must be increased in proportion to the density.

By this theorem were computed the numbers in the 3d column, which, it is evident, agree very nearly with the experimental resistances in the 2d column; excepting in two or three of the small numbers near the beginning, which are of the least consequence. In all other cases, the theorem gives nearly the true resistance. In the 4th or last column, are entered the sines of the angles of the first column, to the radius ·84, in order to compare them with the resistances in the other columns. From whence it appears that those resistances bear no sort of analogy to the sines of the angles, nor yet to the squares, or any other power of the sines. In the beginning of the columns, the sines much exceed the resistances all the way till the angle is between 55° and 60°; after which the sines are less than the resistances all the way to the end, or, till the angle becomes a right angle. See *Hutton's Dictionary*, art. Resistance *of Fluids*, and his 8vo. Tracts, vol. iii. pp. 163—243.

See also a paper by Mr. Vince on this subject in the Phil. Trans. for 1798, or Nicholson's Journal, vol. iii. p. 506.

559. Messrs. *D'Alembert*, *Condorcet*, and *Bossut*, pursuant to the directions of M. *Turgot*, comptroller-general of finances, made, in the year 1775, a series of experiments, in order to perfect internal navigation; and the resistance of fluids was the

principal object of their researches. Their experiments, however, were confined to very small velocities, and to *direct* resistances; so that their results, as might be expected, did not differ sensibly from those furnished by the ordinary theory. But, in 1778, Bossut and Condorcet directed their attention to oblique resistances, and undertook "A new series of experiments, "chiefly intended to discover the law according to which the "resistance experienced by an angular prow diminishes in pro- "portion as the angle of that prow becomes more acute; the "fluid being indefinite, such as the sea."

These experiments were made in a great reservoir situated on the north side of the ancient Boulevards of Paris. The vessels were prismatic (excepting the prow, which was an isosceles wedge), and their motion was produced by the descent of a weight, which, by means of a cord, and different pullies, caused the vessel to advance. During the first instants of the descent of the weight, the motion was accelerated; but the vessel soon acquired a uniform velocity, and the resistance of the water, relative to that velocity, makes the equilibrium with the moving force of the weight due to gravity, &c.

The following table comprises 69 experiments of Bossut on simple angular prows. The first five were made with a boat in form of a rectangular parallelopiped, the length of which was 4 feet, breadth 2 feet, depth in the fluid 2 feet, and height of the part jutting out of the fluid about 7 feet.

The other 64 experiments were made with 14 vessels in form of a prism, the prows being isosceles wedges, having their vertices forwards: these vertical angles increased gradually from 12°, 24°, 36°, &c. to 180°; the length of each vessel was 4 feet, breadth 2 feet, depth in the water 2 feet, height out of the water about 7 feet: all in Paris measure. The titles of the columns require no explanation.

Angle of the prow.	Weight of the mover in pounds.	Time of describing 96 feet uniformly.	Angle of the prow.	Weight of the mover in pounds.	Time of describing 96 feet uniformly.
180°	61·8	78·08″	96°	60·5	63·00″
	112·5	57·51		110·5	46·45
	162·5	47·44		162·5	38·05
	212·5	41·49		212·5	32·66
	262·6	37·32		262·5	29·27
168°	62·5	77·50	84°	60·5	60·55
	112·5	56·95		110·5	44·56
	162·5	47·22		162·5	35·78
	212·5	41·26		212·5	31·25
	262·5	37·12		262·5	27·51
156°	62·5	75·09	72°	62·5	57·50
	112·5	56·15		112·5	42·75
	162·5	46·44		162·5	34·85
	212·5	41·03		212·5	29·65
	262·5	36·52		262·5	25·86
144°	62·5	73·38	60°	60·5	55·45
	112·5	54·75		112·5	40·04
	162·5	45·35		162·5	33·05
	212·5	39·58		212·5	28·25
	262·5	37·57		262·5	24·77
132°	60·5	72·08	48°	62·5	52·51
	111·5	53·25		112·5	38·05
	162·5	43·75		162·5	31·61
	212·5	38·26		212·5	27·56
	262·5	34·30		262·5	24·30
120°	62·0	68·32	36°	62·5	51·15
	112·5	50·84		112·5	36·96
	162·5	41·84		162·5	30·53
	212·5	36·62	24°	61·5	49·48
	262·5	32·77		112·5	35·76
108°	61·5	65·85		162·5	30·23
	112·5	48·75	12°	62·5	49·38
	162·5	39·50		112·5	35·33
	212·5	34·46		162·5	30·01
	262·5	31·05			

The direct resistance on the end of the vessel without an isosceles prow may be represented by any arbitrary number, as 10000 for instance

And if the angle of the prow, or that formed at the fore-part of the vessel or summit of the triangular section, be x, the resistance will be expressed by the formula 10000 $\cos^2 x$. To correct this expression by means of the foregoing table, we may observe that, when the angle x undergoes a variation of 12°, each of the angles at the base of the isosceles prow will vary 6°: calling this latter variation q, M. Bossut deduced from the table the additive correction $3{\cdot}153\left(\frac{x}{q}\right)^{3{\cdot}25}$, x and q being expressed in decimal parts of the radius unity. Thus the whole resistance against the prow, estimated in the opposite direction to that of the motion of the boat, is equal to $10000\cos^2 x + 3{\cdot}153\left(\frac{x}{q}\right)^{3{\cdot}25}$. And this formula will be sufficiently exact when the extreme angle of the prow is greater than 12°: but when that angle is less than 12° the theorem will err in excess.

560. M. Lacroix has inserted in the Bulletin de la Société Philomathique, a comparative view of the experiments of Bossut and Vince, on the resistance of fluids striking on oblique surfaces. He does not appear to have assigned a sufficient reason for the difference of the results of those experiments; but the subject is of so much importance, as to deserve such an examination in its present state as may serve to assist in the prosecution of farther experiments.

The first approximation, says Dr. T. Young, to a determination of the effect of the resistance to a body of a given section, terminated by oblique planes, is to suppose each particle of the fluid to impinge once on the surface, and then to retire for ever: on this supposition, the resistance ought to vary as the square of the cosine of the angle of incidence.

Another part of the resistance is occasioned by the adhesion of the particles of the fluid; this may be supposed to vary, as the product of the secant and the sine of the angle of incidence; that is, as its tangent.

A third part depends on the form of the posterior surface of the body, and upon the unknown irregularities produced in the motions of the particles of the fluid, by the difference of the forms of its anterior part. It may be expected, that this negative pressure will be nearly uniform, when the shape of the posterior part of the body remains unaltered, as in Bossut's experiments; but that, when a thin surface is employed, as in Mr. Vince's apparatus, it will be somewhat diminished by the obliquity of that surface, even supposing the transverse projection of the surface to remain unaltered. This portion, however, may naturally be expected to be liable to great irregularities;

and it appears to be somewhat increased when the thin surface is inclined in a small angle only.

Mr. Romme has remarked, that the facility with which the particles of the fluid can escape before the moving body, is proportional to the angular space of the fluid which remains open to admit them, and that therefore the resistance must vary in proportion to this angle. Without allowing the truth of the observation in its whole extent, we may with propriety inquire whether or no the portion of the pressure derived from impulse may not in part depend on some simple function of the angle of incidence; and whether the whole resistance to an oblique surface may not be considered as composed of a constant portion, a portion varying with the tangent of the angle of incidence, and a third portion proportionate to the square of the cosine, diminished in the ratio of a power, or other function, of the angle of incidence. And it will appear upon inquiry, that if we take one fifth of the radius, increased by one-twenty-fifth of the tangent, and add to it four-fifths of the square of the cosine, diminished in the ratio of the circumference of a circle increased by the angle of incidence, to the simple circumference, we may approach always within about one-fiftieth to the number expressing the oblique resistance, until the angle of incidence becomes greater than 80°. Thus, the direct resistance being unity, and a the angle of incidence, the oblique resistance will $\cdot 2 + \cdot 04 \tan a + 288 \cos^2 a : (360 + a^\circ)$. A formula, somewhat more accurate than this, deduced from experiment only, is $r = \cos^2 a + \cdot 0000004217 a^{3\cdot 18}$: the quantity added to the square of the cosine being a little less than the millionth of the cube of the angle of incidence, expressed in degrees. The result of these two theorems are compared with the experiments of Bossut, and the theorem of Eytelwein (viz. $\cos^2 a + 4$ versin a) in the following table.

Angle.	$\cos^2 a$.	Tan a.	Form. 1.	Form. 2.	Bossut.	Eytelw.
0°	1·0000	·000	1·0000	1·0000	1·0000	1·0000
6	·9890	·105	·9824	·9891	·9893	·9910
12	·9568	·212	·9492	·9580	·9578	·9656
18	·9045	·325	·9022	·9086	·9084	·9241
24	·8346	·445	·8438	·8449	·8446	·8690
30	·7500	·577	·7769	·7710	·7710	·8036
36	·6544	·726	·7049	·6919	·6925	·7308
42	·5523	·900	·6317	·6135	·6148	·6551
48	·4478	1·111	·5606	·5414	·5433	·5802
54	·3455	1·376	·4985	·4816	·4800	·5103
60	·2500	1·732	·4407	·4403	·4404	·4500

Angle.	$\cos^2 a$	Tan a.	Form. 1.	Form. 2.	Bossut.	Eytelw.
66	·1654	2·346	·3924	·4231	·4240	·4026
72	·0955	3·078	·3869	·4344	·4142	·3719
78	·0432	4·705	·4166	·4816	·4063	·3600
84	·0109	9·514	·5875	·5658	·3999	·3693

Mr. Romme found, by numerous experiments, that when the magnitude of the greatest section of a floating body, and its distance from the angular points, were constant, the form of the outline of any section of the body, whether composed of right lines or of curves of any kind, was either wholly, or very nearly, indifferent to the magnitude of the resistance: hence he infers, that in the construction of ships, the curve of the sides ought to be determined from considerations independent of the resistance.

In experiments like those of Mr. Vince, the circumstances are materially different: but the accuracy of Mr. Vince's experiments on water is, in some measure, confirmed by a comparison with those of Schober, which were made in a similar manner on air. The results of both these investigations are here exhibited in a table, and compared with a coarse approximation from this formula $r = 4\cdot + 6 \cos a$, and with the results of Dr. Hutton.

Angle.	Cos a.	Form. 3.	Schober.	Vince.	Hutton.
0°	1·0000	1·0000	1·0000	1·0000	1·000
10	·9848	·9909	·9854	·9787	1·010
20	·9397	·9638	·9808	·9743	1·068
30	·8660	·9196	·9200	·9463	1·003
40	·7660	·8596	·8308	·8729	·963
50	·6428	·7857	·7563	·7869	·830
60	·5000	·7000	·6738	·6610	·662
70	·3420	·6052	·6359	·4913	·453
80	·1736	·5042	·5833	·2779	·304

(Vide *Journals of the Royal Institution.*) Dr. Hutton's experiments, when reduced to surfaces of given transverse projections, indicate at first an increase of resistance as the surface becomes more oblique: a circumstance analogous to what was remarked by Bossut in reference to float-boards, and mentioned already at the end of book iv. ch. 4.

561. Such are the conclusions and rules which have been deduced from some of the most judicious experiments on the resistance of fluids to the motion of bodies passing through

them. Considering the number and importance of these inferences with respect to the magnitude and extent of the experiments from which they were drawn, we cannot but earnestly wish that those who have leisure and opportunity to pursue such enquiries, would still further extend and vary their researches in this department of experimental philosophy. Much valuable knowledge may be expected to result from labours of this kind: for it is, probably, only by reiterated experiments and observations that sufficient data can be obtained to perfect the theory of two momentous branches of mechanical science, viz. those relating to the motion of military projectiles, and to the manœuvres of vessels at sea.

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END OF VOL I.

ERRATA IN VOL. I.

Page 32, line 9, *for* $C : C$, *read* $C' : C$.
—— 36, line 23, *for* $AP . c'$, *read* $AP' . c'$.
—— 33, line 2, *for* $(C' \times SQ)$, *read* $+ (C' \times SQ)$.
—— 94, line 15, *for* Cr. 4., *read* Cor. 4.
—— 120, line 24, take away the sign + before $\frac{w + 2u}{w}$.
—— 155, line 16, for $A \frac{\dot{y}}{}$, *read* $A \frac{\dot{y}}{\dot{x}}$.
—— 158, line 25, *for* $= \frac{\ddot{x}}{} + \frac{\dot{t}}{t}$, *read* $= \frac{\ddot{x}}{\dot{y}} + \frac{\dot{t}}{t}$.
—— 162, line 4 of note, *read* l'Equilibre des
—— 165, line 6, *for* $f \propto BV$, *read* $F \propto BV$.
—— 165, line 11, *for* $\frac{1}{v} \propto B$, *read* $\frac{1}{V} \propto B$.
—— 187, line 6 from bottom, *for* $\left(\frac{\dot{s}}{\dot{t}}\right)$. *read* $\left(\frac{\dot{s}}{t}\right)$
—— 226, line 5 from bottom, *for* force, *read* forces.
—— 228, line 4 from bottom, *for* ay be, *read* may be.
—— 279, line 1, *for* $x = \backsim$, *read* $x = \infty$.
—— 283, line 21, *for* as at its, *read* as its.
—— 284, line 26, *for* $r \surd = \frac{1}{3}$, *read* $r \surd \frac{1}{3} =$.
—— 316, line 26, put $\dot{z}$ for $\dot{x}$ in the denominator.
—— 513, line 14, *for* invertigation, *read* investigation.
—— 547, line 3, *for* $\frac{c^{x} - 1}{bx}$, *read* $\frac{c^{bx} - 1}{c^{bx}}$.

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Whitefriars, London.

IV – 14-11